Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.6, No.1, 2016

On nth - power paranormal operators on Hilbert spaces

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Abstract

In this paper we introduce a new class of operators on a Hilbert space. We call these operators in this class, nth-power paranormal operators. We study this class of operators and give some of their basic properties.
Keywords: paranormal operator , Hilbert space .

0. Introduction.

Let H be a Hilbert space and let B(H) be the algebra of all bounded linear operators on H.

Furuta [2] has defined a bounded linear operator T on a Hilbert space H as paranormal if $\| T^2x\| \geq \| Tx\|^2$ for every unit vector x in H .

In this paper we discuss a new class of operators as follow :

Let T ∈ B(H), T is called nth - power paranormal operator if for some positive integer n, we have $\| T^{2n}x \| \geq \| T^n x \|^2$ for every unit vector x in H.

Moreover, we give a characterization of nth - power paranormal operator (see theorem (1.4)) , and prove some important results about it .

Theorem 0 [4]:- Let T ∈ B(H) . If T is paranormal operator, then T^n is paranormal operator for each n ∈ N .

1. nth - power paranormal operators.

Definition 1.1: Let T ∈ B(H) . T is called nth - power paranormal operator if for some positive integer n, we have $\|T^{2n}x \| \geq \| T^n x \|^2$, for every unit vector x.

Remark 1.2: One can see that every paranormal operator is 1th-power paranormal operator. But the converse is not necessary true in general. For example it T is any nilpotent operator of order m, i.e, T^m=0, then T is mth – power paranormal operator, but it is not necessarily paranormal operator.

We can by theorem (0), conclude the following result.

Corollary 1.3: Let T be a bounded linear operator. If T is nth - power paranormal operator, then T^n is paranormal operator for each n∈N.

We start this section by the following main result which is characterized the nth - power paranormal operator.

Theorem 1.4 [2]: An operator T is nth –power paranormal operator if and only if $T^{*2n} T^{2n} - 2\lambda T^{*n} T^n + \lambda^2 I \geq 0$, for all $\lambda \geq 0$ and positive integer number n.

To prove theorem (1. 4) we need the following lemma.

Lemma 1.5 [3] : Let a and b two positive number, then $a^\beta b^\mu \leq \beta a + \mu b$
holds for β, μ > 0 such that β + μ =1.

The proof of theorem (1.4)

Assume that T is nth – power paranormal operator, then $\|T^{2n}x\| \geq \|T^n x\|^2$,
x ∈ H, ||x|| = 1, n ∈ N.Thus

$$\|T^{2n}(\frac{x}{\|x\|}) \| \geq \|T^n (\frac{x}{\|x\|}) \|^2, \ \|x\|=1, n \in N, x \in H.$$

So that,

$$(\frac{1}{||x||})\ \|T^{2n}x\| \geq (\frac{1}{||x||^2})\ \|T^{n}x\|^2,$$

this implies that

$$\|T^{2n}x\|\ \|x\| \geq \|T^{n}x\|^2,\ x \in H,\ \|x\|=1.$$

Hence,

$$< T^{2n}x, T^{2n}x >^{1/2} < x, x >^{1/2} \geq < T^{n}x, T^{n}x >.$$

Therefore, $<T^{*2n}\, T^{2n}\, x, x>^{1/2}\ <x, x>^{1/2} \geq <T^{*n}\, T^{n}x, x> \ldots..\ (1)$.

But $<T^{*2n}\, T^{2n}x, x>^{1/2}$ and $<x, x>^{1/2}$

are positive , therefore by using lemma (1.4) with β=μ=1/2.

Thus for each λ >0, we have

$$<T^{*2n}\, T^{2n}x, x>^{1/2} <x, x>^{1/2} = (\frac{1}{\lambda}<T^{*2n}\, T^{2n}\ x, x>)^{1/2}\ (\lambda<x, x>)^{1/2}$$

$$\leq \frac{1}{2\lambda} < T^{*2n}\, T^{2n}\ x, x > + \frac{\lambda}{2} < x, x>.$$

Hence, by (1) we have

$$\frac{1}{2\lambda}<T^{*2n}\, T^{2n}\ x, x>+\frac{\lambda}{2}\ <x, x> \geq <T^{*n}\, T^{n}\ x, x>.$$

Therefore,

$$<(\frac{1}{2\lambda}T^{*2n}\, T^{2n} - T^{*n}\, T^{n} + \frac{\lambda}{2})\ x, x> \geq 0\ \ \ldots.\ (2)$$

This implies that

$$\frac{1}{2\lambda}T^{*2n}\, T^{2n} - T^{*n}\, T^{n} + \frac{\lambda}{2} \geq 0.$$

Therefore, $T^{*2n}\, T^{2n} - 2\lambda\, T^{*n}\, T^{n} + \lambda^2 \geq 0$ for all $\lambda > 0$, $n \in N\ldots\ldots$ (3)

The left side of (2) is zero and (3) again holds. Hence,

$$T^{*2n}\, T^{2n} - 2\lambda\, T^{*n}\, T^{n} + \lambda^2 \geq 0, \quad \text{for each } \lambda > 0,\ n \in N.$$

Conversely, if $T^{*2n}\, T^{2n} - 2\lambda\ T^{*n}T^{n} + \lambda^2 \geq 0$, for each $\lambda > 0$, $n\in N$, then

$$\frac{1}{2\lambda}\ T^{*2n}\, T^{2n} - T^{*n}\, T^{n} + \frac{\lambda}{2} \geq 0.$$

Thus $\frac{1}{2\lambda}T^{*2n}\, T^{2n} + \frac{\lambda}{2} \geq T^{*n}\, T^{n}$.

Hence, for each x∈H, we have,

$$\frac{1}{2\lambda}< T^{*2n}\, T^{2n}x, x >+\frac{\lambda}{2}<x, x> \geq < T^{*n}T^{n}\ x, x >.$$

Now,

Put $\lambda = \left(\frac{<T^{*2n}\, T^{2n}x,x>}{<x,x>}\right)^{1/2}$, then

$$\frac{1}{2}< T^{*2n}\, T^{2n}x, x >^{1/2}< x, x >^{1/2} + \frac{1}{2}<T^{*2n}\, T^{2n}x, x >^{1/2}< x, x >^{1/2} \geq\ < T^{*n}\, T^{n}x, x >.$$

Hence, $<T^{*2n}\, T^{2n}x, x>^{1/2}<x, x>^{1/2} \geq < T^{*n}\, T^{n}x, x>$.

Therefore, $<T^{2n}x, T^{2n}x>^{1/2}<x, x>^{1/2} \geq < T^{n}x, T^{n}x>$.

So that $\|T^{2n}x\|\ \|x\| \geq \|T^{n}x\|^2$

Hence, for each unit vector x∈H and positive integer number n, we have

$$\|T^{2n}x\| \geq \|T^{n}x\|^2.$$

Thus T is nth - power paranormal operator.

Following results collects some of basic properties of nth-power paranormal operators.

proposition 1.6: If T∈B(H) is nth - power paranormal operator, then

1- T^* is nth - power paranormal operator.

2- If T^{-1} exist then T^{-1} is nth - power paranormal operator.

3-If S∈B (H) is unitary equivalent to T, then S is nth - power paranormal operator.

4- If M is a closed subspace of H such that M reduces T, then (T/M) is nth-power paranormal operator.

Proof:- Since T is nth-power paranormal operator, then for some positive integer n, we have $\|T^{2n}x\| \geq \|T^{n}x\|^2$ for every unit vector x in H.

1- For all x in H,

$$T^{2n}T^{*2n} - 2\lambda T^n\ T^{*n} + \lambda^2 \geq 0$$

$\Leftrightarrow\ < (T^{2n}T^{*2n} - 2\lambda T^n\ T^{*n} + \lambda^2)\, x, x > \ \geq 0,$ for all $\lambda \in R$

$\Leftrightarrow\ < T^{2n}T^{*2n}\ x, x > - 2\lambda < T^n\ T^{*n}x, x > + \lambda^2 < x, x > \geq 0$

$\Leftrightarrow\ < T^{*2n}x, T^{*2n}x > - 2\lambda < T^{*n}x, T^{*n}x > + \lambda^2 < x, x > \geq 0 .$

$$\|T^{*2n}\|^2 - 2\lambda\|T^{*n}\|^2 + \lambda^2\|x\|^{\,2} \geq 0 .$$

By elementary properties of real quadratic forms: If a > 0 , b and c are real quadratic forms: If a > 0, b and c are real numbers then $at^2 + bt + c \geq 0$ for every real t if and only if $b^2 - 4ac \leq 0$,we get

$$4\|T^{*n}\|^4 \leq 4\|T^{*2n}\|^2\|x\|^{\,2} \text{ for all } x \in H.$$

$$\|T^{*n}\|^2 \leq \|T^{*2n}\|^2\ \ \|x\| \quad \text{for all } x \in H.$$

T^* is nth - power paranormal operator.

2- Since T is nth-power paranormal operator, then

$$\|T^{2n}x\|\|x\| \geq \|T^n x\|^2$$

For each x, $\|x\| = 1,\ n \in N.$ Thus ,

$$\frac{\|x\|}{\|T^n x\|} \geq \frac{\|T^n x\|}{\|T^{2n}x\|}$$

Now replace x by $(T^{-1})^{2n}\, x$ then

$$\|x\|\ \ \|(T^{-1})^{2n}x\| \geq \|(T^{-1})^n x\|^2.$$

for each $x \in H$, $n \in N$. This shows that T^{-1} is nth - power paranormal operator.

3- Since S is unitary equivalent to T , then S= UTU^*. Therefore, $S^n = U\, T^n\, U^*$.

$$\|S^n x\|^2 = \|U\, T^n U^* x\|^2$$
$$\leq \|U\|^2\ \|T^{\,n}\, (U^* x)\|^2$$

(Since T^n is paranormal)

$$= \|U\ (T^n(U^* x))\|^2$$
$$\leq \|U\, T^{2n}(U^* x)\|$$
$$\leq \|S^{2n}x\|$$

4- Let $x \in M$. Then we have

$\|(T/M)^n\, x\|^2 = \|\,(T^n/M)x\|^2 = \|T^n x\|^2 \leq \|T^{2n}x\|\ \ \|x\|$

$= \|(T^{2n}/M)x\,\|\ \|x\| = \|(T/M)^{2n}\, x\|\ \|x\|.$

This implies that (T/M) is nth - power paranormal operator.

Theorem 1.7: If a nth - power paranormal operator T commutes with an isometric operator S, then TS is nth-power paranormal operator.

Proof:- let A=TS, We have for any real number λ that,

$A^{*2n}\ \ A^{2n} - 2\lambda\, A^{*n}\ A^n + \lambda^2\ I =$

$S^{*n}T^{*n}S^{*n}T^{*n}T^nS^nT^nS^n - 2\lambda\ S^{*n}T^{*n}T^nS^n + \lambda^2\ I.$

But $T^nS^n = S^nT^n$ and $S^{*n}S^n = I$, we have $A^{*2n}\ \ A^{2n} - 2\lambda A^{*n}\ \ A^n + \lambda^2 I =$

$T^{*2n}T^{2n} - 2\lambda\, T^{*n}T^n + \lambda^2 I \geq 0,$

so that A is nth - power paranormal operator.

Theorem 1. 8:- Let T be a nth-power paranormal operator ,then

$$\| T^{3n}x \| \geq \|T^{2n}x\|\ \|T^n x\|$$

for each x, $\|x\| = 1$, $n \in N$.

<u>**Proof :-**</u>

$\|T^{3n}\, x\| = \|\, T^n x\,\|\ \|\, T^{2n}\ \left(\frac{T^n x}{\|T^n x\|}\right) \|$

$\geq \|T^n x\| \; \| T^n \left(\frac{T^n x}{||T^n x||}\right) \|^2$

$= \frac{1}{||T^n x||} \quad \|T^{2n} x\|^2$

$= \frac{||T^{2n}x||}{||T^n x||} \quad \|T^{2n} x\|$

$\geq \frac{||T^{2n}x||}{||T^n x||} \quad \|T^n x\|^2$

$= \|T^{2n} x\| \, \|T^n x\|$.

As we desired .

Theorem 1-9 :- Let T be a weighted shift with non zero weights $\{\propto_n\}$ (n=1,2,….) . Then T is a mth-power paranormal operator if and only if

$$|\propto_n| \, |\propto_{n+1}| \ldots |\propto_{n+m-1}| \leq |\propto_{n+m}| \, |\propto_{n+m+1}| \ldots |\propto_{n+2m-1}| \text{ for } n= 1,2,3\ldots$$

Proof :- Let $\{e_n\}_{n=1}^{\infty}$ be an orthnonmal basis of a Hilbert space H .

Suppose T is a mth-power paranormal operator then T^m is paranormal operator Therefore $\|T^m e_n\|^2 \leq \|T^{2m} e_n\|$ (n=1,2,3,…..) .

Note that $\|Te_n\| = \| \propto_n e_{n+1}\| = |\propto_n|$

Here

$$T^m e_n = \propto_n \propto_{n+1} \ldots\ldots \propto_{n+(m-1)} e_{n+m}$$

And

$$T^{2m} e_n = \propto_n \propto_{n+1} \ldots\ldots \propto_{n+(m-1)} \propto_{n+m} \ldots \propto_{n+(2m-1)} e_{n+2m.}$$

For m=1,2,…. . But $\|T^m e_n\|^2 \leq \|T^{2m} e_n\|$ (n=1,2,…..),

and so

$$|\propto_n|^2 |\propto_{n+1}|^2 \ldots |\propto_{n+m-1}|^2 \leq |\propto_n| \, |\propto_{n+1}| \ldots |\propto_{n+m-1}| \, |\propto_{n+m}| \ldots |\propto_{n+2m-1}|.$$

Therefore, for n =1,2,3, . . . ,

$$|\propto_n| \, |\propto_{n+1}| \ldots |\propto_{n+m-1}| \leq |\propto_{n+m}| \, |\propto_{n+m+1}| \ldots |\propto_{n+2m-1}| .$$

Conversely,

Suppose $|\propto_n| \, |\propto_{n+1}| \ldots |\propto_{n+m-1}| \leq |\propto_{n+m}| \, |\propto_{n+m+1}| \ldots |\propto_{n+2m-1}|$

for n= 1,2,3,…. .Then we have

$\|T^{2m} e_n\| - \|T^m e_n\|^2 = \| \propto_n \propto_{n+1} \ldots \propto_{n+m-1} \propto_{n+m} \ldots \propto_{n+2m-1} e_{n+m} \|$

$- \|\propto_n \propto_{n+1} \ldots \propto_{n+m-1} e_{n+m}\|^2$

$= |\propto_n| \, |\propto_{n+1}| \ldots |\propto_{n+m-1}| \, (|\propto_{n+m}| \ldots |\propto_{n+2m-1}| - |\propto_n| \ldots |\propto_{n+m-1}|) \geq 0.$

Therefore , $\|T^m e_n\|^2 \leq \|T^{2m} e_n\|$ (n= 1,2,. . . .) ,

and so T is a mth – power paranormal operator.

Recall that T∈ B(H) is called n-normal operator if $T^n T^* = T^* T^n$ for some positive integer n.

It is well known that every normal operator is paranormal operator. In the next

Theorem we prove that every n-normal operator is nth-power paranormal operator-

Theorem 1.11: - Every n-normal operator T in B(H) is nth-power paranormal operator.

Proof: - since T is n-normal operator, there T^n is normal operator (see [1]) .

Thus we have, $\|T^n x\| = \|T^{*n} x\| \quad \forall x$,

since $\|T^n x\|^2 = \langle T^n x, T^n x \rangle$

$= \langle T^{*n}(T^n x), x \rangle$

$\leq \|T^{*n}(T^n x)\| \, \|x\|$

$= \|T^n(T^n x)\| \, \|x\|$

$= \|T^{2n} x\| \, \|x\|.$

Therefore, T is nth-power paranormal operator .

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