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Commutativity in Prime Gamma Rings with Jordan Left Derivations

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Abstract

Let M be a 2–torsion free prime Γ–ring and let d: M→M be a Jordan left derivation . In this article, we show that under a suitable condition every nonzero Jordan left derivation on M induces the commutativity of M and accordingly d is a left derivation of M.

Key words: Jordan left derivation, prime Γ– ring, left derivation, Γ– ring.

1. Introduction

N. Nobusawa [7] introduced the notion of a Γ–ring as a generalization of classical ring . Barnes [2] generalized the concept of Nobusawa's Γ–ring. Now a days, a Γ–ring due to Barnes is known as a Γ–ring and the Γ–ring due to Nobusawa is known as ΓN–ring. A number of important properties of Γ–rings were introduced by them as well as by Kyuno [5], Luh [6] and others. We begin with the following definition :

Let M and Γ be additive Abelian groups. If there is a mapping $M\times\Gamma\times M\to M$ sending $(x,\alpha,y)\to x\alpha y$ such that

(a) $(x+y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha+\beta)z = x\alpha z + x\beta z$, $x\alpha(y+z) = x\alpha y + x\alpha z$

(b) $(x\alpha y)\beta z = x\alpha(y\beta z)$

for all $x, y, z\in M$ and $\alpha, \beta\in\Gamma$, then M is called a Γ–ring.

For example, let R be a ring with 1 and let $M=M_{m,n}(R)$ and $\Gamma= M_{n,m}(R)$ then M is a Γ–ring with respect to the matrix addition and matrix multiplication.

Note that every ring is a Γ–ring if we take $\Gamma= M$.

The notions of a prime Γ–ring and a completely prime Γ–ring were initiated by Luh [6] and some analogous results corresponding to the prime rings were obtained by him and Kyuno [5], whereas the concept of a strongly completely prime Γ–ring was used and developed by Sapanie and Nakajima in [8].

A Γ–ring M is called a prime Γ–ring if for all $a,b\in M$, $a\Gamma M\Gamma b=0$ implies $a=0$ or $b=0$.

And, M is called a completely prime if $a\Gamma b = 0$ (with $a,b\in M$) implies $a=0$ or $b=0$.

It is noted that every completely prime Γ–ring is a prime Γ–ring.

A Γ–ring M is 2–torsion free if $2a=0$ implies $a=0$ for all $a\in M$. And, a Γ–ring M is said to be commutative if

$a\gamma b = b\gamma a$ holds for all $a,b\in M$ and $\gamma\in\Gamma$.

The notions of derivation and Jordan derivation of a Γ–ring have been introduced by Sapanie and Nakajima [8]. Afterwards, Jun and Kim [4] obtained some significant results due to Jordan left derivation of a classical ring. Y.Ceven [3] worked on left derivations of completely prime Γ–rings and obtained some extensive results of left derivation and Jordan left derivation of a Γ–ring. M. Soyturk [9] investigated the commutativity of prime Γ–rings with the left and right derivations. He obtained some results on the commutativity of prime Γ–rings of characteristic not equal to 2 and 3. Some commutativity results of prime Γ–rings with left derivations were obtained by Asci and Ceran [1].

In this paper, we obtain the commutativity results of 2–torsion free prime Γ–rings with left Jordan derivations and consequently we prove that every Jordan left derivation is a left derivation.

Let M be a Γ–ring. An additive mapping d:M→M is called a ***derivation*** if

$d(a\alpha b) = d(a)\alpha b + a\alpha d(b)$, for all a, b ∈M and α ∈ Γ.

An additive mapping d:M→M is called a ***left derivation*** if

$d(a\alpha b) = a\alpha d(b) + b\alpha d(a)$, for every a, b ∈M and α ∈ Γ

An additive mapping d:M→M is called a ***Jordan derivation*** if

$d(a\alpha a)=d(a)\alpha a+a\alpha d(a)$, for every a∈M and α ∈ Γ.

An additive mapping d:M→M is called a Jordan ***left derivation*** if $d(a\alpha a)=2a\alpha d(a)$, for every a∈M and α∈Γ.

Y. Ceven [3] proved that every Jordan left derivation on a Γ–ring M is a left derivation if M is 2–torsion free completely prime Γ–ring.

Now we define $[a,b]_\alpha = a\alpha b - b\alpha a$. $[a,b]_\alpha$ is called the commutator of a and b with respect to α. We use the condition $a\alpha b\beta c=a\beta b\alpha c$ for all a,b,c∈M and α,β∈Γ and will represented it by (⋆). An example of a left derivation and a Jordan left derivation are given in [3].

2. Main Results

The following Lemma is due to Y.Ceven [3, Lemma 2.1]

2.1 Lemma Let M be a two torsion free Γ– ring satisfying the condition (⋆) and d be a Jordan left derivation on M. Then for all a,b∈ M and for all $\alpha \in \Gamma$:

(i) $d(a\alpha b + b\alpha a) = 2a\alpha d(b) + 2b\alpha d(a)$.

(ii) $d(a\alpha b\beta a) = a\beta a\alpha d(b) + 3a\alpha b\beta d(a) - b\alpha a\beta d(a)$

(iii) $d(a\alpha b\beta c + c\alpha b\beta a) = a\beta c\alpha d(b) + c\beta a\alpha d(b) + 3a\alpha b\beta d(c) + 3c\alpha b\beta d(a) - b\alpha c\beta d(a) - b\alpha a\beta d(c)$.

2.2 Lemma Let M be a 2–torsion free Γ–ring satisfying the condition (⋆) and let d be a Jordan left derivation on M.

Then

(i) $[a,b]_\alpha \beta a\alpha d(a) = a\alpha[a,b]_\alpha \beta d(a)$

(ii) $[a,b]_\alpha \beta(d(a\alpha b) - a\alpha d(b) - b\alpha d(a)) = 0$

(iii) $[a,b]_\alpha \beta d([a,b]_\alpha) = 0$

(iv) $d(a\alpha a\beta b) = a\beta a\alpha d(b) + (a\beta b + b\beta a)\alpha d(a) + a\alpha d([a,b]_\beta)$

Proof (i) By Lemma 2.1(iii)

$d(a\alpha b\beta c + c\alpha b\beta a) = a\beta c\alpha d(b) + c\beta a\alpha d(b) + 3a\alpha b\beta d(c) + 3c\alpha b\beta d(a) - b\alpha a\beta d(c) - b\alpha c\beta d(a)$.

Replacing $a\alpha b$ for c, we get $d((a\alpha b)\beta(a\alpha b) + (a\alpha b)\alpha b\beta a) = a\beta(a\alpha b)\alpha d(b) + (a\alpha b)\beta a\alpha d(b) + 3a\alpha b\beta d(a\alpha b)$

$+3(a\alpha b)\alpha b\beta d(a) - b\alpha a\beta d(a\alpha b) - b\alpha(a\alpha b)\beta d(a)$.

This gives

$2(a\alpha b)\beta d(a\alpha b) + d(a\alpha(b\alpha b)\beta a)$

$= a\beta a\alpha b\alpha d(b) + a\alpha b\beta a\alpha d(b) + 3a\alpha b\beta d(a\alpha b) + 3(a\alpha b)\alpha b\beta d(a) - b\alpha a\beta d(a\alpha b) - b\alpha(a\alpha b)\beta d(a)$.

By using Lemma 2.1(ii)

$-a\alpha b\beta d(a\alpha b) + b\alpha a\beta d(a\alpha b) + a\beta a\alpha d(b\alpha b) + 3a\alpha b\alpha b\beta d(a) - b\alpha b\alpha a\beta d(a)$

$= a\beta a\alpha b\alpha d(b) + a\alpha b\beta a\alpha d(b) + 3a\alpha b\alpha b\beta d(a) - b\alpha a\alpha b\beta d(a)$

Thus $(a\alpha b-b\alpha a)\beta d(a\alpha b)=$ $a\beta a\alpha b\alpha d(b)-a\alpha b\beta a\alpha d(b)-b\alpha b\alpha a\beta d(a)+b\alpha a\alpha b\beta d(a)$

$=a\alpha a\alpha b\beta d(b)-a\alpha b\alpha a\beta d(b)-b\alpha b\alpha a\beta d(a)+b\alpha a\alpha b\beta d(a)=a\alpha(a\alpha b-b\alpha a)\beta d(b)+b\alpha(a\alpha b-b\alpha a)\beta d(a)$….(A)

Replacing a+b for b $(a\alpha b-b\alpha a)\beta(2a\alpha d(a)+d(a\alpha b))=a\alpha(a\alpha b-b\alpha a)\beta d(a+b)+(a+b)\alpha(a\alpha b-b\alpha a)\beta d(a)$

Thus $a\alpha(a\alpha b-b\alpha a)\beta d(b)+b\alpha(a\alpha b-b\alpha a)\beta d(a)=2a\alpha(a\alpha b-b\alpha a)\beta d(a)+a\alpha(a\alpha b-b\alpha a)\beta d(b)+b\alpha(a\alpha b-b\alpha a)\beta d(a)-2(a\alpha b-b\alpha a)\beta a\alpha d(a)$, By using (A)

$(a\alpha b-b\alpha a)\beta a\alpha d(a) = a\alpha(a\alpha b-b\alpha a)\beta d(a)$, since M is 2–torsion free

Hence $[a,b]_\alpha \beta a\alpha d(a) = a\alpha [a,b]_\alpha \beta d(a)$.

(ii) Replacing a+b for a in Lemma 2.2 (i)

$((a+b)\alpha b-b\alpha (a+b))\beta (a+b)\alpha d(a+b) = (a+b)\alpha((a+b)\alpha b-b\alpha (a+b))\beta d(a+b)$

$(a\alpha b-b\alpha a)\beta(a\alpha d(a)+b\alpha d(a)+a\alpha d(b)+b\alpha d(b))=a\alpha(a\alpha b-b\alpha a)\beta(d(a)+d(b))+b\alpha(a\alpha b-b\alpha a)\beta(d(a)+d(b))$

$(a\alpha b-b\alpha a)\beta a\alpha d(a) +(a\alpha b-b\alpha a)\beta b\alpha d(a)+(a\alpha b-b\alpha a)\beta a\alpha d(b) +(a\alpha b-b\alpha a)\beta b\alpha d(b)= a\alpha(a\alpha b-b\alpha a)\beta d(a) + a\alpha(a\alpha b-b\alpha a)\beta d(b) + b\alpha(a\alpha b-b\alpha a)\beta d(a) + b\alpha(a\alpha b-b\alpha a)\beta d(b)$

Now using Lemma 2.2(i), we have

$a\alpha(a\alpha b-b\alpha a)\beta d(a) +(a\alpha b-b\alpha a)\beta b\alpha d(a)+(a\alpha b-b\alpha a)\beta a\alpha d(b) -b\alpha(b\alpha a-a\alpha b)\beta d(b)= a\alpha(a\alpha b-b\alpha a)\beta d(a) + a\alpha(a\alpha b-b\alpha a)\beta d(b) + b\alpha(a\alpha b-b\alpha a)\beta d(a) - b\alpha(b\alpha a-a\alpha b)\beta d(b)$

Thus $(a\alpha b-b\alpha a)\beta(b\alpha d(a)+a\alpha d(b)) = (a\alpha b-b\alpha a)\beta d(a\alpha b)$, By using (A)

Therefore $(a\alpha b-b\alpha a)\beta (d(a\alpha b)-a\alpha d(b)-b\alpha d(a)) = 0$

This gives $[a,b]_\alpha \beta(d(a\alpha b) -a\alpha d(b) - b\alpha d(a))=0$.

(iii) Using Lemma 2.1(i) in 2.2(ii) ,we get

$(a\alpha b - b\alpha a)\beta\,(-d(b\alpha a)+2\,a\alpha d(b)+2b\alpha d(a) - a\alpha d(b) - b\alpha d(a)) = 0$

$(a\alpha b - b\alpha a)\beta\,(d(b\alpha a) - a\alpha d(b) - b\alpha d(a)) = 0$…………….(B)

Taking 2.2(ii)–(B) $(a\alpha b - b\alpha a)\beta d(a\alpha b) - d(b\alpha a) = 0$. Therefore $[a,b]_\alpha\,\beta\, d([a,b]_\alpha) = 0$.

(iv) From Lemma 2.1(i) , we have $d(a\alpha b+b\alpha a) = 2a\alpha d(b)+2b\alpha d(a)$

Replacing $b\beta a$ for b, we get $d(a\alpha b\beta a+b\beta a\alpha a) = 2a\alpha d(b\beta a)+2b\beta a\alpha d(a)$…………...(C)

Again replacing $a\beta b$ for b in Lemma 2.1(i) $d(a\alpha a\beta b + a\beta b\alpha a) = 2a\alpha d(a\beta b)+2a\beta b\,\alpha d(a)$…...(D)

Taking (D) – (C) and using (⋆) , we get

$d(a\alpha a\beta b + a\beta b\alpha a - a\alpha b\beta a - b\beta a\alpha a) = 2a\alpha d(a\beta b - b\beta a)+2(a\beta b - b\beta a)\alpha d(a)$.

Thus $d(a\alpha a\beta b - b\beta a\alpha a) = 2a\alpha d(a\beta b - b\beta a)+2(a\beta b - b\beta a)\alpha d(a)$……….(E)

Now replacing $a\beta a$ for a in Lemma 2.1(i) and using (⋆)

$d(a\beta a\alpha b + b\alpha a\beta a)=2a\beta a\alpha d(b)+2b\alpha d(a\beta a)=2a\beta a\alpha d(b)+4b\alpha a\beta d(a)=2a\beta a\alpha d(b)+4b\beta a\alpha d(a)$…(F)

Taking (E)+(F) $d(2a\alpha a\beta b)=2a\beta a\alpha d(b)+2a\alpha d(a\beta b - b\beta a)+2(a\beta b+b\beta a)\alpha d(a)$

Since M is 2–torsion free, we have $d(a\alpha a\beta b)= a\beta a\alpha d(b)+(a\beta b+b\beta a)\alpha d(a)+a\alpha d([a,b]_\beta)$.

2.3Theorem Let M be a 2–torsion free prime Γ–ring satisfying the assumption (⋆). If there exists a nonzero Jordan left derivation d: M→M, then M is commutative.

Proof: Let us assume that M is non commutative. Lemma 2.2(i) can be written as

$(x\alpha(x\alpha y - y\alpha x) - (x\alpha y - y\alpha x)\alpha x)\beta\, d(x) = 0 \quad \forall\ x,y \subset M$ and $\forall\, \alpha,\beta \subset \Gamma$.

This gives $(x\alpha x\alpha y - 2x\alpha y\alpha x+y\alpha x\alpha x)\beta\, d(x)=0 \quad \forall\ x,y \in M$ and $\forall\, \alpha,\beta \in \Gamma$

Replacing $[a,b]_\gamma$ for x, we have

$[a,b]_\gamma\,\alpha[a,b]_\gamma\,\alpha y\beta\, d([a,b]_\gamma) - 2[a,b]_\gamma\,\alpha y\alpha\,[a,b]_\gamma\,\beta\, d([a,b]_\gamma) + y\alpha\,[a,b]_\gamma\,\alpha[a,b]_\gamma\,\beta\, d([a,b]_\gamma)=0$.

But by Lemma 2.2(iii), we get $[a, b]_\gamma \ \alpha\, [a, b]_\gamma\, \alpha y\beta\, d([a, b]_\gamma) = 0$

For the primeness of M, we have $[a, b] \ \alpha\, [a, b] = 0$ or $d([a, b]_\gamma) = 0$

Suppose that $[a, b] \ \alpha\, [a, b] = 0$, for all $\alpha \in \Gamma$ …………(1)

Using Lemma 2.1 (i) , 2.1(ii), 2.2(iii) and (1) we see that

$W = d([a, b]_\gamma \ \beta x\beta\, [a, b]_\gamma \ \alpha y\alpha\, [a, b]_\gamma + [a, b]_\gamma \ \alpha y\alpha\, [a, b]_\gamma \ \beta\, [a, b]_\gamma \ \beta x)$

$= 2\, [a, b] \ \beta x\beta\, d([a, b]_\gamma \ \alpha y\alpha\, [a, b]_\gamma) + 2[a, b]_\gamma \ \alpha y\alpha\, [a, b]_\gamma \ \beta\, d([a, b]_\gamma \ \beta x)$

$= 2\, [a, b]_\gamma \ \beta x\beta\, [a, b]_\gamma \ \alpha\, [a, b]_\gamma \ \alpha d(y) + 6[a, b]_\gamma \ \beta x\beta\, [a, b]_\gamma \ \alpha y\alpha\, d([a, b]_\gamma)$

$-2\, [a, b]_\gamma \ \beta x\beta\, y\alpha\, [a, b]_\gamma \ \alpha\, d([a,b]_\gamma) + 2\, [a, b]_\gamma \ \alpha y\alpha\, [a, b]_\gamma \ \beta\, d([a, b]_\gamma \ \beta x)$

$= 6[a, b]_\gamma \beta x\beta\, [a, b]_\gamma \alpha y\alpha d([a, b]_\gamma) - 2[a, b]_\gamma \beta x\beta\, y\alpha[a, b]_\gamma \alpha d([a, b]_\gamma) + 2[a, b]_\gamma \ \alpha y\alpha[a, b]_\gamma \beta d([a, b]_\gamma \beta x)$

$= 6[a, b]_\gamma \ \beta x\beta\, [a, b]_\gamma \ \alpha y\alpha\, d([a, b]_\gamma) + 2\, [a, b]_\gamma \ \alpha y\alpha\, [a, b]_\gamma \ \beta\, d([a, b]_\gamma \ \beta x)$...............(2)

On the other hand $W = d([a, b]_\gamma \ \beta\, (x\beta\, [a, b]_\gamma\, \alpha y)\, \alpha\, [a, b]_\gamma)$

$= [a, b]_\gamma \ \alpha[a, b]_\gamma \ \beta\, d(x\beta[a, b]_\gamma \alpha y) + 3[a, b]_\gamma \ \beta x\beta\, [a, b]_\gamma \ \alpha y\alpha d([a, b]_\gamma) - x\, \beta[a, b]_\gamma \ \alpha y\beta\, [a, b]_\gamma \alpha d([a, b]_\gamma)$

$= 3[a, b] \ \beta x\beta\, [a, b]_\gamma \ \alpha y\alpha\, d([a, b]_\gamma)$ .... (3)

Therefore, subtracting (3) from (2), and using (⋆), we get

$3[a, b]_\gamma \ \beta x\alpha[a, b]_\gamma \ \beta y\alpha\, d([a, b]_\gamma) + [a, b]_\gamma \ \alpha y\beta\, 2[a, b]_\gamma \ \alpha\, d([a, b]_\gamma \ \beta x) = 0$ .... (4)

Also we have $V = d([a, b]_\gamma \ \alpha x\beta\, [a, b]_\gamma + x\alpha\, [a, b]_\gamma \ \beta\, [a, b]_\gamma) = d([a, b]_\gamma \ \alpha x\beta\, [a, b]_\gamma)$

$= [a, b]_\gamma \beta\, [a, b]_\gamma \alpha\, d(x) + 3[a, b]_\gamma \alpha x\beta\, d([a, b]_\gamma) - x\alpha\, [a, b]_\gamma \ \beta\, d([a, b]_\gamma) = 3[a, b]_\gamma \alpha x\beta\, d([a, b]_\gamma)$…..(5)

On the other hand

$V = d([a, b]_\gamma \ \alpha x\beta\ [a, b]_\gamma \ + x\alpha[a, b]_\gamma \ \beta\ [a, b]_\gamma \ ) = 2[a, b]_\gamma\, \alpha\ d(x\beta\ [a, b]_\gamma) + 2x\beta\ [a, b]_\gamma \ \alpha\ d([a, b]_\gamma)$

$= 2[a, b] \ \alpha\ d(x\beta\ [a, b])$............ (6)

Comparing (5) and (6) we obtain $3\ [a, b] \ \alpha x\beta d([a, b]_\gamma) = 2[a, b]_\gamma \ \alpha\ d(x\beta\ [a, b]_\gamma)$ ....(7)

Now $[a, b]_\gamma \ \alpha\ d(x\beta\ [a, b]_\gamma \ + [a, b]_\gamma \ \beta x) = [a, b]_\gamma \ \alpha\ (2x\beta\ d([a, b]_\gamma \ ) + 2[a, b]_\gamma \ \beta\ d(x)$

$= 2\ [a, b]_\gamma \ \alpha x\beta\ d([a, b]_\gamma \ ) + 2[a, b]_\gamma \ \alpha\ [a, b]_\gamma \ \beta\ d(x) = 2\ [a, b]_\gamma \ \alpha x\beta\ d([a, b]_\gamma \ )$ .... (8)

Thus $3[a, b]_\gamma \ \alpha\ (d(x\beta\ [a, b]_\gamma \ ) + d([a, b]_\gamma \ \beta x)) = 6[a, b]_\gamma \ \alpha x\beta\ d([a, b]_\gamma \ )= 2.3\ [a, b]_\gamma \ \alpha x\beta\ d([a, b]_\gamma)$

$= 2.2\ [a, b] \ \alpha\ d(x\beta\ [a, b]) = 4\ [a, b]_\gamma \ \alpha\ d(x\beta\ [a, b]_\gamma)$ by using (7)

Therefore we have $3[a, b]_\gamma \ \alpha\ (d(x\beta\ [a, b]_\gamma \ ) + 3[a, b]_\gamma \ \alpha\ d(\ [a, b]_\gamma \ \beta x) = 4\ [a, b]_\gamma \ \alpha\ d(x\beta\ [a, b]_\gamma)$.

This gives $3[a, b] \ \alpha\ d([a, b]_\gamma \ \beta x) = [a, b]_\gamma \ \alpha\ d(x\beta\ [a, b]_\gamma)$ .... (9)

By (9) we have $[a, b]_\gamma \ \alpha\ d((x\beta\ [a, b]_\gamma) + [a, b]_\gamma \ \beta x) = 3[a, b]_\gamma \ \alpha\ d([a, b]_\gamma \ \beta x) + [a, b]_\gamma \ \alpha\ (d([a, b]_\gamma \ \beta x)$

$= 4[a, b] \ \alpha\ d([a, b]_\gamma \beta x)$ ....(10)

We also obtain that $[a, b]_\gamma \ \alpha d((\ x\beta\ [a, b]_\gamma) + [a, b]_\gamma \beta x)=[a, b]_\gamma \alpha 2x\ \beta\ d(\ [a, b]_\gamma) + [a, b]_\gamma \alpha 2[a, b]_\gamma \beta d(x)$

$= 2\ [a, b] \ \alpha x\beta d([a, b]_\gamma)$………………(11)

From (10) and (11), we obtain $4[a, b]_\gamma \ \alpha\ d([a, b]_\gamma \ \beta x) – 2[a, b]_\gamma \ \alpha\ x\beta\ d([a, b]_\gamma \ ) = 0$

Since M is 2– torsion free, we have $2[a, b]_\gamma \ \alpha\ d([a, b]_\gamma \ \beta x) = [a,b]_\gamma \alpha x\beta\ d([a, b]_\gamma\ )$............(12)

From (4) and (12) $3[a, b]_\gamma \ \beta x\alpha\ [a, b]_\gamma \ \alpha y\beta\ d([a, b]_\gamma) + [a, b]_\gamma \ \alpha y\beta\ [a, b]_\gamma \ \alpha x\beta\ d([a, b]_\gamma\ ) = 0$.......(13)

Replacing $y\alpha[a, b]_\gamma \ \beta y$ for x in (12), $2[a, b]_\gamma \alpha d([a, b]_\gamma \beta y\alpha[a, b]_\gamma \beta y)=[a, b]_\gamma \alpha y\beta[a, b]_\gamma \alpha y\beta d([a, b]_\gamma)$

$2[a, b]_\gamma \ \alpha 2[a, b]_\gamma \ \beta y\alpha\ d\ ([a, b]_\gamma \ \beta y) = [a, b]_\gamma \ \alpha y\beta\ [a, b]_\gamma \ \alpha y\beta\ d([a, b]_\gamma)$

$4[a, b]_\gamma \ \alpha\ [a, b]_\gamma \ \beta y\alpha\ d([a, b]_\gamma \ \beta y) = [a, b]_\gamma \ \alpha y\beta\ [a, b]_\gamma \ \alpha y\beta\ d([a, b]_\gamma)$

Therefore $[a, b] \ \alpha y\beta\ [a, b] \ \alpha y\beta\ d([a, b]_\gamma) = 0$............. (14)

Now replacing y by x + y, $[a, b]_\gamma \ \alpha\ (x + y)\ \beta\ [a, b]_\gamma \ \alpha(x + y)\beta\ d([a, b]_\gamma)= 0$

$[a, b]_\gamma \ \alpha x\beta\ [a, b]_\gamma \ \alpha x\beta\ d([a, b]_\gamma) + [a, b]_\gamma \ \alpha x\beta\ [a, b]_\gamma \ \alpha y\beta\ d([a, b]_\gamma)$

$+ [a, b]_\gamma \ \alpha y\beta\ [a, b]_\gamma \ \alpha x\beta\ d([a, b]_\gamma) + [a, b]_\gamma \ \alpha y\beta\ [a, b]_\gamma \ \alpha y\beta\ d([a, b]_\gamma) = 0.$

This gives $[a, b]_\gamma \ \alpha x\beta\ [a, b]_\gamma \ \alpha y\beta\ d([a, b]_\gamma) + [a, b]_\gamma \alpha y\beta\ [a, b]_\gamma \ \alpha x\beta\ d([a, b]_\gamma) = 0.$

Therefore, $[a, b]_\gamma \ \alpha x\beta\ [a, b]_\gamma \ \alpha y\beta\ d([a, b]_\gamma) + [a, b]_\gamma \ \alpha y\beta\ [a, b]_\gamma \ \alpha x\beta\ d([a, b]_\gamma) = 0$ …… (15)

Subtracting (15) from (13) ,we obtain $2[a, b]_\gamma \ \alpha x\beta\ [a, b]_\gamma \ \alpha y\beta\ d([a, b]_\gamma) = 0.$

Since M is 2–torsion free, we have $[a, b]_\gamma \ \alpha x\beta\ [a, b]_\gamma \ \alpha y\beta\ d([a, b]_\gamma) = 0$, for every $y \in M$.

Since M is prime and non commutative, we have $d\ ([a, b]_\gamma) = 0$, $\forall\ a,b \in M$ and $\forall \gamma \in \Gamma$.

This implies that $d(a\gamma b - b\gamma a) = 0$. Therefore $d(a\gamma b) = d(b\gamma a)$, $\forall\ a,b \in M$ and $\forall \gamma \in \Gamma$……(16)

$d((b\alpha a)\beta a+a\beta(b\alpha a))=d((b\alpha a)\beta a)+d(a\beta(b\alpha a))=d((b\alpha a)\beta a)+d((b\alpha a)\beta a)=2d((b\alpha a)\beta a)$ by using (16)

Thus $2d((b\alpha a)\beta a)= d((b\alpha a)\beta a + a\beta(b\alpha a))= d((b\alpha a)\beta a)+d(a\beta(b\alpha a))$

This implies $d(b\alpha a\beta a)= a\alpha a\beta d(b)+3a\beta b\alpha d(a)– b\beta a\alpha d(a)$…(17), by using Lemma 2.1(ii).

On the other hand $d(a\alpha(b\beta a)+(b\beta a)\alpha a)=2(a\alpha d(b\beta a) + b\beta a\alpha d(a))$ ……………(18)

and $d(a\alpha(a\beta b)+(a\beta b)\alpha a)=2(a\alpha d(a\beta b)+(a\beta b)\alpha d(a))$ ……………(19)

Taking(19) –(18), $d(a\alpha a\beta b–b\beta a\alpha a)=2a\alpha\, d([a,b]_\beta)+2[a,b]_\beta\alpha\, d(a),\forall a,b\in M,\alpha,\beta\in\Gamma$……(20)

Now putting $a\beta a$ for a in Lemma 2.1(i), we have

$d(a\beta a\alpha b+b\alpha a\beta a)=2(a\beta a\alpha\, d(b)+ b\alpha\, d(a\beta a))= 2(a\beta a\alpha\, d(b)+ b\alpha\, 2a\beta\, d(a))$.

Using (⋆) above equation reduces to $d(a\alpha a\beta b+b\beta a\alpha a)=2(a\alpha a\beta d(b)+2b\beta a\alpha\, d(a))$…(21)

Subtracting (20) from (21) and using (⋆), we get

$d(2b\alpha a\beta a)=2a\beta a\alpha d(b)+4b\alpha a\beta d(a)–2a\alpha\, d([a,b]_\beta)–2[a,b]_\beta\,\alpha\, d(a))$

$d(b\alpha a\beta a)=a\beta a\alpha d(b)+2b\alpha a\beta d(a)– a\alpha\, d([a,b]_\beta)– a\beta b\alpha d(a)+ b\beta a\alpha d(a)$, M is 2–torsion free

$= a\beta a\alpha d(b)+3b\alpha a\beta d(a)– a\alpha\, d([a,b]_\beta)– a\beta b\alpha d(a)$, by using (⋆)

$= a\alpha a\beta d(b)+3b\beta a\alpha d(a)– a\beta b\alpha d(a)$…………(22)

From (22)and(17), $a\alpha a\, \beta d(b)+3a\beta b\alpha d(a)– b\beta a\alpha\, d(a)= a\alpha a\beta d(b)+3b\beta a\alpha d(a)– a\beta b\alpha d(a)$.

This gives $–3(b\beta a – a\beta b)\alpha\, d(a) – (b\beta a – a\beta b)\alpha\, d(a) = 0$

Then we have $-3[b,a]_\beta\, \alpha\, d(a) - [b,a]_\beta\, \alpha\, d(a) = 0$.

Since M is 2–torsion free, so $[b,a]_\beta\, \alpha\, d(a) = 0$……………….(23)

Now putting $b\gamma x$ for b in (23), we get $[b\gamma x, a]_\beta\, \alpha\, d(a) = 0$

$([b,a]_\beta\, \gamma x + a[\gamma,\beta]_a x + b\gamma\, [x,a]_\beta)\alpha\, d(a) = 0$

Since $a[\gamma,\beta]_a x = a(\gamma a\beta - \beta a\gamma)x = a\gamma a\beta x - a\beta a\gamma x = 0$, by using (⋆).

Thus $([b,a]_\beta\, \gamma x + b\gamma\, [x,a]_\beta)\alpha\, d(a) = 0$

$[b,a]_\beta\, \gamma x\, \alpha\, d(a) + b\gamma\, [x,a]_\beta\, \alpha\, d(a) = 0$

$[b,a]_\beta\, \gamma x\, \alpha\, d(a) = 0$, $\forall\ a,b \in M$ and $\forall\ \alpha, \beta, \gamma \in \Gamma$

Since M is prime , thus d(a)=0. Hence we conclude that if d≠0, then M is commutative.

2.3Theorem If M is a 2–torsion free prime Γ–ring satisfying the assumption (⋆) then every Jordan left derivation on M is a left derivation on M.

Proof: Since M is commutative. Thus $a\alpha b = b\alpha a$, for all $a,b \in M$ and $\alpha \in \Gamma$.

By Lemma 2.1(i), we have $2d(a\alpha b)=2a\alpha\, d(b)+2b\alpha\, d(a)= 2(a\alpha d(b)+ b\alpha\, d(a))$

Since M is 2–torsion free, we get $d(a\alpha b) = a\alpha d(b)+ b\alpha\, d(a)$.

This completes the proof.

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