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Multiplicity of Solutions for a Reactive Variable Viscous Couette Flow under Arrhenius Kinetics

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Abstract

This paper investigates the properties of solution for a reactive temperature dependent viscous Couette flow through parallel plates with non-uniform temperature. We showed that the problem has two distinct solutions and the solution breaks down for some values of the Frank-kameneskii parameter. Effect of various parameters in the model are presented and discussed.

Keywords: Arrhenius kinetics, Couette flow, variable viscosity, Bratu-type equation

1. Introduction

Studies on combustion problem has been an area of active research over the past few decades, this is because it has many practical applications. Safety speaking is the major interest on the study of combustion which is to prevent unwanted thermal ignition, pressure build up and detonation as described in (Williams, 1999). However, in many cases of interest, exposure of reactive fluids to sufficiently hot surfaces cannot be avoided. This has led to significant amount of work in literature (Kobo and Makinde (2010), Chinyoka and Makinde (2011), Makinde and Anwar Beg (2010) and Lacey (1998)) in order to ensure safety of lives, properties and quality of the final product from the study of reactive hydromagnetic channel flow. Other notable work on combustion includes (Goldshtein and Zinoviev (1991), Boling and Peicheng (1999) and Palymskiy, Fomin and Hieronymus (1999)) and many more.

It is known that combustion models have multiple solutions, single solution or no solution depending on the value of the Frank-kameneskii parameter as described in Buckmaster and Ludford (1983). Based on the fact by using asymptotic techniques, multiple solutions were presented for the steady reactive-diffusive problem for a non-isothermal permeable pellet with first-order Arrhenius kinetic as stated in Kapila and Matkowsky (1979) and Kapila et al (1980). Recently, Mohammed (2006) applied the maximum principle to proof the existence of multiple solutions to a steady reactive-diffusive problem for a non-isothermal permeable pellet with first-order Arrhenius kinetics.

In this paper attention is focused on the properties of solution for a reactive temperature dependent viscous Couette flow under Arrhenius kinetics. In order to prove the existence of multiple solutions of the coupled non-linear problem, the original problem is modified by using a suitable transformation and analytical solution in the form of Adomian decomposition method is applied to the modified problem, we showed the upper and the lower branches of the solutions exist.

The rest of the paper is organized as follows: section 2 provides adequate information on the formulation and non-dimensionalization of the problem. In section 3, the main results are presented. Section 4 presents the results and discussions while section 5 concludes the work.

2. Mathematical Formulation

Consider a reactive, incompressible. The flow is considered steady and is fully developed between two infinite parallel isothermal plates. Neglecting the reactant consumption, the dimensionless flow governing equations are following [2],

$$\frac{d}{dy}\left(e^{-\frac{T}{1+\beta T}}\frac{du}{dy}\right)=0; \quad (1)$$

together with

$$\frac{d^2T}{dy^2}+Bre^{-\frac{T}{1+\beta T}}\left(\frac{du}{dy}\right)^2+\lambda e^{\frac{T}{1+\beta T}}=0 \quad (2)$$ such that

$$u(0)=0,\ u(1)=1 \quad (3)$$

$$T(0)=0,\ T(1)=0 \quad (4)$$

For many cases of interest and by Frank-Kameneskii approximation, we take $\beta=0$. A good example of such approximation is the Bratu – type equation for combustion in Aregbesola (2003) such that (1) and (2) reduce to

$$\frac{d}{dy}\left(e^{-T}\frac{du}{dy}\right)=0; \quad (5)$$

$$\frac{d^2T}{dy^2}+Br\,e^{-T}\left(\frac{du}{dy}\right)^2+\lambda e^{T}=0 \quad (6)$$

Upon integration of (5) we get

$$\frac{du}{dy}=me^{T} \quad (7)$$ where m is a constant to be determined by using the boundary condition (3).

Substituting (7) in (6) leads to

$$\frac{d^2T}{dy^2}+\gamma e^{T}=0 \quad (8)$$

Where λ is the Frank – Kameneski parameter such that $\gamma=\left(Brm^2+\lambda\right)$ (9)

3. Method of Solution

To show that (8) has two solutions if $\gamma<\gamma_c$, one solution if $\gamma=\gamma_c$ and no solution if $\gamma>\gamma_c$, we introduce the transformation

$$w(y)=e^{-T} \quad (10)$$

that is,

$$T=-\ln w(y) \quad (11a)$$

and

$$T' = -\frac{w'(y)}{w(y)} \tag{11b}$$

Differentiating again,

$$T'' = -\left[\frac{w(y)w''(y)-(w'(y))^2}{[w(y)]^2}\right] \tag{11c}$$

Substituting (11a), (11b), and (11c) into (9), we have

$$-\left[\frac{w(y)w''(y)-(w'(y))^2}{[w(y)]^2}\right]+\gamma\left(\frac{1}{w(y)}\right)=0 \tag{12}$$

Writing (12) in a better form, we have

$$w(y)w''(y)-(w'(y))^2-\gamma w(y)=0 \tag{13}$$

Subject to the following boundary conditions

$$w(0)=1 \quad and \quad w(1)=1 \tag{14}$$

Introducing a series solution $S(y)$ that satisfies (14) in the form

$$w(y)=1+S(y), \tag{15a}$$

$$w'(y)=S'(y) \quad \text{and} \tag{15b}$$

$$w''(y)=S''(y) \tag{15c}$$

where $S(y)$ is a series solution that is needed to seek the two solutions. Using (15a), (15b) and (15c) in (13), we deduced as follows:

$$[1+S(y)]S'''(y)-[S'(y)]^2-\gamma[1+S(y)]=0$$ (16) Subject to

$$S(0)=S(1)=0 \tag{17}$$

Simplifying (16) in a proper way, we have

$$S''(y)=\gamma[1+S(y)]+[S'(y)]^2-[S(y)(S''(y))] \tag{18}$$

Integrating (18), we obtain the integral equation

$$S'(y)=A+\int_0^y \gamma[1+S(y)]+[S'(y)]^2-[S(y)(S''(y))]dy \tag{19a}$$

Where $S^{'}(0) = A$, is a constant to be determined by using the boundary condition.

Integrating (19a) again, we have

$$S(y) = Ay + \int_0^y \int_0^y \gamma[1 + S(y)] + [S^{'}(y)]^2 - [S(y)(S^{''}(y))] dy\, dy \qquad (19b)$$

We define a series of the form

$$S(y) = \sum_{n=0}^{\infty} S_n(y) \qquad (20)$$

Substituting (20) in (19b) we get

$$\sum_{n=0}^{\infty} S_n(y) = A\,y + \int_0^y \int_0^y \gamma dy dy + \int_0^y \int_0^y \left(\left[\sum_{n=0}^{\infty} S_n'(y) \right]^2 + \gamma \sum_{n=0}^{\infty} S_n(y) - \sum_{n=0}^{\infty} S_n(y) \sum_{n=0}^{\infty} S_n''(y) \right) dy dy \quad (21)$$

We let the nonlinear terms be represented as

$$\sum_{n=0}^{\infty} B_n = \left[\sum_{n=0}^{\infty} S_n'(y) \right]^2 , \qquad (22)$$

$$\sum_{n=0}^{\infty} C_n = \sum_{n=0}^{\infty} S_n(y) \sum_{n=0}^{\infty} S_n''(y) \qquad (23)$$

Using (22) and (23) in (21) and taking the Taylor's expansion we obtained few terms for B_n and C_n

$$\begin{aligned} B_0 &= (S_0'(y))^2 \\ B_1 &= 2S_0'(y)S_1'(y) \\ B_2 &= 2S_2'(y)S_0'(y) + (S_1'(y))^2 \\ B_3 &= 2S_3'(y)S_0'(y) + 2(S_1'(y)S_2'(y)) \\ &\dots \end{aligned} \qquad (24)$$

and

$$\begin{aligned} C_0 &= S_0(y)S_0''(y) \\ C_1 &= S_0(y)S_1''(y) + S_1(y)S_0''(y) \\ C_2 &= S_0(y)S_2''(y) + S_1(y)S_1''(y) + S_2(y)S_0''(y) \\ C_3 &= S_0(y)S_3''(y) + S_1(y)S_2''(y) + S_2(y)S_1''(y) + S_3(y)S_0''(y) \\ &\dots\dots \end{aligned} \qquad (25)$$

Then the zeroth component of (21) can be written following new modification in Wazwaz and El – Sayed (2001)

$$S_0(y)=\int_0^y\int_0^y \gamma dydy \tag{26}$$

$$S_1(y)=Ay+\int_0^y\int_0^y\left(B_0+\gamma\sum_{n=0}^{\infty}S_0(y)-C_0\right)dydy \tag{27}$$

To obtain other terms of the series, we use the recurrence relation

$$S_{n+1}(y)=\int_0^y\int_0^y\left(B_n+\gamma\sum_{n=1}^{\infty}S_n(y)-C_n\right)dydy \tag{28}$$

Obtaining few terms of the series leads to

$$S_0(y)=\frac{\gamma y^2}{2} \tag{29}$$

$$S_1(y)=Ay+\frac{\gamma^2 y^4}{12} \tag{30}$$

$$S_2(y)=\frac{\gamma^3 y^6}{180}+\frac{A\gamma y^3}{3} \tag{31}$$

$$S_3(y)=\frac{\gamma^4 y^8}{5040}+\frac{A\gamma^2 y^5}{30}+\frac{A^2 y^2}{2} \tag{32}$$

Where A is a constant to be determined later and required to seek the two solutions of the problem. Then the partial sum

$$S(y)=\sum_{n=0}^{3}S_n(y)=Ay+\left(\frac{\gamma}{2}+\frac{A^2}{2}\right)y^2+\frac{A\gamma}{30}y^3+\frac{\gamma^2}{12}y^4+\frac{A\gamma^2}{30}y^5+\frac{\gamma^3}{180}y^6+\frac{\gamma^4}{5040}y^8 \tag{33}$$

It is important to note that the accuracy of the series can be drastically improved by computing more terms of the series.

Invoking the boundary condition $S(1)=0$ gives

$$A_{1,2}=-\left(1+\frac{\gamma}{3}+\frac{\gamma^2}{30}\right)\pm\sqrt{1-\frac{\gamma}{3}+\frac{\gamma^2}{90}+\frac{\gamma^3}{90}+\frac{\gamma^4}{1400}} \tag{34}$$

Hence we write the two solutions as follows

$$S1(y)=A_1 y+\left(\frac{\gamma}{2}+\frac{A_1^2}{2}\right)y^2+\frac{A_1\gamma}{30}y^3+\frac{\gamma^2}{12}y^4+\frac{A_1\gamma^2}{30}y^5+\frac{\gamma^3}{180}y^6+\frac{\gamma^4}{5040}y^8 \tag{35}$$

and

$$S2(y)=A_2 y+\left(\frac{\gamma}{2}+\frac{A_2^2}{2}\right)y^2+\frac{A_2\gamma}{30}y^3+\frac{\gamma^2}{12}y^4+\frac{A_2\gamma^2}{30}y^5+\frac{\gamma^3}{180}y^6+\frac{\gamma^4}{5040}y^8 \tag{36}$$

Back-solving, (15a) becomes

$$w_1(y)=1+S1(y)=1+A_1y+\left(\frac{\gamma}{2}+\frac{A_1^2}{2}\right)y^2+\frac{A_1\gamma}{30}y^3+\frac{\gamma^2}{12}y^4+\frac{A_1\gamma^2}{30}y^5+\frac{\gamma^3}{180}y^6+\frac{\gamma^4}{5040}y^8 \quad (37)$$ and

$$w_2(y)=1+S2(y)=1+A_2y+\left(\frac{\gamma}{2}+\frac{A_2^2}{2}\right)y^2+\frac{A_2\gamma}{30}y^3+\frac{\gamma^2}{12}y^4+\frac{A_2\gamma^2}{30}y^5+\frac{\gamma^3}{180}y^6+\frac{\gamma^4}{5040}y^8 \quad (38)$$

$$T_1(y)=-Log(w_1(y))=-Log\left[1+A_1y+\left(\frac{\gamma}{2}+\frac{A_1^2}{2}\right)y^2+\frac{A_1\gamma}{30}y^3+\frac{\gamma^2}{12}y^4+\frac{A_1\gamma^2}{30}y^5+\frac{\gamma^3}{180}y^6+\frac{\gamma^4}{5040}y^8\right] \quad (39)$$ and

$$T_2(y)=-Log(w_2(y))=-Log\left[1+A_2y+\left(\frac{\gamma}{2}+\frac{A_2^2}{2}\right)y^2+\frac{A_2\gamma}{30}y^3+\frac{\gamma^2}{12}y^4+\frac{A_2\gamma^2}{30}y^5+\frac{\gamma^3}{180}y^6+\frac{\gamma^4}{5040}y^8\right] \quad (40)$$

Therefore, (39) and (40) give the two solutions of temperature distribution with the channel.
To find the velocity of the fluid flow, we reconsider (8) and integrate to obtain

$$u(y)=m\int_0^y e^T dy \quad (41)$$

Where m is to be evaluated using $u(1)=1$, the two solutions of the velocity are as follows:

$$u_1(y)=m_1\int_0^y Exp(-Log(w_1(y)))dy=m_1\int_0^y\left(\frac{1}{w_1(y)}\right)dy \quad (42)$$ and

$$u_2(y)=m_2\int_0^y Exp(-Log(w_2(y)))dy=m_2\int_0^y\left(\frac{1}{w_2(y)}\right)dy \quad (43)$$

It is obvious that (41) cannot be solved in algebraic way therefore the exponential function needed to be Taylors' expanded to be able to get the two solutions for velocity profiles.

4. Discussion of Results

Figure 1 displays the upper and lower solutions for $\gamma = 0.5$. It showed that the maximum values of the upper and lower solutions occurred at the centre line of the channel, that is, when $y = 0.5$. In general, the temperature is maximal at the centre and decreases as we move towards the boundaries in the channel.

The lower and upper solutions of velocity profile are shown in figure 2. The two solutions satisfy the boundary conditions. It is observed that the fluid velocity is zero at the lower region and increases gradually towards the upper moving plate. The lower solution increases linearly while the upper solution increases nonlinearly across the channel.

The temperature profile of the upper solution with various values of γ is shown in figure 3. It is noticed that the temperature is almost the same before the centre line and decreases as γ increases after the centre line. The fluid behaviour as secondary solution is highly noticed.

Figure 4 shows the typical variations of the fluid temperature of lower solution which is similar to [2]. The fluid temperature increases with increasing values of γ. This is due to an increase in heat generation within the fluid due to exothermic reaction as shown in figure 4.

The upper solution of the velocity profiles for various values of γ is shown in figure 5. It is observed that, the parameter γ increases causing the velocity profile to increase nonlinearly across the channel to a maximum of $y = 1$.

Figure 6 displays the lower solution of the velocity profiles for various values of γ. It is good to note that, the velocity remained unchanged for various values of γ as the profile maintain a linear flow that is equivalent to $y = u(y)$ which is standard Couette flow.

5. Conclusion

We have studied the multiplicity and properties of solutions for a reactive variable viscous Couette flow through parallel plates with non-uniform temperature under Arrhenius Kinetics. It is shown that there exist two distinct upper and lower solutions and the solution breaks down for some values of the Frank-kameneskii parameter for temperature and velocity profiles using Adomian decomposition method with suitable transformation to the modified problems.

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Nomenclature

L	Channel characteristic length	E	Activation energy
G	Reaction Kinetic	R	Universal gas constant
ρ	Fluid density	ε	Channel aspect ratio
λ	Frank – Kamenettski parameter	β	Activation energy parameter
T	Fluid temperature (k)	T_o	plate surface temperature (k)
C_o	Reactant species initial concentration	C_p	Specific heat at constant pressure
μ	Viscosity	Re	Reynolds number
U	Velocity scale (ms^{-1})	Pe	Peclet Number
k	Thermal conductivity coefficient ($Wm^{-1}k^{-1}$)	Q	Heat generation term (W)
Br	Brinkman number	Q_o	Dimensional heat generation coefficient
A	Reaction rate constant	u	Axial velocity (ms^{-1})
v	Normal velocity (ms^{-1})	x, y	Coordinate system (m)

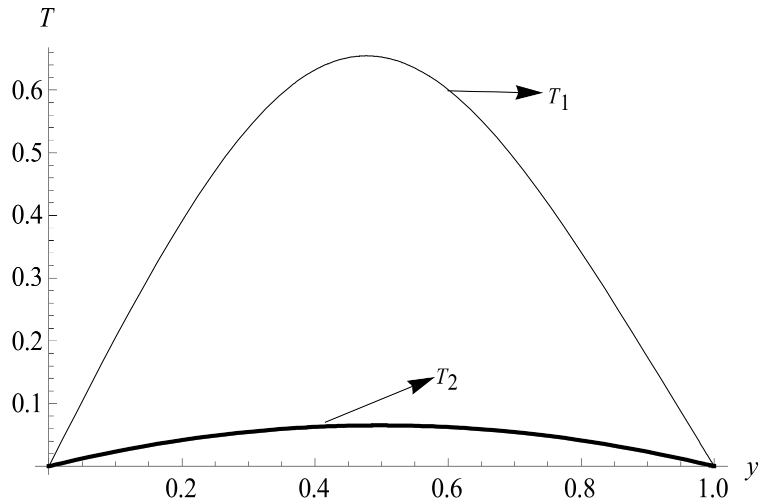

Figure 1: Graph showing upper and lower solutions for temperature when γ = 0.5

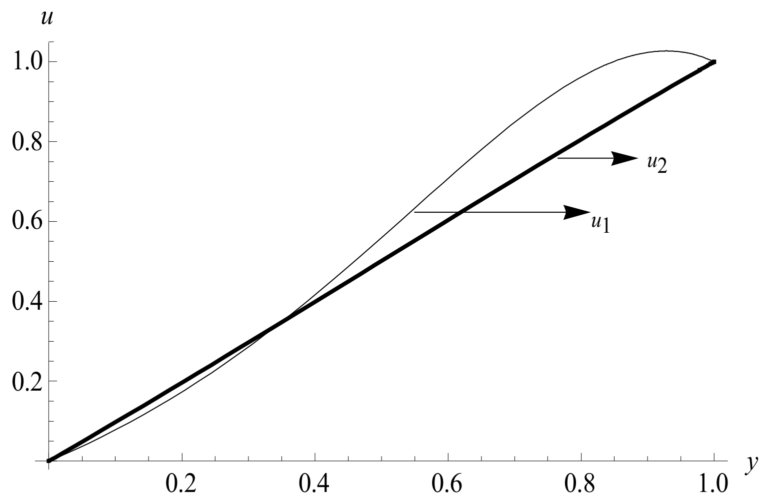

Figure 2: Graph showing upper and lower solutions for velocity when γ = 0.5

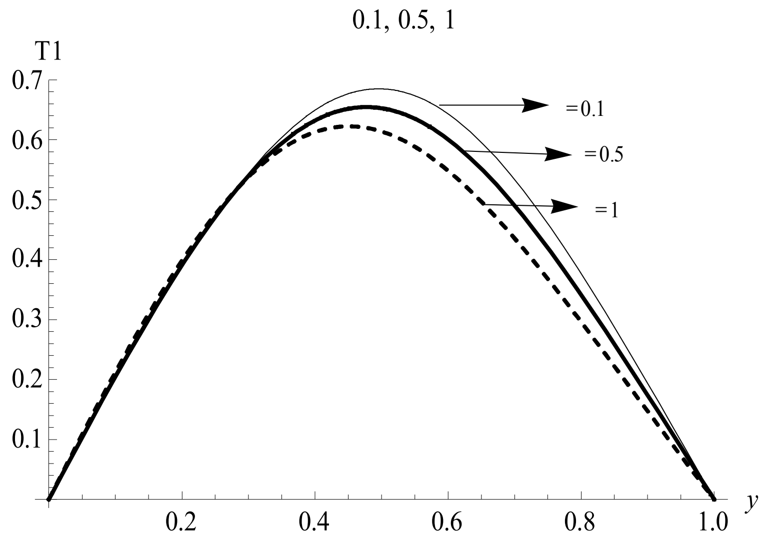

Figure 3: Upper solution of temperature profiles for various values of γ

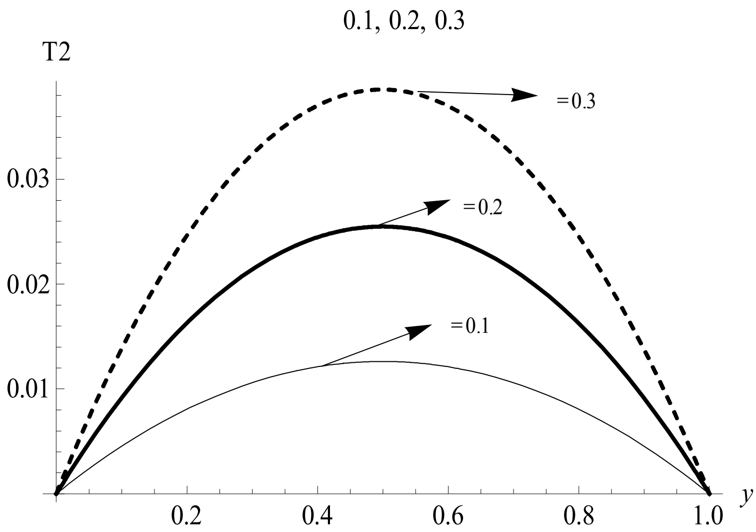

Figure 4: Lower solution of temperature profiles for various values of γ

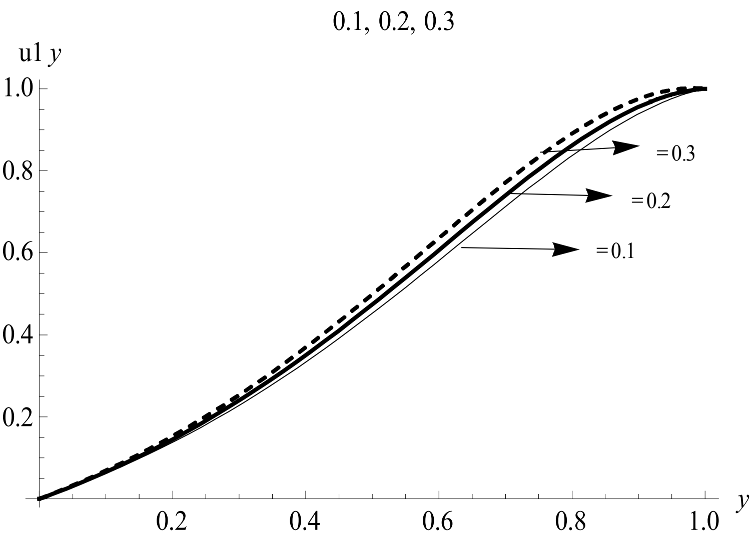

Figure 5: Upper solution of velocity profiles for various values of γ

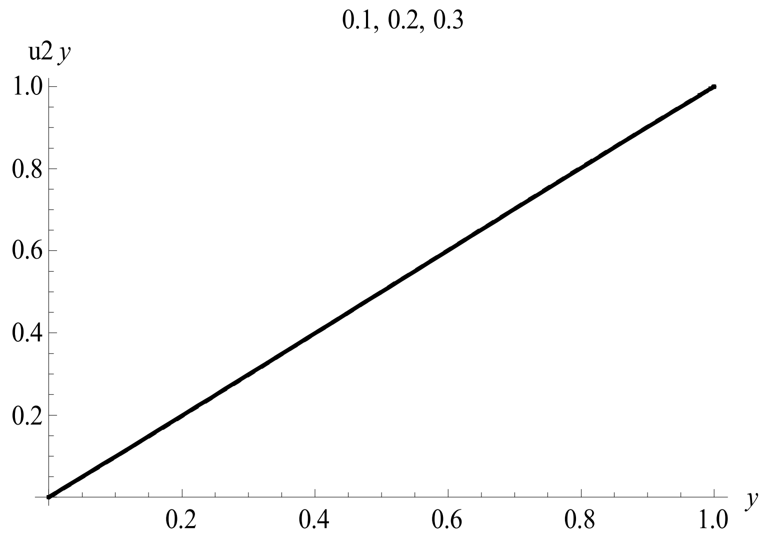

Figure 6: Lower solution of velocity profiles for various values of γ