Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.6, No.1, 2016

Γ^*- derivation and Jordan Γ^*- derivation on Prime Γ-ring with Involution

Ali Kareem Kadhim

School of Mathematical Sciences
Universiti Sains Malaysia, 11800 USM
Penang, Malaysia

Hajar Sulaiman

School of Mathematical Sciences
Universiti Sains Malaysia, 11800 USM
Penang, Malaysia

Abdul-Rahman Hameed Majeed

Department of Mathematics
University of Baghdad
Baghdad, Iraq

Abstract

Let M be a 2-torsion free prime Γ-ring with involution satisfying the condition that $a\alpha b\beta c = a\beta b\alpha c$ ($a,\ b,\ c \in M$ and $\alpha, \beta \in \Gamma$). In this paper we will give the relation between Γ^*-derivation and Jordan Γ^*-derivation. Also we will prove that if d is a non-zero Jordan Γ^*-derivation such that $d(x\alpha y) = d(y\alpha x)$ for all $x, y \in M$ and $\alpha \in \Gamma$, then M is a commutative Γ-ring with involution.

Keywords: semiprime Γ-ring with involution, Jordan derivation, Γ^*-derivation
AMS : 16W10 , 17C50

1 Introduction

The notion of Γ-rings was first introduced by Nobusawa [14] who also showed that Γ-rings are more general than rings. Bernes[17] slightly weakened the conditions in the definition of Γ-ring in the sense of Nobusawa. Bernes[17], Kyuno[16], Luh[5], Ceven[18], Hoque and Paul[8, 10, 11] and others established a large number of important basic properties on Γ-rings in various ways and determined some more remarkable results of Γ-rings. We start with some definitions. Let M and Γ be additive abelian groups. If there exists a mapping $M \times \Gamma \times M \longrightarrow M$ defined by $(x, \alpha, y) \longrightarrow (x\alpha y)$ which satisfies the conditions

(i) $x\alpha y \in M$.

(ii) $(x + y)\alpha z = x\alpha z + y\alpha z,\ x(\alpha + \beta)y = x\alpha y + x\beta y,\ x\alpha(y + z) = x\alpha y + x\alpha z.$

(iii) $(x\alpha y)\beta z = x\alpha(y\beta z)$

then M is called a Γ-ring (see [5],[9]). Let M be a Γ-ring. Then an additive subgroup U of M is called a left (right) ideal of M if $M\Gamma U \subset U (U\Gamma M \subset U)$. If U is both a left and a right ideal, then we say U is an ideal of M. Suppose again that M is a Γ-ring. Then M is said to be 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in M$. An ideal P_1 of a Γ-ring M is said to be prime if for some ideals A and B of M , $A\Gamma B \subseteq P_1$ implies $A \subseteq P_1$ or $B \subseteq P_1$. An ideal P_2 of a Γ-ring M is said to be semiprime if for any ideal U of M, $U\Gamma U \subseteq P_2$ implies $U \subseteq P_2$. A Γ-ring M is said to be prime if $a\Gamma M\Gamma b = (0)$ with $a, b \in M$, implies $a = 0$ or $b = 0$ and semiprime if $a\Gamma M\Gamma a = (0)$ with $a \in M$ implies $a = 0$. Furthermore, M is said to be a commutative Γ-ring if $x\alpha y = y\alpha x$ for all $x, y \in M$ and $\alpha \in \Gamma$. The set Z(M)=$\{x \in M : x\alpha y = y\alpha x$ for all $\alpha \in \Gamma,\ y \in M\}$ is called the center of the Γ-ring M. If M is a Γ-ring, then $[x, y]_\alpha = x\alpha y - y\alpha x$ is known as the commutator of x and y with respect to α, where $x,\ y \in M$ and $\alpha \in \Gamma$. We make the following basic commutator identities:

$$[x\alpha y, z]_\beta = [x, z]_\beta \alpha y + x[\alpha, \beta]_z y + x\alpha[y, z]_\beta \quad (1)$$

$$[x, y\alpha z]_\beta = [x, y]_\beta \alpha z + y[\alpha, \beta]_x z + y\alpha[x, z]_\beta \quad (2)$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Now, we consider the following assumption:

(A) $x\alpha y\beta z = x\beta y\alpha z$ for all $x,\ y,\ z \in M$ and $\alpha,\ \beta \in \Gamma$.
According to assumption (A), the above commutator identities reduce to $[x\alpha y, z]_\beta = [x, z]_\beta \alpha y + x\alpha[y, z]_\beta$ and $[x, y\alpha z]_\beta = [x, y]_\beta \alpha z + y\alpha[x, z]_\beta$. which we will extensively used.

Bernes[17], Luh[5], Kyuno[16], Hoque and Paul[10] studied the structure of Γ-rings and obtained various generalizations of corresponding parts in ring theory. Note that during the last few decades many authors have studied derivations in the context of prime and semiprime rings and Γ-rings with involution ([6],[7],[12],[15],[4]). The notion of derivation and Jordan derivation on a Γ-ring were defined by Sapanci and Nakajima[13].

Definition 1.1. *[18] An additive mapping $D : M \longrightarrow M$ is called a derivation if $D(x\alpha y) = D(x)\alpha y + x\alpha D(y))$, which holds for all $x, y \in M$ and $\alpha \in \Gamma$.*

Definition 1.2. *[18] An additive mapping $D : M \longrightarrow M$ is called a Jordan derivation if $D(x\alpha x) = D(x)\alpha x + x\alpha D(x))$, which holds for all $x \in M$ and $\alpha \in \Gamma$.*

Definition 1.3. *[18] A Γ-ring M is called a completely prime if $a\Gamma b = 0$ implies that $a = 0$ or $b = 0$, where $a, b \in M$*

Remark 1. *[18] Every completely prime Γ-ring is prime.*

Definition 1.4. *An additive mapping $(x\alpha x) \to (x\alpha x)^*$ on a Γ-ring M is called an involution if $(x\alpha y)^* = y^*\alpha x^*$ and $(x\alpha x)^{**} = x\alpha x$ for all $x, y \in M$ and $\alpha \in \Gamma$. A Γ-ring M equipped with an involution is called a Γ-ring M with involution (also known as Γ^*-ring).*

Definition 1.5. *An additive mapping $d : M \longrightarrow M$ is called Γ^*-derivation if $d(x\alpha y) = d(x)\alpha y^* + x\alpha d(y)$ which holds for all $x, y \in M$ and $\alpha \in \Gamma$.*

Definition 1.6. *An additive mapping $d : M \longrightarrow M$ is called a Jordan Γ^*-derivation if $d(x\alpha x) = d(x)\alpha x^* + x\alpha d(x)$ which holds for all $x \in M$ and $\alpha \in \Gamma$.*

Example 1 Let R be a commutative ring with chR=2 . Define $M = \left\{\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} : a, b \in R\right\}$ and $\Gamma = \left\{\begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} : \alpha \in R\right\}$, then M is a Γ-ring under addition and multiplication of matrices.

Define a mapping $d : M \to M$ by $d\left(\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\right) = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$

To show that d is a Γ^*-derivation, let

$$x = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}, \; y = \begin{bmatrix} c & d \\ 0 & c \end{bmatrix}, \; y^* = \begin{bmatrix} -c & d \\ 0 & -c \end{bmatrix},$$

then $d(x\alpha y) = d(x)\alpha y^* + x\alpha d(y)$. Hence, d is a Γ^*-derivation

It is clear that every Γ^*-derivation is a Jordan Γ^*-derivation, but the converse in general is not true.

Example 2 Let M be a $\Gamma-$ring and let $a \in M$ such that $a\Gamma a = (0)$ and $x\alpha a\beta x = 0$ for all $x \in M$ and $\alpha, \beta \in \Gamma$, but $x\alpha a\beta y \neq 0$ for some $x, y \in M$ such that $x \neq y$.
Define a map $d : M \to M$ by $d(x) = x\alpha a + a\alpha x^*$ for all $x \in M$ and $\alpha \in \Gamma$, then d is a Jordan Γ^*-derivation but not Γ^*-derivation.

In this paper we will give the relation between Γ^*- derivation and Jordan Γ^*-derivation. Also we will prove that if d is a non-zero Jordan Γ^*-derivation such that $d(x\alpha y) = d(y\alpha x)$ for all $x, y \in M$ and $\alpha \in \Gamma$, then M is a commutative $\Gamma-$ring with involution.

2 Γ^*-derivation and Jordan Γ^*-derivation

To prove our main results, we need the following lemmas.

Lemma 2.1. *Let M be a non-commutative prime $\Gamma-$ring with involution satisfying assumption A, let $d : M \to M$ be a Γ^*-derivation, then d=0.*
Proof. *We have*

$$d(x\alpha y) = d(x)\alpha y^* + x\alpha d(y) \tag{3}$$

for all $x, y \in M$ and $\alpha \in \Gamma$. Replacing y by $y\beta z$ in (3) we get

$$d(x\alpha(y\beta z)) = d(x)\alpha z^*\beta y^* + x\alpha d(y)\beta z^* + x\alpha y\beta d(z) \tag{4}$$

for all $x, y \in M$ and $\alpha, \beta \in \Gamma$ and

$$d((x\alpha y)\beta z)) = d(x)\alpha y^*\beta z^* + x\alpha d(y)\beta z^* + x\alpha y\beta d(z) \tag{5}$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. By compare (4) and (5), we get

$$d(x)\alpha[z^*, y^*]_\beta = 0 \tag{6}$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, then by using [3], we get $d = 0$.

Lemma 2.2. *Let M be a semiprime $\Gamma-$ring satisfying assumption A, and suppose that $a \in M$ centralizes all $[x, y]_\alpha$ for all $x, y \in M$ and $\alpha \in \Gamma$. Then $a \in Z(M)$.*

Proof. *The proof of this lemma can be found in [3] (Lemma 2.4).*

Lemma 2.3. *Let M be a 2-torsion free non-commutative prime $\Gamma-$ring with involution satisfying assumption (A)and suppose there exists an element $a \in M$ such that $a\alpha[x, y]^*_\beta = [x, y]_\beta \alpha a$ for all $x, y \in M$ and $\alpha, \beta \in \Gamma$, then $a = 0$.*

Proof. We have

$$a\alpha[x, y]^*_\beta = [x, y]_\beta \alpha a \tag{7}$$

for all $x, y \in M$ and $\alpha, \beta \in \Gamma$. Replace y by $x\delta y$ in (7)

$$\begin{aligned} a\alpha[x, x\delta y]^*_\beta &= [x, x\delta y]_\beta \alpha a \\ a\alpha(y^*\delta[x, x]^*_\beta + [x, y]^*_\beta \delta x^*) &= (x\delta[x, y]_\beta + [x, x]_\beta \delta y)\alpha a \end{aligned}$$

for all $x, y \in M$ and $\alpha, \beta, \delta \in \Gamma$, then after reduces, we obtain

$$a\alpha[x, y]^*_\beta \delta x^* = x\delta[x, y]_\beta \alpha a \tag{8}$$

for all $x, y \in M$ and $\alpha, \beta, \delta \in M$. By using relation (7), (8), we get

$$\begin{aligned} &[x, y]_\beta \alpha a\delta x^* = x\delta[x, y]_\beta \alpha a \\ &[x, y]_\beta \alpha x\delta a = x\delta[x, y]_\beta \alpha a \\ &[x, y]_\beta \alpha x\delta a - x\delta[x, y]_\beta \alpha a = 0 \end{aligned}$$

The substitution $z = [z_1, z_2]_\gamma$ where $z_1, z_2 \in M$ for x in above relation

$$[z, y]_\beta \alpha z\delta a - z\delta[z, y]_\beta \alpha a = 0$$

gives

$$[[z, y]_\beta, z]_\delta \alpha a = 0 \tag{9}$$

for all $z, y \in M$ and $\alpha, \beta, \delta \in \Gamma$. Putting $y\gamma w$ for y in (9)

$$\begin{aligned} &[[z, y\gamma w]_\beta, z]_\delta \alpha a = 0 \\ &[y\gamma[z, w]_\beta + [z, y]_\beta \gamma w, z]_\delta \alpha a = 0 \\ &[y\gamma[z, w]_\beta, z]_\delta \alpha a + [[z, y]_\beta \gamma w, z]_\delta \alpha a = 0 \\ &y\gamma[[z, w]_\beta, z]_\delta \alpha a + ([y, z]_\beta \gamma[z, w]_\delta)\alpha a + [z, y]_\beta \gamma[w, z]_\delta \alpha a + [[z, y]_\beta, z]_\delta \gamma w\alpha a = 0 \\ &y\gamma[[z, w]_\beta, z]_\delta \alpha a + [z, y]_\beta \gamma[w, z]_\delta \alpha a + [z, y]_\beta \gamma[w, z]_\delta \alpha a + [[z, y]_\beta, z]_\delta \gamma w\alpha a = 0 \end{aligned}$$

for all $z, y \in M$ and $\alpha, \beta, \delta \in \Gamma$, then by using (9), we get

$$[[z, y]_\beta, z]_\delta \gamma w\alpha a + 2[z, y]_\beta \gamma[w, z]_\delta \alpha a = 0 \tag{10}$$

for all $z, y, w \in M$ and $\alpha, \beta, \delta, \gamma \in \Gamma$. Replace w by $s = [w_1, w_2]_\beta$ for all $w_1, w_2 \in M$ in (10)

$$[[z, y]_\beta, z]_\delta \gamma s\alpha a + 2[z, y]_\beta \gamma[s, z]_\delta \alpha a = 0$$

for all $z, y, s \in M$ and $\alpha, \beta, \delta, \gamma \in \Gamma$. By using relation (7), we get

$$[[z, y]_\beta, z]_\delta \gamma a \alpha [w_1, w_2]^*_\beta + 2[z, y]_\beta \gamma [s, z]_\delta \alpha a = 0$$

for all $z, y, s \in M$ and $\alpha, \beta, \delta, \gamma \in \Gamma$. By using relation (9), we get

$$2[z, y]_\beta \gamma [s, z]_\delta \alpha a = 0$$

for all $z, y, s \in M$ and $\alpha, \beta, \delta, \gamma \in \Gamma$. Since M is 2-torsion free, we obtain

$$[z, y]_\beta \gamma [s, z]_\delta \alpha a = 0 \tag{11}$$

for all $z, y, s \in M$ and $\alpha, \beta, \delta, \gamma \in \Gamma$. Putting in relation(11) $y\gamma r$ for y, for all $y, r \in M$ and $\alpha \in \Gamma$

$$\begin{aligned} [z, y\gamma r]_\beta \gamma [s, z]_\delta \alpha a &= 0 \\ (y\gamma [z, r]_\beta + [z, y]_\beta \gamma r)\gamma [s, z]_\delta \alpha a &= 0 \\ y\gamma [z, r]_\beta \gamma [s, z]_\delta \alpha a + [z, y]_\beta \gamma r \gamma [s, z]_\delta \alpha a &= 0 \end{aligned}$$

for all $z, y, r \in M$ and $\alpha, \beta, \delta, \gamma \in \Gamma$. By using relation (11), we get

$$[[z_1, z_2]_\alpha, y]_\beta \gamma r \gamma [s, [z_1, z_2]_\alpha]_\delta \alpha a = 0. \tag{12}$$

for all $z_1, z_2, y, r, s \in M$ and $\alpha, \beta, \delta, \gamma \in \Gamma$. Linearization (12) on z_1 yields

$$[[b_1, z_2]_\alpha, y]_\beta \gamma r \gamma [s, [b_2, z_2]_\alpha]_\delta \alpha a = -[[b_2, z_2]_\alpha, y]_\beta \gamma r \gamma [s, [b_1, z_2]_\alpha]_\delta \alpha a \tag{13}$$

for all $b_1, b_2, z_2, y, r, s \in M$ and $\alpha, \beta, \delta, \gamma \in \Gamma$. Then we have from (13) that

$$\begin{aligned} &[[b_1, z_2]_\alpha, y]_\beta \gamma r \gamma [s, [b_2, z_2]_\alpha]_\delta \alpha a \gamma r \gamma [[b_1, z_2]_\alpha, y]_\beta \gamma r \gamma [s, [b_2, z_2]_\alpha]_\delta \alpha a = \\ &-[[b_1, z_2]_\alpha, y]_\beta \gamma (r \gamma [s, [b_2, z_2]_\alpha]_\delta \alpha a \gamma r \gamma [[b_2, z_2]_\alpha, y]_\delta \gamma r) \gamma [s, [b_1, z_2]_\alpha]_\beta \alpha a = 0 \end{aligned} \tag{14}$$

for all $b_1, b_2, z_2, y, r, s \in M$ and $\alpha, \beta, \delta, \gamma \in \Gamma$. Then by using (12), we obtain

$$[[b_1, z_2]_\alpha, y]_\beta \gamma r \gamma [s, [b_2, z_2]_\alpha]_\delta \alpha a \gamma r \gamma [[b_1, z_2]_\alpha, y]_\beta \gamma r \gamma [s, [b_2, z_2]_\alpha]_\delta \alpha a = 0$$

for all $b_1, b_2, z_2, y, r, s \in M$ and $\alpha, \beta, \delta, \gamma \in \Gamma$. Since M is a prime ring, we get

$$[[b_1, z_2]_\alpha, y]_\beta \gamma r \gamma [s, [b_2, z_2]_\alpha]_\delta \alpha a = 0 \tag{15}$$

for all $b_1, b_2, z_2, y, r, s \in M$ and $\alpha, \beta, \delta, \gamma \in \Gamma$. Now replace z_2 by $x_1 + x_2$ in (15) for all $x_1, x_2 \in M$, we get (see how (15) was obtained from (12))

$$[[b_1, x_1]_\alpha, y]_\beta \gamma r \gamma [s, [b_2, x_2]_\alpha]_\delta \alpha a = 0 \tag{16}$$

for all $b_1, b_2, x_1, x_2, y, r, s \in M$ and $\alpha, \beta, \delta, \gamma \in \Gamma$. Putting $u = [b_1, x_1]_\alpha$ and $q = [b_2, x_2]_\alpha$, the relation (16) leads to

$$[u, y]_\beta \gamma r \gamma [s, q]_\delta \alpha a = 0 \tag{17}$$

for all $u, y, s, r, q \in M$ and $\alpha, \beta, \delta, \gamma \in \Gamma$. Since M is a non-commutative prime Γ-ring with involution, then by using Lemma 2.2, we get

$$[[w_1, w_2]_\beta, q]_\delta \alpha a = 0 \tag{18}$$

for all $w_1, w_2, q \in M$ and $\alpha, \beta, \delta, \gamma \in \Gamma$. The substitution $w_1 \gamma w_2^*$ for w_2 in (18)

$$\begin{aligned}
&[[w_1, w_1 \gamma w_2^*]_\alpha, q]_\delta \alpha a = 0\\
&[w_1 \gamma [w_1, w_2^*]_\alpha + [w_1, w_1]_\alpha \gamma w_2^*, q]_\delta \alpha a = 0\\
&[w_1 \gamma [w_1, w_2^*]_\alpha, q]_\delta \alpha a = 0\\
&w_1 \gamma [[w_1, w_2^*]_\alpha, q]_\delta \alpha a + [w_1, q]_\delta \gamma [w_1, w_2^*]_\alpha \alpha a = 0
\end{aligned}$$

for all $w_1, w_2^*, q, a \in M$ and $\alpha, \beta, \delta, \gamma \in \Gamma$. By using relation (18) and assumption (A), we obtain

$$[w_1, q]_\delta \gamma [w_1, w_2^*]_\alpha \alpha a = 0$$

for all $w_1, w_2^*, q, a \in M$ and $\alpha, \beta, \delta, \gamma \in \Gamma$. By using relation (7) we get

$$[w_1, q]_\beta \gamma a \gamma [w_2, w_1^*]_\delta = 0 \tag{19}$$

for all $w_1, w_2, q \in M$ and $\alpha, \beta, \delta, \gamma \in \Gamma$. Replace w_2 by $r \alpha w_2$ in (19)

$$\begin{aligned}
&[w_1, q]_\beta \gamma a \gamma [r \alpha w_2, w_1^*]_\delta = 0\\
&[w_1, q]_\beta \gamma a \gamma (r \alpha [w_2, w_1^*]_\delta + [r, w_1^*]_\delta \alpha w_2) = 0\\
&[w_1, q]_\beta \gamma a \gamma r \alpha [w_2, w_1^*]_\delta + [w_1, q]_\beta \gamma a \gamma [r, w_1^*]_\delta \alpha w_2 = 0
\end{aligned}$$

for all $w_1, r, q, a \in M$ and $\alpha, \beta, \delta, \gamma \in \Gamma$. Then by using (19) we get

$$[w_1, q]_\beta \gamma a \alpha r \gamma [w_2, w_1^*]_\delta = 0 \tag{20}$$

for all $w_1, w_2, q, r \in M$ and $\alpha, \beta, \delta, \gamma \in \Gamma$. From the relation (20) one obtains (see how (15) was obtained from its previous relation)

$$[w_1, q]_\beta \gamma a \alpha r \gamma [w_2, t]_\delta = 0 \tag{21}$$

for all $w_1, w_2, q, r, t \in M$ and $\alpha, \beta, \delta, \gamma \in \Gamma$. Since M is non-commutative prime Γ-ring with involution, we get

$$[w_1, q]_\beta \gamma a = 0 \tag{22}$$

for all $w_1, q \in M$ and $\alpha, \beta, \delta, \gamma \in \Gamma$. Replace w_1 by $w_1 \alpha r$ in (22)

$$[w_1 \gamma r, q]_\delta \gamma a = w_1 \gamma [r, q]_\beta \gamma a + [w_1, q]_\delta \gamma r \gamma a = 0$$

for all $w_1, q, r \in M$ and $\alpha, \beta, \delta, \gamma \in \Gamma$. Then by using relation (22) we obtain

$$[w_1, q]_\beta \alpha r \gamma a = 0 \tag{23}$$

for all $w_1, q, r \in M$ and $\alpha, \beta, \delta, \gamma \in \Gamma$. Since M is a non-commutative prime Γ-ring with involution, then from relation (23) we get $a = 0$.

Lemma 2.4. *If M is a completely prime Γ-ring with 2-torsion free satisfying assumption (A), then every Jordan derivation on M is a derivation on M.*

Proof. By using [1], we have

$$d(x\alpha y) = x\alpha d(y) + y\alpha d(x) \tag{24}$$

for all $x, y \in M$ and $\alpha \in \Gamma$. By using [2], we have

$$d(x\alpha y) = d(x)\alpha y + d(y)\alpha x \tag{25}$$

for all $x, y \in M$ and $\alpha \in \Gamma$. If we combine the relation (24) and (25), then we get

$$d(x\alpha y) = d(x)\alpha y + x\alpha d(y) \tag{26}$$

Lemma 2.5. *Let M be a 2-torsion free prime $\Gamma-$ring with involution satisfying assumption (A) and let $d : M \to M$ be a non-zero Jordan Γ^*-derivation, then M is commutative if and only if d is a Γ^*-derivation.*

Proof. Let M be a commutative Γ-ring with involution , since d is a non-zero Jordan Γ^*-derivation we have

$$d(x\alpha x) = d(x)\alpha x^* + x\alpha d(x) \tag{27}$$

for all $x \in M$ and $\alpha \in \Gamma$. Linearization of (27) yields

$$d(x\alpha y + y\alpha x) = d(x)\alpha y^* + x\alpha d(y) + d(y)\alpha x^* + y\alpha d(x) \tag{28}$$

for all $x, y \in M$ and $\alpha \in \Gamma$. Replace y by $x\delta y + y\beta x$ in (28) and since M is 2-torsion free we get

$$d(x\alpha y\beta x) = d(x)\alpha y^*\beta x^* + x\alpha d(y)\beta x^* + x\alpha y\beta d(x) \tag{29}$$

for all $x, y \in M$ and $\alpha, \beta \in \Gamma$. Putting $x + z$ for x in (29) we obtain

$$\begin{aligned} d(x\alpha y\beta z + z\alpha y\beta x) = d(x)\alpha y^*\beta z^* + x\alpha d(y)\beta z^* + x\alpha y\beta d(z) + \\ d(z)\alpha y^*\beta x^* + z\alpha d(y)\beta x^* + z\alpha y\beta d(x) \end{aligned} \tag{30}$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Replace x by $x\delta y$ and y by z in relation (28) we get

$$\begin{aligned} d(x\delta y\alpha z + z\alpha x\delta y) = d(x)\alpha y^*\beta z^* + x\alpha d(y)\beta z^* + d(z)\alpha y^*\delta x^* + x\alpha y\beta d(z) + \\ +z\alpha d(x)\beta y^* + z\alpha x\beta d(y) \end{aligned} \tag{31}$$

for all $x, y, z \in M$ and $\alpha, \beta, \delta \in \Gamma$. Since M is commutative, comparing the relation (30) and (31) we get

$$B(x, y)\alpha z^* + z\alpha B(y, x) = 0 \tag{32}$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, where $B(x, y)$ stands for $d(x\alpha y) - d(x)\alpha y^* - x\alpha d(y)$, since $B(x, y) = -B(y, x)$, then from the relation (32) we obtain

$$B(x, y)\alpha(z^* - z) = 0 \tag{33}$$

for all $x, y, z \in M$ and $\alpha \in \Gamma$. Right multiplication the relation (33) by r we get

$$B(x, y)\alpha r\beta(z^* - z) = 0 \tag{34}$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Since M is prime, then either d is a Γ^*-derivation, or $z = z^*$ for all $z \in M$. If $z = z^*$ for all $z \in M$, then from the relation (27) we obtain

$$d(x\alpha x) = d(x)\alpha x + x\alpha d(x) \tag{35}$$

for all $x \in M$ and $\alpha \in \Gamma$. Now by using Lemma(2.4) we get

$$d(x\alpha y) = d(x)\alpha y + x\alpha d(y) \tag{36}$$

for all $x, y \in M$ and $\alpha \in \Gamma$. Then by using the above relation and since $y = y^*$ for all $y \in M$, we conclude that d is a Γ^*-derivation. To prove the converse, assume d is a non-zero Γ^*-derivation, then by using Lemma 2.1 we get M is a commutative Γ^*-ring.

Theorem 2.6. *Let M be a 2-torsion free prime $\Gamma-$ring with involution satisfies assumption (A) and let $d : M \to M$ be a non-zero Jordan Γ^*-derivation such that $d([x, y]_\alpha) = 0$ for all $x, y \in M$ and $\alpha \in \Gamma$, then M is a commutative Γ^*-ring.*

Proof. We have

$$d(x\alpha y) = d(y\alpha x) \tag{37}$$

for all $x, y \in M$ and $\alpha \in \Gamma$. Replace y by $y\beta z + z\beta y$ in (37) we obtain

$$d(x\alpha y\beta z + z\beta y\alpha x) = d(y\beta z\alpha x + x\alpha z\beta y) \tag{38}$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Using relation (30) we obtain

$$\begin{aligned} d(x\alpha y\beta z + z\beta y\alpha x) = {} & d(x)\alpha y^*\beta z^* + x\alpha d(y)\beta z^* + x\alpha y\beta d(z) + \\ & d(z)\beta y^*\alpha x^* + z\beta d(y)\alpha x^* + z\beta y\alpha d(x) \end{aligned} \tag{39}$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$ and

$$\begin{aligned} d(y\beta z\alpha x + x\alpha z\beta y) = {} & d(y)\beta z^*\alpha x^* + y\beta d(z)\alpha x^* + y\beta z\alpha d(x) + \\ & d(x)\alpha z^*\beta y^* + x\alpha d(z)\beta y^* + x\alpha z\beta d(y). \end{aligned} \tag{40}$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. According to (39), (40) and (38) we get

$$\begin{aligned} & d(x)\alpha[y^*, z^*]_\beta + [z, y]_\beta\alpha d(x) + x\alpha(d(y)\beta z^* + y\beta d(z) - d(z)\beta y^* - z\beta d(y)) \\ & -(d(y)\beta z^* + y\beta d(z) - d(z)\beta y^* - z\beta d(y))\alpha x^* = 0 \end{aligned} \tag{41}$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Putting in (41) $[a, b]_\alpha$ for x, we obtain

$$A(y, z)\gamma[a, b]^*_\alpha = [a, b]_\alpha\gamma A(y, z) \tag{42}$$

for all $y, z \in M$ and $\alpha, \beta \in \Gamma$, where $A(y, z)$ stand by $d(y)\beta z^* + y\beta d(z) - d(z)\beta y^* - z\beta d(y)$. Now if M is a non-commutative prime Γ-ring with involution, then by using Lemma 2.3 we get from (42)

$$A(y, z) = 0 \tag{43}$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Using relations (37) and (28) we get

$$2d(y\beta z) = d(y)\beta z^* + y\beta d(z) + d(z)\beta y^* + z\beta d(y) \tag{44}$$

for all $y, z \in M$ and $\beta \in \Gamma$. Since M is a 2-torsion free, then using (43) and (44) we conclude that d is a Γ^*-derivation, then by using Lemma 2.1 we get $d = 0$, which is a contradiction. Then the proof is complete.

Corollary 2.7. *Let M be a 2-torsion free prime Γ-ring with involution and let $d : M \to M$ be a non-zero Jordan Γ^*-derivation such that $d([x, y]_\alpha) = 0$ for all $x, y \in M$ and $\alpha \in \Gamma$, then d is a Γ^*-derivation .*

Proof. By using Theorem 2.6 we get M is commutative prime Γ-ring with involution, also by using Lemma 2.5 we get d is Γ^*-derivation.

3 Acknowledgments

This work is supported by the School of Mathematical Sciences, Universiti Sains Malaysia on Short-term Grant 304/PMATHS/6313171.

References

[1] A.C. Pual, Commutativity in prime gamma-rings with Jordan left derivation , *Mathematical Theory and Modeling,* **2** (2012), 96 - 107.

[2] Frahan D. Shiya, Jordan Right derivation on completely prime Γ-rings , *Journal of AL-Qadisiyah for pure science (quarterly) ,* **13** (2008),1-5.

[3] I.S. Rakhimov, K.K.Dey and A.C. Pual, On commutativity of completely prime gamma-rings , *Malaysian journal of mathematical science,* **7** (2013), 283 - 295.

[4] J. Vukman and I.Kosi-Uibi, On centralizers of semiprime rings with involution, *Studio. Scientiarm Mathematicarum Hungarica,* **43** (2006), 61 - 67.

[5] L. Luh, On the theory of simple Gamma rings, *Michigan Math. J.,* **16** (1969), 65 - 75.

[6] M. Ashraf and S. Ali, On $(\alpha, \beta)^*$-derivation in H^*-algebra , *Advance in algebra,* **2** (2009), 23 - 31.

[7] M. Bresar and J. Vukman, On some additive mapping in ring with involution, *Aequationes Math.,* **38** (1989), 178 - 185.

[8] M.F. Hoque and A.C. Paul,An Equation Related to Centralizers in Semiprime Gamma Rings , *Annals of Pure and Applied Mathematics,* **1**(2012), 84 - 90.

[9] M.F. Hoque and A.C. Paul, Generalized derivations on semiprime gamma rings with involution, *Palestine Journal of Mathematics,* **3** (2014), 235 - 239.

[10] M.F. Hoque and A. C. Paul, On Centralizers of Semiprime Gamma ring, *International Mathematical Forum,* **6** (2011), 627 - 638.

[11] M.F. Hoque and A. C. Paul, Prime Gamma ring with centralizing and commuting generalized derivation , *International journal of algebra,* **7** (2013), 645 - 651.

[12] M.F. Hoque and N.Rahman, The Jordan θ-centralizers of semiprime gamma rings with involution, *International journal of Math. combin.,* **4** (2013), 16 - 31.

[13] M. Sapanci and A. Nakajima, Jordan derivations on completely prime gamma rings , *Math. Japonica,* **46** (1997), 47 - 51.

[14] N. Nobusawa, On the Generalization of the Ring Theory, *Osaka J. Math.,* **1** (1964), 81 - 89.

[15] S. Ali and A. Fosner, On Jordan (α, β)-derivations in rings, *International journal of algebra,* **1** (2010), 99 - 108.

[16] S. Kyuno, On prime gamma rings, *Pacific J. Math.,* **75** (1978), 185 - 190.

[17] W.E. Bernes, On the $\Gamma-$rings of Nobusawa, *Pacific J.Math.,* **18** (1966), 411- 422.

[18] Y. Ceven, Jordan left derivations on completely prime Γ-rings, *C.U.Fen-Edebiyat Fakultesi,Fen Bilimleri Dergisi* **23**(2003),39-43.