Endo SS-Coprime Modules

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Abstract

The purpose of this paper is to introduce and investigate the notion of endo strongly S-coprime modules, where an *R*-module *M* is called an endo strongly S-coprime module (briefly endo SS-coprime) if, for all $f, g \in End(M)$, $\text{Im}(f \circ g)$ is small in M implies $f = 0$ or $g = 0$. We give some properties of endo SS-coprime modules. Several of various relations between such modules and other classes are obtained. Moreover, we give some equivalent statements for endo SS-coprime modules. We also introduce the notion of semi-endo strongly S-coprime modules as generalization of endo strongly S-coprime modules, where an *R*-module M is called semi-endo strongly S-coprime (briefly semi-endo SS-coprime) if, for each $f \in End(M)$, $\text{Im}(f \circ f)$ is small in M implies $f = 0$. Some results of such modules are given .

Key Words : Endo SS-coprime modules ; Semi-endo SS-coprime modules ; T-noncosingular modules ; SS-coprime modules; SSS-coprime modules ; S-coprime modules .

1. Introduction

Throughout this article, let *M* be a left module as a commutative ring with identity. We denote the ring of all endomorphisms of *M* by $End(M)$ and the Jacobson radical of M by $Rad(M)$.We will denote the left annihilator of M in $S = End(M)$ by $l_S(M)$, and the direct summand N of M by $N \leq^{\oplus} M$. Recall that a submodule $N \leq M$ is called small and denoted by $N \ll M$ if, $N + K \neq M$ for every proper submodule K of *M*, [2]. Following [10], an *R*-module *M* is called T-noncosingular if, for every nonzero endomorphism φ of *M*, $\text{Im}\,\varphi$ is not small in *M*. Hadi I.M-A in [7] introduce the notion of strongly S-coprime module (briefly, SS-coprime), where a module *M* is called strongly S-coprime (briefly SS-coprime) if, for all $a, b \in R$, $abM \ll M$ implies $aM = 0$ or $bM = 0$. In section 3 of this paper we further investigate the notion of endo strongly S-coprime module (briefly endo SS-coprime), where an *R*-module *M* is called endo SS-coprime if for each $f, g \in End(M)$ with $\text{Im}(f \circ g) \ll M$ implies $f = 0$ or $g = 0$.We show that in general the direct sum of endo SS-coprime modules is not endo SS-coprime module. We also prove that endo SS-coprime is inherited by direct summands. We prove some results concerning these types of modules. It is shown that a divisible R-module M over no zero divisor ring $S = End(M)$, is faithful endo SS-coprime. For a multiplication module M, we prove that the concepts endo SS-coprime module and a SS-coprime module are coincide. In section 4, the concept of semi-endo SS-coprime modules is presented as generalization of endo SS-coprime modules, where an R-module M is called semi-endo SS-coprime if for any $f \in End(M)$ with $\operatorname{Im}(f \circ f) \ll M$ implies $f = 0$. Most of properties of endo SS-coprime modules generalized to semi-endo SS-coprime module. Several properties of semi-endo SS-coprime modules and some connections between semi-endo SS-coprime modules and other related concepts are given. It is proved that, an R-module M is semi-endo SS-coprime if and only if $l_S(M)$ is a semiprime ideal of S and M is T-noncosingular, where $S = End(M)$.

2. Definitions and Notation

Definition 2.1 An R-module M is called S-coprime if, $ann_R M = ann_R \frac{M}{N}$ for every small submodule N of M [8]. Equivalently, M is S-coprime if whenever $r \in R$, $rM \ll M$ implies $rM = 0$.

Definition 2.2 An R-module M is called strongly S-coprime (briefly SS-coprime) if, for all $a, b \in R$, $abM \ll M$ implies $aM = 0$ or $bM = 0$ [7].

Definition 2.3 An R-module M is called semi-strongly S-coprime (briefly SSS-coprime) if, for all $r \in R$, $r^2 M \ll M$ implies $rM = 0$ [7].

Definition 2.4 An R-module M is said to be T-noncosingular if, for each nonzero endomorphism φ of M, $\operatorname{Im}\varphi$ is not small in M [10].

Remark 2.5 By [7], we have the following implications:

SS-coprime $\Rightarrow$ S-coprime,
SS-coprime $\Rightarrow$ SSS-coprime,
T-noncosingular $\Rightarrow$ S-coprime.

3. Endo strongly S-coprime modules

In this section, the class of endo strongly S-coprime modules is defined and investigated. First we obtain some properties of this kind of modules. Also relations between such modules and some other classes of modules will be studied .

Definition 3.1 An R-module M is called endo strongly S-coprime (briefly endo SS-coprime) if, for each $f, g \in End(M)$ with $\operatorname{Im}(f \circ g) \ll M$ implies $f = 0$ or $g = 0$.

Remarks and Examples 3.2

(1) It is clear that every endo SS-coprime module is T-noncosingular, but not conversely, as the following example shows : it is clear that the Z-module Z_6 is T-noncosingular. Assume $f, g \in End(Z_6)$ defined by $f(\bar{x}) = 2\bar{x}$, $g(\bar{x}) = 3\bar{x}$ for all $\bar{x} \in Z_6$. Now, we have $\operatorname{Im}(f \circ g) = (0) \ll M$, but neither f nor g is zero. This means that the Z-module Z_6 is not endo SS-coprime.

(2) Let M be an R-module, $S = End(M)$. Then M is endo SS-coprime if and only if M is T-noncosingular and $l_S(M)$ is prime.

Proof. It is obvious . □

(3) Every endo SS-coprime module is SS-coprime.

Proof. Suppose that M is an endo SS-coprime module. Let $a, b \in R$, $abM \ll M$. Define the endomorphisms f, g on M by $f(m) = am$ and $g(m) = bm$ for all $m \in M$. Then $\operatorname{Im}(f \circ g) = f(bM) = abM \ll M$, but M is endo SS-coprime, so either $f = 0$ or $g = 0$ and this implies $aM = 0$ or $bM = 0$. □

The converse is not true in general, for example : consider the Z-module $Z_{2^\infty} \oplus Z_2$. See [7, Rem.and.Ex. 2.2(3)], $Z_{2^\infty} \oplus Z_2$ is SS-coprime Z-module but not T-noncosingular, thus by (1), $Z_{2^\infty} \oplus Z_2$ is not endo SS-coprime.

By (3) and Remark 2.5, we have the following.

Corollary 3.3 Every endo SS-coprime module is S-coprime and SSS-coprime.

Proposition 3.4 Let M be an R-module, $\overline{R} = R/annM$. Then M is an endo SS-coprime R-module if and only if M is an endo SS-coprime $\overline{R}$-module.

Proof. It is obvious . □

Proposition 3.5 If M_1 and M_2 are two isomorphic R-modules. Then M_1 is endo SS-coprime if and only if M_2 is endo SS-coprime.

Proposition 3.6 Let M be an R-module, $S = End(M)$. Then S is an endo SS-coprime S-module implies S has no zero divisors.

Proof. Suppose S is an endo SS-coprime S-module. Let $f, g \in S$ such that $f \circ g = 0$, thus $Sf.Sg \ll S$, so either $Sf = 0$ or $Sg = 0$, hence $f = 0$ or $g = 0$, and so S has no zero divisors . □

A submodule N of M is said to be stable if, $f(N) \subseteq N$ for each R-homomorphism $f : N \to M$. A module M is called fully stable in case each submodule of M is stable [1]. An R-module M is called multiplication if for every submodule N of M there exists an ideal I of R such that $N = IM$ [3].

The following two corollaries are immediately.

Corollary 3.7 Let M be a fully stable R-module, $S = End(M)$ .Then S is an endo SS-coprime S-module implies S is an integral domain.

Proof. Since M is fully stable, thus by [1, Prop. 1.2.1] $S = End(M)$ is a commutative ring. Hence the result is follow by Proposition 3.6 . □

Corollary 3.8 Let M be a multiplication R-module, $S = End(M)$ .Then S is an endo SS-coprime S-module implies S is an integral domain.

Proof. If M is multiplication, thus by [13, Prop. 1.1] $S = End(M)$ is a commutative ring. Hence the result is follow by Proposition 3.6 . □

Proposition 3.9 Let M be a (multiplication or fully stable) R-module. If M is endo SS-coprime then S is an integral domain, where $S = End(M)$.

Proof. Let $f, g \in End(M)$ such that $f \circ g = 0$, then $\mathrm{Im}(f \circ g) \ll M$, but M is endo SS-coprime, implies $f = 0$ or $g = 0$. Thus S has no zero divisors. Since M is (multiplication or fully stable), then the result is obtained . □

Hadi I.M-A in [7], presented the following result.

Lemma 3.10 Let M be a multiplication R-module. Then M is an SS-coprime R-module if and only if M is an SS-coprime S-module, where $S = End(M)$.

The next result follows directly.

Proposition 3.11 Let M be a multiplication R-module. Then M is an endo SS-coprime module if and only if M is a SS-coprime module.

Corollary 3.12 Let R be a ring. Then R is endo SS-coprime if and only if R is SS-coprime.

Proof. Since R is a commutative ring with identity, then R is multiplication. Hence, the result obtained by Proposition 3.11 . □

Recall that an R-module M is called a scalar module if, for all $\varphi \in End(M)$, there exists $r \in R$ such that $\varphi(m) = rm$ for all $m \in M$ [14].

We noticed that every endo SS-coprime module is a SS-coprime module but not conversely (see Rem. and.Ex. 3.2(3)). In the next result we present condition under which the converse is satisfied.

Proposition 3.13 Let M be a scalar R-module. If M is a SS-coprime module, then M is endo SS-coprime.

Proof. Let $f, g \in End(M)$, $\mathrm{Im}(f \circ g) \ll M$ .Since M is scalar, so there exist $a, b \in R$ such that $f(m) = am$ and $g(m) = bm$ for all $m \in M$ .Then $abM = \mathrm{Im}(f \circ g)$ is small in M, but M is SS-coprime, so either $aM = 0$ or $bM = 0$ this implies $f = 0$ or $g = 0$. □

The following two results are characterizations of endo SS-coprime modules.

Proposition 3.14 Let M be an R-module, $S = End(M)$. Then M is an endo SS-coprime module if and only if, for each ideals A , B of S, $ABM \ll M$ implies that $AM = 0$ or $BM = 0$.

Proof. Assume that M is an endo SS-coprime module. Let A , B be ideals of S , $ABM \ll M$ and $BM \neq 0$, so there exists $g \in B$ such that $g(M) \neq 0$. For each $f \in A$, $\mathrm{Im}(f \circ g) \leq ABM \ll M$, but M is endo SS-coprime and $g \neq 0$, thus $f = 0$ for all $f \in A$. Hence $AM = 0$.

Conversely, let $f, g \in End(M)$ with $\mathrm{Im}(f \circ g) \ll M$. Then $(Sf.Sg)M \ll M$, so by assumption, $(Sf)M = 0$ or $(Sg)M = 0$, and hence $f = 0$ or $g = 0$. □

Proposition 3.15 Let M be an R-module, $S = End(M)$. Then M is endo SS-coprime if and only if for each $f, g \in S$, $\mathrm{Im}(f \circ g) \ll M$ implies $(f(M) :_S M) = l_S(M)$ or $(g(M) :_S M) = l_S(M)$.

Proof. Assume M is an endo SS-coprime R-module. Let $f, g \in End(M)$ with $\mathrm{Im}(f \circ g) \ll M$, so $f = 0$ or $g = 0$ and hence $(f(M) :_S M) = (0 :_S M) = l_S(M)$ or $(g(M) :_S M) = (0 :_S M) = l_S(M)$.

Conversely, if $f, g \in End(M)$, $\mathrm{Im}(f \circ g) \ll M$, thus by hypothesis, $(f(M) :_S M) = l_S(M)$ or $(g(M) :_S M) = l_S(M)$. But $f \in (f(M) :_S M)$ and $g \in (g(M) :_S M)$, so either $f \in l_S(M)$ or $g \in l_S(M)$; that is, either $f = 0$ or $g = 0$. □

Proposition 3.16 Every direct summand of an endo SS-coprime module is also endo SS-coprime.

Proof. Let M be an endo SS-coprime module, and let $N \leq^{\oplus} M$, then $M = N \oplus K$ for some submodule K of M. Let $f, g \in End(N)$, $\mathrm{Im}(f \circ g) \ll N$. Consider the endomorphisms φ, ψ of M , $\varphi(n+k) = f(n)$ and $\psi(n+k) = g(n)$ for all $n \in N$. Notice that φ, ψ are well-defined. Now, $\mathrm{Im}(\varphi \circ \psi) = \mathrm{Im}(f \circ g) \ll N$ implies that $\mathrm{Im}(\varphi \circ \psi) \ll M$, but M is endo SS-coprime, so either $\varphi = 0$ or $\psi = 0$ this mean that, $f = 0$ or $g = 0$. Hence N is endo SS-coprime . □

Remarks 3.17

(1) A homomorphic image of endo SS-coprime module is not necessarily endo SS-coprime, for example: we know that in the Z-module Z, the zero submodule is the only small, so it is clear that Z as a Z-module is endo SS-coprime. Consider the natural epimorphism $\pi : Z \to Z_4$, then $\mathrm{Im}\,\pi = Z_4$ is not endo SS-coprime, to see this : let $\varphi, \psi \in End(Z_4)$ such that $\varphi(\bar{x}) = \bar{x}$ and $\psi(\bar{x}) = 2\bar{x}$ for all $\bar{x} \in Z_4$. Then $\mathrm{Im}(\varphi \circ \psi) = \varphi(\mathrm{Im}\,\psi) = \varphi(\{\bar{0}, \bar{2}\}) = \{\bar{0}, \bar{2}\} \ll Z_4$, but neither φ nor ψ is zero. Also, this example show that, the endo SS-coprime property does not always transfer from a module to each of factor modules.

(2) The direct sum of endo SS-coprime modules need not be endo SS-coprime module, for example: we know that Z_6 as Z-module is not endo SS-coprime, but we

have $Z_6 \cong Z_2 \oplus Z_3$ and each of Z_2 and Z_3 are endo SS-coprime.

Proposition 3.18 Let *M* be an *R*-module. If $M \oplus M$ is an endo SS-coprime *R*-module, then *M* is so.

Proof. Since $M \cong M \oplus (0) \leq^{\oplus} M \oplus M$ and $M \oplus M$ is endo SS-coprime, so by Proposition 3.16, $M \oplus (0)$ is an endo SS-coprime module, and hence *M* is an endo SS-coprime module . □

The converse is not true in general, as the following shows: we know *Z* as *Z*-module is endo SS-coprime. Consider the *Z*-module $Z \oplus Z$ .Let $f, g \in End(Z \oplus Z)$ are defined by $f(x, y) = (x,0)$, $g(x, y) = (0, y)$ for all $(x, y) \in Z \oplus Z$. Then $\operatorname{Im}(f \circ g) = f((0) \oplus Z) = (0,0)$ which is small in $Z \oplus Z$, but $f \neq 0$ and $g \neq 0$. Then $Z \oplus Z$ is not endo SS-coprime as *Z*-module.

To prove the following Proposition, we need the following Lemma.

Lemma 3.19 Let *M* be an *R*-module, $S = End(M)$. Then *M* is a *T*-noncosingular module if and only if $(N :_S M) = l_S(M)$ for any $N \ll M$.

Proof. Assume that *M* is a *T*-noncosingular module. Let $f \in (N :_S M)$, then $f(M) \leq N \ll M$, so $f(M)$ is small in *M*, thus $f = 0$; that is $f \in l_S(M)$. Therefore $(N :_S M) = l_S(M)$.

Conversely, let $\varphi \in End(M)$ with $\operatorname{Im}\varphi \ll M$, put $N = \varphi(M)$ .Thus $\varphi \in (N :_S M) = l_S(M)$,this implies $\varphi = 0$. □

An *R*-module *M* is called small prime if, for every nonzero small submodule *N* of *M*, $ann_R N = ann_R M$ [12]. Also, a module *M* is called endo-small prime if, $l_S(N) = l_S(M)$ for all $N \ll M$, where $S = End(M)$ [9]. Equivalently, a module *M* is endo-small prime if, for any $x \in M$, $\langle x \rangle \ll M$ and $f(x) = 0$ implies $x = 0$ or $\operatorname{Im} f = 0$.

Remark 3.20 If *M* is an endo-small prime module then $l_S(M)$ is a prime ideal in $S = End(M)$.

Proof. Let $f \circ g \in End(M)$ with $f \circ g(M) = 0$ .Thus, for any $x \in M$, $\langle x \rangle \ll M$ and $f \circ g(x) = 0$, implies $f(g(x)) = 0$ and $\langle g(x) \rangle \ll M$,and hence $g(x) = 0$ or $\operatorname{Im} f = 0$, so $x = 0$ or $\operatorname{Im} g = 0$ or $\operatorname{Im} f = 0$, thus $\operatorname{Im} g = 0$ or $\operatorname{Im} f = 0$. Thus the result obtained . □

Now, recall the following definition.

Definition 3.21 An *R*-module *M* is called S-relatively divisible if, for all $f \in End(M)$, $f(M) \cap N = f(N)$ for all $N \leq M$.

Proposition 3.22 Let *M* be a S-relatively divisible and endo-small prime module then *M* is an endo SS-coprime module, provided that *M* has a nonzero $x \in M$ and $\langle x \rangle \ll M$.

Proof. First we shall prove that *M* is *T*-noncosingular. Assume that there exists $f \in (N :_S M)$ and $f \notin l_S(M)$, $S = End(M)$; that is $f(M) \neq 0$. As *M* is endo-small prime, $l_S(N) = l_S(M)$ for all $N \ll M$ .Hence $f(N) \neq 0$. But $f(M) \cap f(N) = f^2(N)$, so $f(N) = f^2(N)$, this implies that, for any $n \in N$, $f(n) = f^2(n_1)$ for some $n_1 \in N$. It follows that $f(n - f(n_1)) = 0$. But, we have $n - f(n_1) \in N$, so $\langle n - f(n_1) \rangle \leq N \ll M$, hence $\langle n - f(n_1) \rangle \ll M$. But *M* is endo-small prime, we get $l_S M = l_S(\langle n - f(n_1) \rangle)$ implies $f \in l_S(M)$ which is a contradiction. Thus, $(N :_S M) = l_S(M)$ for all $N \ll M$. Therefore, *M* is *T*-noncosingular, by Lemma 3.19. On the other hand, *M* is an endo-small prime module, so by Remark 3.20, $l_S(M)$ is a prime ideal.Thus the result obtained by [Rem.and.Ex. 3.2(2)] . □

Recall that an *R*-module *M* is called F-regular if, for each submodule *N* of *M*, $IN = N \cap IM$ for every ideal *I* of *R* [5]. An *R*-module *M* is called prime if, for all nonzero submodule *N* of *M*, $ann_R N = ann_R M$ [4].

Corollary 3.23 Let M be an F-regular module over $S = End(M)$. If M is endo-small prime, then M is endo SS-coprime.

Corollary 3.24 Let M be an R-module, $S = End(M)$. If S a regular ring, then the following statements are equivalent.

(i) M is an endo-small prime R-module.
(ii) M is an endo SS-coprime R-module.
(iii) M is a prime as S-module.

Proof. $(i) \Rightarrow (ii)$ If S is a regular ring , $S/ann_S(x)$ is also a regular ring, hence M is F-regular as S-module, so by previous Corollary, M is endo SS-coprime.

$(ii) \Rightarrow (iii)$ Since M is an endo SS-coprime module, then $l_S(M)$ is a prime ideal of S, so $S/l_S(M)$ has no zero divisors. But S is a regular ring, then $S/l_S(M)$ is a regular ring. It follows that $\overline{S} = S/l_S(M)$ is a division ring. Hence M is a prime $\overline{S}$ -module, and so M is a prime as S-module.

$(iii) \Rightarrow (i)$ It is obvious . □

Proposition 3.25 Let M be a divisible module over the ring $S = End(M)$, where S has no zero divisors, then M is a faithful endo SS-coprime module.

Proof. Assume that $f, g \in End(M)$, $\operatorname{Im}(f \circ g) \ll M$. If $f \circ g \neq 0$ this implies $f \circ g(M) = M$, since M is a divisible S-module, so $M \ll M$ which is a contradiction. Thus $f \circ g = 0$, but S has no zero divisors, so either $f = 0$ or $g = 0$, and hence M is an endo SS-coprime module . □

Recall that an R-module M is called small retractable if, $Hom(M, N) \neq 0$ for each $N \ll M$ [7] .

Remark 3.26 Let M be a small retractable and scalar R-module. If M is an endo SS-coprime R-module, then $Rad(M) = 0$.

Proof. By Remark 3.2(1), M is an endo SS-coprime module, implies that M is a T-noncosingular module, and so by Remark 2.5, M is S-coprime, but M is small retractable and scalar, hence $Rad(M) = 0$, by [7, Prop. 2.22] . □

Proposition 3.27 Let M be an R-module. Then M is a T-noncosingular module, wherever $Hom(M, N) = 0$ for each $N \ll M$.

Proof. Assume that $f \in End(M)$, $\operatorname{Im} f \ll M$. Define $g : M \to \operatorname{Im} f$ by $g(m) = f(m)$ for all $m \in M$ .Hence $g \in Hom(M, \operatorname{Im} f)$, $\operatorname{Im} f \ll M$ and so by assumption $g = 0$. Hence $f = 0$. Then M is T-noncosingular . □

Corollary 3.28 Let M be an R-module with $l_S(M)$ is a prime ideal of $S = End(M)$. If $Hom(M, N) = 0$ for all $N \ll M$, then M is endo SS-coprime.

Proof. It follows directly by previous Proposition and [Rem.and.Ex.3.2(2)] . □

Proposition 3.29 Let M be an R-module, $S = End(M)$. Then M is an endo SS-coprime module if and only if $Hom(M, N) = 0$ for every $N \ll M$, and $l_S(M)$ is a prime ideal of S .

Proof. If M is an endo SS-coprime R-module. Let $f \in Hom(M, N)$, $N \ll M$. Thus $\operatorname{Im} f \leq N \ll M$, but M is an endo SS-coprime module implies that M is a T-noncosingular module, and so $f = 0$. Therefore $Hom(M, N) = 0$. Moreover, since M is an endo SS-coprime R-module, then $l_S(M)$ is a prime ideal in S, by [Rem.and.Ex.3.2(2)].

Conversely, if $Hom(M, N) = 0$ for each $N \ll M$, then by Proposition 3.27, M is T-noncosingular. But $l_S(M)$ is prime, thus by [Rem.and.Ex.3.2(2)], M is an endo SS-coprime module . □

A nonzero module M is called hollow if, every proper submodule is small in M [6] .

However, the following result gives a condition under which the concepts of endo SS-coprime module and T-noncosingular module are coincide.

Proposition 3.30 Let M be a hollow R-module.Then M is an endo SS-coprime module if and only M is a T-noncosingular module.

Proof. Assume that M is a T-noncosingular R-module. Let $f, g \in End(M)$, $\operatorname{Im}(f \circ g) \ll M$. So either $\operatorname{Im} f$ or $\operatorname{Im} g$ is a proper submodule of M. If $\operatorname{Im} f \subset M$, then $\operatorname{Im} f \ll M$ and hence $f = 0$. Similarly, $g = 0$. Thus M is an endo SS-coprime module.

Conversely, follows by [Rem.and.Ex.3.2(1)] . □

Lemma 3.31 Let M be a module and A be a nilideal of $S = End(M)$. If M is a T-noncosingular module, then $AM = 0$.

Proof. Let $f \in A$, we claim that $\operatorname{Im} f \ll M$. Assume that $\operatorname{Im} f + N = M$ for some submodule N of M. Thus, for all $n \in Z_+$, $f^n(M) + N = M$. But f is a nilpotent element, so $f^n = 0$ for some $n \in Z_+$, then $N = M$, and so $\operatorname{Im} f \ll M$. Thus $f = 0$ for any $f \in A$, since M is T-noncosingular. Therefore $AM = 0$. □

Proposition 3.32 Let M be an R-module and A , B be two ideals of $S = End(M)$ such that AB is a nilideal. If M is an endo SS-coprime R-module, then $AM = 0$ or $BM = 0$.

Proof. Since M is an endo SS-coprime module, then M is T-noncosingular module and hence by above Lemma, $ABM = 0$,so $ABM \ll M$. But, M is an endo SS-coprime module, so by Proposition 3.14, $AM = 0$ or $BM = 0$. □

Recall that a ring R is semilocal provided that $R/J(R)$ is a semi-simple ring .

Proposition 3.33 Let M be an R-module, $S = End(M)$ be a semilocal ring and $J(S)$ is a nilideal. Then M is a T-noncosingular R-module if and only if M is a semisimple R-module.

Proof. If M is a T-noncosingular R-module. Since S is a semilocal ring, then $S/J(S)$ is semisimple and hence $M/J(S)M$ is semisimple, by [2,Cor.15.18]. But $J(S)$ is a nilideal, thus by Lemma 3.31, $J(S)M = 0$ and hence M is semisimple.

Conversely, since M is a semisimple module, then the zero submodule is the only small submodule of M, this implies that M is T-noncosingular . □

Proposition 3.34 Let M be a scalar faithful R-module. Then R is an endo SS-coprime ring if and only if $S = End(M)$ is an endo SS-coprime ring.

Proof. Since M is a scalar faithful R-module, then by [11, Lemma 6.1] $S = End(M) \cong R$. Hence the result follows by Proposition 3.5 . □

Proposition 3.35 Let M be an R-module such that $S = End(M)$ is a regular ring with out zero divisors, then M is endo SS-coprime.

Proof. Let $f, g \in End(M)$, $\operatorname{Im}(f \circ g) \ll M$. Since S is a regular ring, so there exists $h \in S$ such that $f \circ g = (f \circ g) \circ h \circ (f \circ g)$, and hence $(f \circ g) \circ h$ is an idempotent element of S, so that $\operatorname{Im}((f \circ g) \circ h)$ is a direct summand of M. But, $\operatorname{Im}((f \circ g) \circ h) \leq \operatorname{Im}(f \circ g) \ll M$, thus $\operatorname{Im}((f \circ g) \circ h) \ll M$ this implies that $\operatorname{Im}((f \circ g) \circ h) = 0$, and hence either $f \circ g = 0$ or $h = 0$. But, $f \circ g = 0$ implies either $f = 0$ or $g = 0$, since S has no zero divisors. Also, if $h = 0$ then $f \circ g = 0$, and so $f = 0$ or $g = 0$. □

Proposition 3.36 Let M be a multiplication finitely generated faithful module over a PID R. Then M is endo SS-coprime if and only if $Rad(M) = 0$.

Proof. If M is an endo SS-coprime R-module, then M is T-noncosingular, but M is multiplication finitely generated faithful module over a PID R, thus by [10, Cor. 2.9] $Rad(M) = 0$.

Conversely, since $Rad(M) = 0$,so by[10, Cor. 2.9] M is T-noncosingular, means for all $f, g \in End(M)$,

$\mathrm{Im}(f \circ g) \ll M$ implies $f \circ g = 0$. But M is a finitely multiplication faithful, then M is scalar faithful, thus $S \cong R$,and so S has no zero divisors. Hence $f \circ g = 0$, implies that $f = 0$ or $g = 0$. □

4. Semi-Endo SS-coprime modules

In this section, we define and study semi-endo SS-coprime modules which is a generalization of endo SS-coprime modules. We give the relations between such modules and other types of modules .

Definition 4.1 An R-module M is called a semi-endo SS-coprime module (briefly semi-endo SS-coprime) if, for each $f \in End(M)$, $\mathrm{Im}(f \circ f) \ll M$ implies $f = 0$.

We shall investigate the relation between semi-endo SS-coprime and other classes of modules.

Remarks and Examples 4.2

(1) It is clear that every endo SS-coprime module is semi-endo SS-coprime, but the converse is not true in general, as the following example shows : Z-module Z_6 is semi-endo SS-coprime, but it is not endo SS-coprime. In fact, if $f \in End(Z_6)$, $f^2(Z_6) \ll Z_6$ this implies that $f^2(Z_6) = 0$, and since (0) is a semiprime submodule of Z_6, hence $f = 0$.

(2) Every semi-endo SS-coprime module is T-noncosingular.

Proof. Let M be a semi-endo SS-coprime module and $f \in End(M)$, $\mathrm{Im}\, f \ll M$. But $\mathrm{Im}(f \circ f) \le \mathrm{Im}\, f$, thus $\mathrm{Im}(f \circ f) \ll M$, and so $f = 0$. □

(3) Let M be an R-module, $S = End(M)$ .Then M is a semi-endo SS-coprime module if and only if M is T-noncosingular and $l_S(M)$ is a semiprime ideal of S.

Proof. It is obvious . □

(4) Let M be an R-module and let $S = End(M)$ be a chained ring. Then M is endo SS-coprime if and only if M is semi-endo SS-coprime .

(5) Every semi-endo SS-coprime module is SSS-coprime.

Proof. Let M be a semi-endo SS-coprime module. Let $r \in R$, $r^2 M \ll M$. Consider $\varphi : M \to M$ by $\varphi(m) = rm$ for all $m \in M$. Thus $\varphi^2(M) = \varphi(rM) = r^2 M \ll M$, but M is semi-endo SS-coprime, thus $rM = \mathrm{Im}\varphi = 0$. Hence M is a SSS-coprime module . □

The converse is not true in general, for example : consider the Z-module $Z_{2^\infty} \oplus Z_2$,then it is SS-coprime and not T-noncosingular see [7, Rem.and.Ex.2.2(3)], this implies $Z_{2^\infty} \oplus Z_2$ is SSS-coprime but not semi-endo SS-coprime.

(6) If M is a semi-endo SS-coprime module, then M is SS-coprime and hence M is S-coprime, whenever $ann_R M$ is a prime ideal.

Proof. It follows by(2) and [Rem.and.Ex. 3.2 (2),(3)].□

The next result gives characterizations of semi-endo SS-coprime modules.

Proposition 4.3 Let M be an R-module, $S = End(M)$. Then the following statements are equivalent.

(i) M is a semi-endo SS-coprime R-module.

(ii) For any ideal A of S, $A^2 M \ll M$ implies $AM = 0$.

(iii) For any ideal A of S, and $n \in Z_+$. If $A^n M \ll M$ implies $AM = 0$.

Proof. It is easy, so is omitted . □

Proposition 4.4 Let M be an R-module, $\overline{R} = R/annM$. Then M is a semi-endo SS-coprime R-module if and only if M is a semi-endo SS-coprime $\overline{R}$ -module.

Proof. It is obvious . □

Proposition 4.5 If M_1 and M_2 are two isomorphic R-modules. Then M_1 is semi-endo SS-coprime if and only if M_2 is semi-endo SS-coprime.

Proposition 4.6 Let M be a scalar R-module. If M is SSS-coprime, then M is semi-endo SS-coprime.

Proof. Let $\varphi \in End(M)$ with $\varphi^2(M) \ll M$. Since M is scalar, so there exists $r \in R$ such that $\varphi(m) = rm$ for all $m \in M$ .Thus $r^2M = \varphi^2(M)$ is small in M , but M is an SSS-coprime module, thus $rM = 0$ which implies $\varphi = 0$. □

Proposition 4.7 Let M be an R-module, $S = End(M)$. Then M is semi-endo SS-coprime if and only if for all $f \in S$, $\mathrm{Im}(f \circ f) \ll M$ implies $(f(M) :_S M) = l_S(M)$.

Proof. Assume that M is a semi-endo SS-coprime R-module. Let $f \in End(M)$, $\mathrm{Im}(f \circ f) \ll M$, so $f = 0$ and hence $(f(M) :_S M) = (0 :_S M) = l_S(M)$.

Conversely, let $\varphi \in End(M)$ with $\varphi^2(M) \ll M$, so by hypothesis, $(\varphi(M) :_S M) = l_S(M)$. But we have $\varphi \in (\varphi(M) :_S M)$, thus $\varphi(M) = 0$, hence $\varphi = 0$ and M is a semi-endo SS-coprime R-module . □

Proposition 4.8 Let M be a semi-endo SS-coprime R-module and let N be a direct summand of M. Then N is semi-endo SS-coprime .

Proof. Let M be a semi-endo SS-coprime R-module. Assume that $N \leq^{\oplus} M$, then $M = N \oplus K$ for some submodule K of M. Let $\varphi \in End(N)$, $\varphi^2(N) \ll N$. Consider the endomorphism $\psi : M \to M$ defined by $\psi(n+k) = \varphi(n)$ for $n \in N$. Now, $\psi^2(M) = \varphi^2(N)$ is small in N, this implies $\psi^2(M) \ll M$, but M is semi-endo SS-coprime, so $\psi = 0$, and hence $\varphi = 0$.□

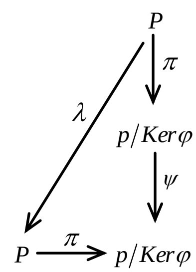
Remark 4.9 A homomorphic image of semi-endo SS-coprime module is not necessarily semi-endo SS-coprime module, for example: it is well known that Z as Z-module is endo SS-coprime, so it is a semi-endo SS-coprime module. Consider the natural epimorphism $\pi : Z \to Z_4$. It is clear that $\mathrm{Im}\,\pi = Z_4$ is not SSS-coprime and hence it is not semi-endo SS-coprime as Z-module, by [Rem.and.Ex. 4.2(5)]. In particular case, this example show that, the factor of semi-endo SS-coprime module need not be semi-endo SS-coprime module.

Proposition 4.10 Let M be an R-module. If $M \oplus M$ is a semi-endo SS-coprime R-module, then M is so.

Proof. By Proposition 4.8, $M \oplus (0)$ is a semi-endo SS-coprime module of $M \oplus M$ .But $M \oplus (0) \cong M$,so M is semi-endo SS-coprime, by Proposition 4.5 . □

Proposition 4.11 Let M be a module has a projective cover $\varphi : p \to M$. If P is semi-endo SS-coprime, then so is M.

Proof. Since M is has a projective cover $\varphi : p \to M$, then φ is an epimorphism and $Ker\varphi \ll P$, thus we have $P/Ker\varphi \cong M$ .It is enough to show that $P/Ker\varphi$ is semi-endo SS-coprime. Assume $\psi \in End(P/Ker\varphi)$, $\psi^2(P/Ker\varphi) \ll P/Ker\varphi$. Consider $\pi : p \to P/Ker\varphi$ the natural epimorphism. Since P is projective, so there exists a homomorphism $\lambda : P \to P$ such that $\psi \circ \pi = \pi \circ \lambda$.



So $\psi^2 \circ \pi = \pi \circ \lambda^2$. Hence, $\pi \circ \lambda^2(P) = \psi^2(P/Ker\varphi)$ is small in $P/Ker\varphi$,and hence $\dfrac{\lambda^2(P) + Ker\varphi}{Ker\varphi} \ll \dfrac{P}{Ker\varphi}$, and since $Ker\varphi \ll P$, thus $\lambda^2(P) \ll P$ .But P is semi-endo SS-coprime, hence $\lambda = 0$, and so $\psi \circ \pi = 0$. Thus $\psi = 0$. □

Corollary 4.12 Let R be a ring. Then the following statements are equivalent.

(i) Every projective R-module is semi-endo SS-coprime.
(ii) Every R-module has a projective cover is semi-endo SS-coprime.

Proof. $(i) \Rightarrow (ii)$ It follows by previous Proposition .

$(ii) \Rightarrow (i)$ Let M be a projective R-module. Consider the identity mapping $i: M \to M$, $Ker i = 0 \ll M$, thus M has a projective cover. Hence by (ii), M is semi-endo SS-coprime . □

Theorem 4.13 Let M be a multiplication R-module. Then M is a SSS-coprime R-module if and only if M is a semi-endo SS-coprime R-module .

Proof. Since M is a SSS-coprime R-module, then by [7, Th. 3.10] M is a SSS-coprime as S-module, where $S = End(M)$. This implies that M is a semi-endo SS-coprime as R-module . □

Proposition 4.14 Let M be a scalar faithful R-module. Then R is a semi-endo SS-coprime ring if and only if $S = End(M)$ is a semi-endo SS-coprime ring .

Proof. Since M is a scalar faithful R-module, so $S \cong R$. Hence the result is obtained . □

Remark 4.15 Let M be an R-module, $S = End(M)$ be a semilocal ring with $J(R)$ is a nilideal. If M is a semi-endo SS-coprime S-module, then M is $\mathcal{T}$-noncosingular and hence M is semisimple, by Proposition 3.33.

For every module M, let $S(M) = \{\varphi \in End(M):$ $\operatorname{Im}\varphi^2 \ll M$ }. It is easy to see that $S(M)$ is an ideal of $End(M)$ .By the semi-endo SS-coprime submodule of M we mean $\overline{Z}_S(M) = \bigcap_{\varphi \in S(M)} Ker\varphi$.

Proposition 4.16 Let M be an R-module. Then M is semi-endo SS-coprime if and only if $\overline{Z}_S(M) = M$.

Proof. Suppose that M is a semi-endo SS-coprime module. Then, for each $\varphi \in S(M)$, $\varphi = 0$ and hence $M = Ker\varphi = \bigcap_{\varphi \in S(M)} Ker\varphi = \overline{Z}_S(M)$.

Conversely, assume $\overline{Z}_S(M) = M$. Let $\varphi \in End(M)$ and $\operatorname{Im}\varphi^2 \ll M$, hence $\varphi \in S(M)$. By hypothesis, we have $M = \overline{Z}_S(M) = \bigcap_{\varphi \in S(M)} Ker\varphi$. Thus $M \subseteq Ker\varphi$; that is $\varphi(M) = 0$. Hence M is semi-endo SS-coprime . □

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