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Dominator Chromatic Number of Circular-Arc Overlap Graphs

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Abstract

A dominator coloring problem of the graph G is a proper coloring of the graph, where every vertex of the graph inevitably dominates an entire color class. It is observed that the number of color classes selected is of minimum order and when only it suits to be the dominator chromatic number. The present paper brings to the fore, the findings related to dominator chromatic number of some special classes of circular-arc overlap graphs, upholding the concepts on the bounds of dominator chromatic number and the relation between the chromatic number and the dominator chromatic number.

Keywords: Circular-arc overlap graphs, independent set, chromatic number, dominator chromatic number

1. Introduction

A dominating set S of a graph G with vertex set V (G) and edge set E (G) is a subset of vertex set V such that every vertex of V outside S is dominated by some vertex of S. The domination number, $\gamma(G)$, is the minimum size of the dominating sets of G. Dominating set is NP - complete on arbitrary graphs and on several classes of graphs. The topic of dominating set has long been of interest to researchers (O.Ore 1962).

A graph coloring is a mapping f from set of vertices V (G) to the set of colors, say C. The coloring is a proper coloring if no two adjacent vertices are assigned with the same color. A k-coloring of G is a proper coloring of G that uses at most k colors. The chromatic number of G, $\chi(G)$, is the minimum number of colors required to properly color the graph. Graph k–colorability is NP – complete in the general case although the problem is solvable in polynomial time in many cases (M. R. Garey and D. S. Johnson 1978).

Dominator coloring of a graph is a proper coloring of a graph in which each vertex of the graph dominates every vertex of some color class. The dominator chromatic number of the graph G, $\chi_d(G)$, is the minimum number of colors needed for a dominator coloring of a graph.

Graphs presented in this article are all finite and connected circular**-** arc overlap graphs. Circular-arc overlap graphs are a new class of overlap graphs introduced by Kashiwa Bara and Masuda, defined for a set of arcs on a circle. A graph is a circular - arc graph, if it is the intersection graph of a finite set of arcs on a circle. A circular-arc overlap graph is a specific enclosure of circular - arc graph; it is an overlap graph defined for a set of arcs on a circle. That is**,** there exists one arc for each vertex of G and two vertices in G are adjacent in G, if and only if the corresponding arcs intersect and one is not contained in the other. A representation of a graph with arcs helps in the solving of combinatorial problems on the graph.

Resolving the dominator chromatic number of some special classes of circular- arc overlap graphs has been the main focus of the present paper. Nature of the arcs of the graph, bounds for dominator chromatic number and the relation between the chromatic number and the dominator chromatic number paved the way for present findings.

Motivation for the generation of ideas for presenting this paper is owed to (R. Gera et al. 2006) and is inspired by (E. Cockayne. S. Hedetniemi and S. Hedetniemi 1979) and (S. M. Hedetniemi et al. 2006).

2 Preliminary results

2.1 Bounds for dominator chromatic number of a graph

Let G be a connected circular – arc overlap graph of order $k \geq 2$. Although, if each vertex is assigned with a different color, $\chi_d(G) = n$. Moreover, as G is a connected graph, at least two vertices are adjacent. In such cases, $\chi_d(G) \geq 2$. Thus $2 \leq \chi_d(G) \leq n$.

2.2 Relation between chromatic number and dominator chromatic number

For any graph G,

$$\chi\,(G) \leq \chi_d(G) \text{ (R. Gera et al.2006).}$$

Every dominator coloring of a graph G is a proper coloring, but every proper coloring need not be a dominator coloring. Hence the result follows.

2.3 Relation between domination number, chromatic number and dominator chromatic number

For any connected graph G, from the definitions of domination number, chromatic number and dominator chromatic number, it follows that

Max $\{\chi\,(G), \gamma\,(G)\} \leq \chi_d(G) \leq \chi\,(G) + \gamma(G)$ (R. M. Gera 2007).

3 Dominator chromatic numbers of some special classes of circular-arc overlap graphs

Theorem 3.1: Let $A = \{A_1, A_2, \ldots, A_k\}$ be a circular – arc overlap family corresponding to a circular – arc overlap graph G. Then there exists an integer m, where $1 \leq m < k$ such that the arc A_i contains the arc A_{i+1} for i = 1, 2, …., m-1, m+1, m+2, …, k-1 and the arc A_i dominates each of the arcs A_{m+1}, A_{m+2}, …….., A_k for i = 1, 2, …, m if and only if the dominator chromatic number of the graph G is 2.

Proof: Let there exist an integer m, where $1 \leq m < k$ satisfying the conditions mentioned in the theorem. Then, the arc A_1 contains the arc A_2, the arc A_2 contains the arc A_3, ……., the arc A_{m-1} contains the arc A_m. It follows that

the arc A_1 contains the arcs $A_2, A_3, \ldots, A_m$,

the arc A_2 contains the arcs $A_3, A_4, \ldots, A_m$

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the arc A_{m-1} contains the arc A_m.

That is the arc A_i doesn't overlap any one of the arcs that belong to the set of arcs $\{A_1, A_2, \ldots, A_m\}$ for

i = 1, 2, …., m. Therefore the set of arcs $\{A_1, A_2, \ldots, A_m\}$ is an independent set of arcs. With the similar type of argument, it can be proved that the set of arcs $\{A_{m+1}, A_{m+2}, \ldots, A_k\}$ is also an independent set of arcs. All the arcs can be properly colored with two colors. Implies, $\chi\,(G) \leq 2$ ……………… (1)

For any connected graph the chromatic number is greater than or equal to 2.

Here G is a connected graph. Therefore, $\chi\,(G) \geq 2$ ………………. (2)

From (1) and (2),

$$\chi\,(G) = 2$$

with two color classes $\{A_1, A_2, \ldots, A_m\}$ and $\{A_{m+1}, A_{m+2}, \ldots, A_k\}$. Due to the second condition in the hypothesis it is clear that every arc in the set of arcs $\{A_1, A_2, \ldots, A_m\}$ dominates every arc belonging to the set of arcs $\{A_{m+1}, A_{m+2}, \ldots, A_k\}$. That is every arc in one color class dominates every arc in the other color class. Every arc dominates an entire color class. Consequently, $\chi_d\,(G) \leq 2$. But $\chi_d\,(G) \geq \chi\,(G) = 2$

Hence, χ_d (G = 2.

Conversely, let $\chi_d\,(G) = 2$. There exist two classes of arcs such that every arc in one color class dominates every arc in the other color class. That is each arc in one color class overlaps every arc in the other class. Hence there must exist an integer m satisfying the conditions mentioned in the theorem.

Theorem 3.2: Let $A = \{A_1, A_2, \ldots\ldots, A_k\}$ be a circular – arc overlap family corresponding to a

Circular – arc overlap graph G. Then there exist integers m and n, where $1 \leq m < n < k$ such that

(i) the arc A_i contains the arc A_{i+1} for $i = 1, 2, \ldots, m-1, m+1, m+2, \ldots, n-1, n+1, n+2, \ldots, k-1$;

(ii) the arc A_i dominates

each of the arcs $A_{m+1}, A_{m+2}, \ldots\ldots, A_k$ for $i = 1, 2, \ldots\ldots, m$ and

each of the arcs $A_1, A_2, \ldots\ldots, A_m, A_{n+1}, A_{n+2}, \ldots\ldots, A_k$ for $i = m+1, m+2, \ldots, n$

if and only if the dominator chromatic number of the graph G is 3.

Proof: Let the circular – arc overlap family $A = \{A_1, A_2, \ldots, A_k\}$corresponding to the circular – arc overlap graph G satisfy the conditions mentioned in the theorem. It follows that the sets of arcs $\{A_1, A_2, \ldots, A_m\}$, $\{A_{m+1}, A_{m+2}, \ldots\ldots, A_n\}$ and $\{A_{n+1}, A_{n+2}, \ldots\ldots, A_k\}$ are independent sets of arcs.

Implies, $\chi(G) \leq 3$ ………………(1)

Moreover arc A_1 dominates every arc belonging to the set of arcs

$\{A_{m+1}, A_{m+2}, \ldots, A_k\}$;

the arc A_{m+1} dominates every arc belonging to the set of arcs

$\{A_1, A_2, \ldots\ldots, A_m, A_{n+1}, A_{n+2}, \ldots, A_k\}$ and

the arc A_{n+1} dominates every arc belonging to the set of arcs

$\{A_1, A_2, \ldots\ldots, A_m, A_{m+1}, A_{m+2}, \ldots, A_n\}$.

It follows that if color C_1 is assigned to A_1, different colors, say C_2 and C_3 are to be assigned to the arcs A_{m+1} and A_{n+1} respectively. Then

$$\chi(G) \geq 3 \ \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Consequently, $\chi(G) = 3$, the sets of arcs $\{A_1, A_2, \ldots, A_m\}$, $\{A_{m+1}, A_{m+2}, \ldots\ldots, A_n\}$ and $\{A_{n+1}, A_{n+2}, \ldots, A_k\}$ being the three color classes of G. From the second condition in the hypothesis it is clear that every arc in one color class dominates every arc in the other color class. Every arc dominates an entire color class. Consequently, $\chi_d(G) \leq 3$.

But, $\chi_d(G) \geq \chi(G) = 3$. Hence, $\chi_d(G) = 3$.

Conversely, let $\chi_d(G) = 3$. There exist three classes of arcs such that every arc in one color class dominates every arc in the other color class. That is each arc in one color class overlaps every arc in the other color class. Hence there must exist integers m and n satisfying the conditions mentioned in the theorem.

Generalized Theorem 3.3: Let $A = \{A_1, A_2, \ldots\ldots, A_k\}$ be a circular – arc overlap family corresponding to a circular – arc overlap graph G. Then there exist integers $m_1, m_2, \ldots\ldots, m_{p-1}$ where

$1 \leq m_1 < m_2 <, \ldots, < m_{p-1} < k$ such that

(i) the arc A_i contains the arc A_{i+1}

for $i = 1, 2, \ldots, m_1-1, m_1+1, m_1+2, \ldots, m_2-1, m_2+1, m_2+2, \ldots, m_3-1, \ldots\ldots, m_{p-1}-1, m_{p-1}+1, m_{p-1}+2, \ldots, k-1$

(ii) the arc A_i dominates

each of the arcs $Am_1+1, Am_1+2, \ldots\ldots, A_k$ for $i = 1, 2, \ldots\ldots, m_1$;

each of the arcs $A_1, A_2, \ldots, Am_1, Am_2+1, Am_2+2, \ldots, A_k$ for $i = m_1+1, m_1+2, \ldots, m_2$;

……………………………………………………………………………………;

each of the arcs $A_1, A_2, \ldots Am_1, Am_1+1, Am_1+2, \ldots, Am_2, \ldots, Am_{p-2}+1, Am_{p-2}+2, \ldots\ldots, Am_{p-1}$

for $i = m_{p-1}+1, m_{p-1}+2, \ldots, k$

if and only if the dominator chromatic number of the graph G is p.

Proof: Let the circular – arc overlap family $A = \{A_1, A_2, \ldots, A_k\}$corresponding to the circular – arc overlap graph G satisfy the conditions mentioned in the theorem. From condition (i) it follows that the sets of arcs $\{A_1, A_2, \ldots\ldots, Am_1\}$, $\{Am_1+1, Am_1+2, \ldots\ldots, Am_2\}$, ……., $\{Am_{p-1}+1, Am_{p-1}+2, \ldots, A_k\}$ are independent sets of arcs. Implies

$$\chi(G) \leq p \ \ldots\ldots\ldots (1)$$

The arc A_1 dominates every arc belonging to the sets of arcs $\{Am_1+1, Am_1+2, \ldots, Am_2\}$, $\{Am_2+1, Am_2+2, \ldots, Am_3\}$, ……………………, $\{Am_{p-1}+1, Am_{p-1}+2, \ldots, A_k\}$;

the arc Am_1+1 dominates every arc belonging to the sets of arcs $\{A_1, A_2, \ldots, Am_1\}$, $\{Am_2+1, Am_2+2, \ldots, Am_3\}, \{Am_3+1, Am_3+2, \ldots, Am_4\}, \ldots, \{Am_{p-1}+1, Am_{p-1}+2, \ldots, A_k\}$;

……………………………………………………………………………… ;

the arc $Am_{p-1}+1$ dominates every arc belonging to the sets of arcs $\{A_1, A_2, \ldots, Am_1\}$, $\{Am_1+1, Am_1+2, \ldots, Am_2\}, \ldots, \{Am_{p-2}+1, Am_{p-2}+2, \ldots, Am_{p-1}\}$.

That is the arcs A_1, Am_1+1, Am_2+1, …….., $Am_{p-1}+1$ mutually dominate each other. To properly color the graph at least p different colors are needed. So, $\chi(G) \geq p$

Consequently, $\chi(G) = p$ Where the sets $\{A_1, A_2, \ldots, Am_1\}$, $\{Am_1+1, Am_1+2, \ldots, Am_2\}$, $\{Am_2+1, Am_2+2, \ldots, Am_3\}$, ……………, $\{Am_{p-1}+1, Am_{p-1}+2, \ldots, A_k\}$ are p different color classes of G. It follows from the remaining conditions that every arc in one color class dominates every arc in the other color class. Every arc dominates an entire color class. Consequently, $\chi_d(G) \leq p$. But, $\chi_d(G) \geq \chi(G) = p$. Hence, $\chi_d(G) = p$

Conversely, let $\chi_d(G) = p$. There exist p classes of arcs such that every arc in one color class dominates at least one color class. That is each arc in one color class overlaps every arc in some other color class. Hence there must exist integers m_1, $m_2, \ldots, m_{p-1}$ satisfying the conditions mentioned in the theorem.

3.3.1 Illustration: Let the circular arc overlap family $A = \{A_1, A_2, \ldots, A_{10}\}$ corresponding to the circular-arc overlap graph G be as in Figure 1. From Figure 1 it is clear that the circular arc family satisfies the conditions mentioned in the theorem for $m_1=1$, $m_2=3$ and $m_3=6$. Hence the dominator chromatic number of the graph is 4.

Verification: Clearly the sets of arcs $\{A_1\}$, $\{A_2, A_3\}$, $\{A_4, A_5, A_6\}$ and $\{A_7, A_8, A_9, A_{10}\}$ are independent sets of arcs. The graph can be properly colored with four colors. Thus, $\chi(G) \leq 4$. Also every arc in one set dominates every other arc in the remaining sets. Minimum four colors are needed to properly color the graph. Thus, $\chi(G) \geq 4$. Implies $\chi(G) = 4$ with 4 color classes $\{A_1\}$, $\{A_2, A_3\}$, $\{A_4, A_5, A_6\}$ and

$\{A_7, A_8, A_9, A_{10}\}$. Every arc in one color class dominates each and every arc in the remaining color classes. That is every arc dominates an entire color class. Hence, $\chi_d(G) \leq 4$.

But

$$\chi_d(G) \geq \chi(G) = 4.$$

Implies,

$$\chi_d(G) = 4.$$

Theorem 3.4: Let the circular – arc overlap family $A = \{A_1, A_2, \ldots, A_k\}$ corresponding to a connected circular – arc overlap graph G be such that

(i) No three arcs form a cycle

(ii) Whenever an arc A_i dominates two arcs A_p and A_q, there is no mutual domination between the arcs dominated by the arc A_p and the arc A_q and

(iii) Whenever two non- overlapping arcs A_i and A_j dominate two mutually non-overlapping arcs A_p and A_q respectively, there is no mutual domination between the arcs dominated by the arc A_p and the arc A_q

Then, $2 \leq \chi_d(G) \leq [\, k/2 \,] + 1$

Proof: Partition the set of arcs in A into two subsets B and C as follows:

First include the arc A_1 in the set B and the arcs that dominate the arc A_1, say $C_1, C_2, \ldots, C_p$ in the set C. In the first stage of the proof $B = \{A_1\}$ and $C = \{C_1, C_2, \ldots, C_p\}$. Obviously, B is an independent set. By condition (i), no two arcs that are dominated by the same arc can dominate each other. It follows that C is an independent set. If $B \cup C = A$, Then B and C are independent sets.

If $B \cup C \neq A$, then include the arcs say $B_1, B_2, \ldots, B_q$ that are dominated by atleast one of the arcs of C in the set B so that $B = \{A_1\} \cup \{B_1, B_2, \ldots, B_q\}$.

By condition (i) and (ii), it follows that B is an independent set. In the second stage of the proof,

$B = \{A_1, B_1, B_2, \ldots, B_q\}$ and $C = \{C_1, C_2, \ldots, C_p\}$. If $B \cup C = A$, Then B and C are independent sets. If $B \cup C \neq A$, then include the arcs $C`_1, C`_2, \ldots, C`_m$, that are dominated by at least one of the newly added arcs of B in the set C. Consequently

$C = \{C1, C_2, \ldots, C_p\} \cup \{C`_1, C`_2, \ldots, C`_m\}$

By condition (i) (ii) and (iii) it follows that C is an independent set. In the third stage of the proof

$B=\{A_1, B_1, B_2, \ldots, B_q\}$ and $C=\{C_1, C_2, \ldots, C_p, C`_1, C`_2, \ldots C`_m\}$

If $B \cup C = A$, Then B and C are independent sets. If $B \cup C \neq A$, then repeat the process until $B \cup C = A$. Finally it follows that, the circular- arc overlap family of the graph G can be partitioned into two independent sets B and C so that $B \cup C = A$.

Consequently, $\chi(G) = 2$. Implies $\chi_d(G) \geq 2$. Let $|B| = m$ and $|C| = n$. Two cases may arise.

Case (i): Let $m \leq n$.

Assign colors 1, 2, …, m to the vertices of B, such that no two vertices get the same color and color all the vertices of C with $(m + 1)^{th}$ color. Every vertex in the set C dominates at least one color class in the set of color classes { 1, 2, …, m }as the graph G is a connected graph. Every color class corresponding to the arcs in the set B consists of only one arc. It follows that, every arc of B dominates its own color class. Thus, $\chi_d(G) \leq m + 1$. But $|B| \leq [k/2]$. Hence $\chi_d(G) \leq [k/2] + 1$.

Case (ii): Let $n < m$.

Assign colors 1, 2, …, n to the vertices of C and $(n + 1)^{th}$ color to the vertices of B which yields a dominator coloring. Thus, $\chi_d(G) \leq n + 1$. But $|C| \leq [k/2]$. Hence, $\chi_d(G) \leq [k/2] + 1$. Therefore the result follows.

4 Conclusion

Initially the paper has outlined the prime points regarding the bounds of the dominator chromatic number and initiated into exploring the domination chromatic number of some particular circular- arc overlap graphs. This can be considered innovation in the arena of theory related to circular- arc overlap graphs. In future, efforts in the paper eventually open up many an avenue in the field of research on circular –arc overlap graphs.

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Figure 1. Circular-arc overlap graph with χ_d (G) = 4