PRINCIPAL IDEAL GRAPHS OF RECTANGULAR BANDS

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Abstract

Let S be a finite regular semigroup. We define the principal left ideal graph of S as the graph ${}_SG$ with $V({}_SG) = S$ and two vertices a and b ($a \neq b$) are adjacent in ${}_SG$ if and only if $Sa \cap Sb \neq \{\}$. The principal right ideal graph is defined accordingly and is denoted by G_S. In this paper we describe the structures of principal ideal graphs of rectangular bands. We see that if $S = I \times \wedge$ is a rectangular band, then ${}_SG$ is a disconnected graph with $|\wedge|$ components in which each component is complete with $|I|$ vertices, while G_S is a disconnected graph with $|I|$ components in which each component is complete with $|\wedge|$ vertices. We also describe the number of edges in ${}_SG$ and G_S when S is a rectangular band.

Keywords: Rectangualar bands, principal ideal graphs, connected graphs, complete graphs.

1. Introduction

Semigroups are the first and simplest type of algebra to which the methods of universal algebra is applied. During the last three decades, Graph Theory has established itself as an important mathematical tool in a wide variety of subjects. The use of graph theory has become widespread in the algebraic theory of semigroups. In 1964, Bosak [1] studied certain graph over semigroups. In 1975, Zelinka [14] studied intersection graphs of nontrivial subgroups of finite abelian groups. The well known study of directed graph considering the elements of a group as its vertex set is the Cayley digraph [2, 8, 12, 13]. Recently Kelarev and Quinn [9, 10] defined two interesting classes of directed graphs, namely, divisibility graph and power graphs on semigroups. The divisibility graph Div(S) of a semigroup S is a directed graph with vertex set S and there is an edge (arc) from u to v if and only if $u \neq v$ and u/v, i.e., the ideal generated by v contains u. On the other hand the power graph, Pow(S) of a semigroup S is a directed graph in which the set of vertices is again S and for $a,b \in S$ there is an arc from a to b if and only if $a \neq b$ and $b = a^m$ for some positive integer m. In 2005, Frank De Mayer and Lisa De Mayer studied zero divisor graphs of semigroups [4]. In 2009, Ivy Chakrabarty, Shamik Ghosh and M K Sen introduced undirected power graphs [7].Following this, we define a new type of graphs on semigroups called the 'Principal Ideal Graphs of Semigroups'. Here we characterise the principal ideal graphs of Rectangular bands.

2. Preliminaries

In the following we give certain definitions and results from graph theory and semigroup theory as given in [5], [11] and [3], [6] respectively, which are used in the sequel.

A graph $G = (V, E)$ consists of a set of objects $V = \{v_1, v_2, \ldots\}$, called vertices and another set $E = \{e_1, e_2, \ldots\}$ whose elements are called edges such that each e_k is identified with an unordered pair (v_i, v_j) of vertices. Two graphs G and G' are said to be isomorphic, denoted by $G \cong G'$, if there is a one-to-one correspondance between their vertices and between their edges such that the incidence relationship is preserved. A graph G is said to be connected if there exists atleast one path between any pair of vertices in G, otherwise G is called disconnected. A graph in which there exists an edge between every pair of vertices is called a complete graph. A complete graph of n vertices is denoted by K_n.

A semigroup S is a non empty set S together with an associative binary operation on S. An element $e \in S$ is said to be an *idempotent* if $e^2 = e$. A semigroup in which every element is an idempotent is called a *band*. Let S be a semigroup and let 1 be an object not in S. Extend the binary operation on S to $S \cup \{1\}$ by $x \cdot 1 = 1 \cdot x = x$ for all $x \in S$

We define $S' = \begin{cases} S & \text{if } 1 \in S \\ S \cup \{1\} & \text{if } 1 \notin S \end{cases}$

We define relations known as Green's relations L and R on S as follows:

$$\mathsf{L} = \{(a,b) \in S : S'a = S'b\}$$

$$\mathsf{R} = \{(a,b) \in S : aS' = bS'\}$$

$\mathcal{H} = \mathcal{L} \cap \mathcal{R}$

An element x of a semigroup S is said to be regular if there exists an element $x' \in S$ such that $xx'x = x$. A semigroup S is said to be regular if all elements of S are regular. A rectangular band is a band S which satisfies $xyx = x$ for all $x, y \in S$. Equivalently $xyz = xz$ for all $x, y, z \in S$ and $x^2 = x$ for all $x \in S$.

We can also describe a rectangular band as given below:(cf.[6])

Given arbitrary non-empty sets I and $\wedge$, we can define a semigroup operation on $I \times \wedge$ by setting;

$(i,\lambda)(j,\mu) = (i,\mu)$ for $(i,\lambda),(j,\mu) \in I \times \wedge$. The resulting semigroup is a rectangular band. For we have,

$(i,\lambda)(i,\lambda) = (i,\lambda)$ and

$(i,\lambda)(j,\mu)(i,\lambda) = (i,\mu)(i,\lambda) = (i,\lambda)$ for all $(i,\lambda),(j,\mu) \in S$.

Infact any rectangular band is isomorphic to one we have described above (cf.[6]).

Both left zero semigroup and right zero semigroup are special cases of a rectangular band. If $|I| = 1$, then S is a right zero semigroup and if $|\wedge| = 1$, then S is a left zero semigroup(cf.[6]). Also note that, a rectangular band is always regular.

The following proposition related to Green's relations is used in the sequel.

Proposition 2.1 (cf.[6]) Let $S = (I \times \wedge)$ be a rectangular band and (i,λ), $(j,\mu) \in S$. Then $(i,\lambda)\mathcal{L}(j,\mu)$ iff $\lambda = \mu$ and $(i,\lambda)\mathcal{R}(j,\mu)$ iff $i = j$. Each $\mathcal{H}$ class of a rectangular band S contains exactly one element.

3. Prinicpal ideal graphs of Rectangular bands

In the following proposition, we characterise the $\mathcal{L}$ relations on S in terms of edges in ${}_S G$.

Proposition 3.1 Let S be a rectangular band and $a, b \in S$. Then $a\mathcal{L}b$ if and only if there exists an edge between a and b in ${}_S G$.

Proof: Let S be a rectangular band and $a, b \in S$. Suppose that $a\mathcal{L}b$. Then $Sa = Sb$ (cf. [6]) and hence

$Sa \cap Sb \neq \{\}$. Therefore there exists an edge between a and b in ${}_S G$.

Conversely assume that, there exists an edge between a and b in ${}_S G$. Let $a = (i, \lambda)$ and $b = (j, \mu)$. Since there exists an edge between a and b, we have $Sa \cap Sb \neq \{\}$. Let $(m, n) \in Sa \cap Sb$. Then $(m, n) = (i', \lambda')(i, \lambda)$ and $(m, n) = (j', \mu')(j, \mu)$ for some $(i', \lambda'), (j', \mu') \in S$. Thus we have, $(i', \lambda')(i, \lambda) = (j', \mu')(j, \mu)$. Hence $(i', \lambda) = (j', \mu)$. This shows that $\lambda = \mu$ and hence $a\mathsf{L}b$ by Proposition 2.1.

In a similar manner, we can characterise the R relations on S in terms of edges in G_S as given in the following proposition.

Proposition 3.2 Let S be a rectangular band and $a, b \in S$. Then $a\mathsf{R}b$ if and only if there exists an edge between a and b in G_S.

The next proposition shows that, the principal left ideal graph ${}_S G$ and the principal right ideal graph G_S of a rectangular band S have disjoint edge sets. That is, if two elements are adjacent in ${}_S G$, then they are not adjacent in G_S.

Proposition 3.3 Let S be a rectangular band. Then $E({}_S G) \cap E(G_S) = \{\}$, where $E(G)$ denotes the set of all edges of a graph G.

Proof: If possible, let $E({}_S G) \cap E(G_S) \neq \{\}$. Then there exist elements a, b in S such that there is an edge between a and b in both ${}_S G$ and G_S. Hence, by Proposition 3.1 and Proposition 3.2, we have $a\mathsf{L}b$ and $a\mathsf{R}b$. Thus, we have $a\mathsf{H}b$ in S. Since any H class of a rectangular band contains exactly one element(cf. Proposition 2.1), we have $a = b$. Since we consider loopless graphs, it follows that $E({}_S G) \cap E(G_S) = \{\}$.

The following corollary is a consequence of Proposition 3.3.

Corollary 3.4 Let S be a rectangular band. Then ${}_S G$ is an induced subgraph of G'_S, where G'_S is the complement of G_S.

Proof: Clearly $|V({}_S G)|=|V(G'_S)|$. Now let $(x,y)\in E({}_S G)$. Then by Proposition 3.3, $(x,y)\notin E(G_S)$. Hence $(x,y)\in E(G'_S)$. Since this is true for any two elements x,y in S, we have $E({}_S G)\subseteq E(G'_S)$. Hence ${}_S G$ is an induced subgraph of G'_S.

In a similar manner, we have the following result.

Corollary 3.5 Let S be a rectangular band. Then G_S is an induced subgraph of ${}_S G'$, where ${}_S G'$ is the complement of ${}_S G$.

The next lemma describes the number of elements which are $\mathcal{L}$ related to a for $a\in S$.

Lemma 3.6 Let S be a rectangular band $I\times\wedge$ and $a\in S$. Then a is $\mathcal{L}$ related to $|I|$ elements.

Proof: Let $S=\{(a_i,b_j): a_i\in I, b_j\in\wedge\}$. If $a=(a_r,b_s)$, then by Proposition 3.1, $x=(a_i,b_j)\mathcal{L}a$ iff $b_j=b_s$. Hence a is $\mathcal{L}$ related to the elements (a_i,b_s) for i = 1,2,..., $|I|$. Thus the element a is $\mathcal{L}$ related to $|I|$ elements.

In a similar manner, we have the following lemma for $\mathcal{R}$ related elements in S.

Lemma 3.7 Let S be a rectangular band $I\times\wedge$ and $a\in S$. Then a is $\mathcal{R}$ related to $|\wedge|$ elements.

The following two lemmas help us to characterise the principal left ideal graphs of a rectangular band.

Lemma 3.8 Let S be a rectangular band $I\times\wedge$ and $a\in S$. Then

(i) ${}_{L_a}G$, the induced subgraph of ${}_S G$ with vertex set L_a (the L-class containing a) is complete.

(ii) For $a,b\in S$ and $b\notin L_a$, ${}_{L_a}G$ and ${}_{L_b}G$ are disjoint.

(iii) ${}_S G=\bigcup_{L_a} {}_{L_a}G$, the disjoint union of ${}_{L_a}G$.

(iv) If x is a vertex in ${}_{L_a}G$ then $\deg(x)=|I|-1$.

Proof: (i) Let $x,y\in L_a$ for $a\in S$. Then we have $x\mathcal{L}a$, $y\mathcal{L}a$ and hence $x\mathcal{L}y$. Hence, by Proposition 3.1, there exists an edge between x and y in ${}_S G$. Since this is true for any two elements $x,y\in L_a$, we see that the induced subgraph ${}_{L_a}G$ of ${}_S G$ is complete.

(ii) Let $a,b\in S$ and $b\notin L_a$. If possible let $x\in V({}_{L_a}G)\cap V({}_{L_b}G)$. Then $x\mathcal{L}a$ and $x\mathcal{L}b$. Since $\mathcal{L}$ is an equivalence relation, $a \mathcal{L} b$ and thus $b \in L_a$. This is a contradiction to the hypothesis that $b \notin Ł_a$. Hence we conclude that ${}_{L_a}G$ and ${}_{L_b}G$ are disjoint.

(iii) By definition $V({}_S G) = S$. Also,

$V(\bigcup_{L_a} {}_{L_a}G) = \bigcup_{L_a} V({}_{L_a}G) = \bigcup_{L_a} L_a = S$

Also, for distinct elements $a, b \in S$, there is an edge between a and b in ${}_S G$ if and only if $a \mathcal{L} b$. But by (i) and (ii) this is possible, if and only if there exists an edge between a and b in ${}_{L_a}G$. This happens if and only if there is an edge between a and b in $\bigcup_{L_a} {}_{L_a}G$. Hence, we have $E({}_S G) = E(\bigcup_{L_a} {}_{L_a}G)$.

Therefore ${}_S G = \bigcup_{L_a} {}_{L_a}G$, the disjoint union of ${}_{L_a}G$.

(iv) Let x be a vertex in the induced subgraph ${}_{L_a}G$ of the graph ${}_S G$. Then, by Lemma 3.6, x is $\mathcal{L}$ related to $|L_a| = |I|$ elements and hence there are $|I|-1$ edges from x, since ${}_S G$ is a loopless graph. Hence $deg(x) = |I|-1$.

Conversely, we have the following theorem.

Lemma 3.9 Let G be a finite disjoint union of finite complete graphs $\{G_i : i \in A\}$ and $|V(G_i)| = |V(G_j)|$ for all $i \neq j$. Then there is a rectangular band S such that ${}_S G \cong G$.

Proof: Let $|V(G_i)| = |V(G_j)| = n$ for all i, j. Let $|A| = m$. Take $I = \{a_1, a_2, \cdots, a_n\}$ and $\wedge = \{b_1, b_2, \cdots, b_m\}$.

Then $S = \{(a_i, b_j) : a_i \in I, b_j \in \wedge\}$ is a rectangular band, where the multiplication is defined as

$(a_i, b_j)(a_k, b_s) = (a_i, b_s)$ for all $(a_i, b_j), (a_k, b_s) \in S$

Also we have,

$S(a_i, b_1) = \{(a_i, b_1) : a_i \in I\}$

$S(a_i,b_2)=\{(a_i,b_2):a_i\in I\}$

$S(a_i,b_3)=\{(a_i,b_3):a_i\in I\}$

$S(a_i,b_{m-1})=\{(a_i,b_{m-1}):a_i\in I\}$

$S(a_i,b_m)=\{(a_i,b_m):a_i\in I\}$

Hence there are m components in ${}_SG$. Also, from Lemma 3.8, we see that each of these components is complete with n vertices. Thus ${}_SG\cong G$.

Now, we have the following important characterisation of ${}_SG$, when S is a rectangular band.

Theorem 3.10 Let $S=I\times\wedge$ be a rectangular band. Then the principal left ideal graph ${}_SG$ is a disconnected graph with $|\wedge|$ components in which each component is complete with $|I|$ vertices.

Proof: Let $S=I\times\wedge$ be a rectangular band. By Lemma 3.8 (iii), we have ${}_SG=\bigcup_{L_a}{}_{L_a}G$, the disjoint union of ${}_{L_a}G$. Again, by Lemma 3.8 (i) and (iv), each ${}_{L_a}G$ is complete with $|I|$ vertices. Also, in the proof of Lemma 3.9, we have seen that ${}_SG$ has $|\wedge|$ components. Hence it follows, that ${}_SG$ has $|\wedge|$ components in which each component is complete with $|I|$ vertices.

Similar to lemma 3.8, and lemma 3.9 we have the following lemmas, which help us to characterise the principal right ideal graph G_S.

Lemma 3.11 Let S be a rectangular band $I\times\wedge$ and $a\in S$. Then

(i) G_{R_a}, the induced subgraph of G_S with vertex set R_a, the R-class containing a is complete.

(ii) For $a,b\in S$ and $b\notin R_a$, G_{R_a} and G_{R_b} are disjoint.

(iii) $G_S=\bigcup_{R_a}G_{R_a}$, the disjoint union of G_{R_a}.

(iv) If x is a vertex in G_{R_a} then $\deg(x)=|\wedge|-1$.

Lemma 3.12 Let G be a finite disjoint union of finite complete graphs $\{G_r:r\in B\}$ and

$|V(G_r)|=|V(G_t)|$ for $r \neq t$. Then there is a rectangular band S such that $G_S \cong G$.

Now, we have the characterisation of G_S, when S is a rectangular band.

Theorem 3.13 Let S = $I \times \wedge$ be a rectangular band. Then the principal right ideal graph G_S is a disconnected graph with $|I|$ components in which each component is complete with $|\wedge|$vertices.

Now, we have the following main theorem.

Theorem 3.14 Let H be a finite disjoint union of finite complete graphs $\{H_\lambda : \lambda \in \wedge\}$ and let G be a finite disjoint union of finite complete graphs $\{G_i : i \in I\}$ such that $|V(H_\lambda)|=|I|$ for all $\lambda \in \wedge$ and $|V(G_i)|=|\wedge|$ for all $i \in I$. Then there is a rectangular band S such that ${}_S G \cong H$ and $G_S \cong G$.

Proof: Let H be a finite disjoint union of finite complete graphs $\{H_\lambda : \lambda \in \wedge\}$ such that $|I|=|V(H_\lambda)|$ for all $\lambda \in \wedge$. Consider the rectangular band $S = I \times \wedge$. Now, by Lemma 3.8, we have ${}_S G = \bigcup_{L_a} {}_{L_a} G$, the disjoint union of ${}_{L_a} G$, where each induced subgraph ${}_{L_a} G$ is complete. Also for $a = (i, \lambda) \in S$, $V({}_{L_a} G)$ = L_a = $\{(i, \lambda) : i \in I\}$. Hence it follows that, $|V({}_{L_a} G)|$ = $|L_a|$ = $|I|$ = $|V(H_\lambda)|$. Also, the number of components in ${}_S G$ is $|\wedge|$. Hence we have ${}_S G \cong H$.

Now let G be a finite disjoint union of finite complete graphs $\{G_i : i \in I\}$ such that $|V(G_i)|=|\wedge|$ for all $i \in I$. Consider the rectangular band $S = I \times \wedge$. Now, by Lemma 3.11, we have $G_S = \bigcup_{R_a} G_{R_a}$, the disjoint union of G_{R_a}, where each induced subgraph G_{R_a} is complete.

Also, for $a = (i, \lambda) \in S$, $V(G_{R_a})$ = R_a = $\{(i, \lambda) : \lambda \in \wedge\}$. Hence it follows that, $|V(G({}_{R_a})|$ = $|R_a|$ = $|\wedge|$ = $|V(G_i)|$. Also, the number of components in G_S is $|I|$. Hence we have $G_S \cong G$. This completes the proof of the theorem.

The following two corollaries are immediate .

Corollary 3.15 Let S = $I \times \wedge$ be a rectangular band. If $|I|$ = n and $|\wedge|$ = 1, then ${}_S G \cong K_n$.

Corollary 3.16 Let S = $I \times \wedge$ be a rectangular band. If $|I|$ = 1 and $|\wedge|$ = m, then $G_S \cong K_m$.

Remark 3.17 In Corollary 3.15, ${}_SG$ is the principal left ideal graph of a left zero semigroup, while in Corollary 3.16, G_S is the principal right ideal graph of a right zero semigroup.

Finally, we describe the number of edges of ${}_SG$ and G_S for a rectangular band S.

Theorem 3.18 Let S = $I \times \wedge$ be a rectangular band with $|I|$ = n and $|\wedge|$ = m. Then ${}_SG$ has $\frac{mn(n-1)}{2}$ edges and G_S has $\frac{nm(m-1)}{2}$ edges.

Proof: Let S = $I \times \wedge$ be a rectangular band with $|I|$ = n and $|\wedge|$ = m. In Theorem 3.10, we have seen that the principal left ideal graph ${}_SG$ is a disconnected graph with $|\wedge|$ components in which each component is complete with $|I|$ vertices. Hence each component has $\frac{n(n-1)}{2}$ edges. Since there are m such components, the total number of edges in ${}_SG$ is $\frac{mn(n-1)}{2}$. Also from Theorem 3.13, it follows that, the principal right ideal graph G_S is a disconnected graph with $|I|$ components in which each component is complete with $|\wedge|$ vertices. Hence each component has $\frac{m(m-1)}{2}$ edges. Since there are n such components, the total number of edges in G_S is $\frac{nm(m-1)}{2}$.

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