

Fully Stable Banach Algebra Module

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Abstract

The object of this paper is to introducea class of module which is a fully stable Banach Algebra module.

Introduction

Given a Banachspace E is aBanach left A-module if E is a left A −module, and $\|a.x\| \leq \|a\|\|x\|(a \in A, x \in E)$[1] .Recall that a submoduleN of an R −module M is said to be stable, if $f(N) \subseteq N$ for each R −homomorphism$f: N \rightarrow M$. In case each submodule of it is stable ,M is called a fully stable module [2],throughout this paper we introduce the concept of full stability for modules . A Banachalgebra module M is called fully stable Banach A-module if for every submodule N of M and for each multiplier $\theta: N \longrightarrow M$ such that $\theta(N) \subseteq N$.Structure of fully stableBanach A- module in term of their elements is considered see(2.5) Studying Baer criterion gives another characterization of fully stable Banach$A$-modulein proposition (2. 8)

1. Preliminaries

In this section the fundamental basic concepts and primitive results are given.

Definition (1.1) [2]

A submoduleN of an R −module M is said to be stable, if $f(N) \subseteq N$ for each R −homomorphism $f: N \longrightarrow M$,In case each submodule of it is stable , M is called a fully stable module.

Examples (1.2) [2]

a) The Z −module Z of all integers is not fully stable .For, define

$\theta: 2Z \longrightarrow Zby\theta(2n) = 3n$ for each $n \in Z$

Clearly ,θ is a Z −homomomrphism .But $\theta(2Z) \nsubseteq$2Z

b) Let M be an$R$ −module .For any ideal I of R ,

$ann_M(I) = \{m \in M | \, Im = (0)\}$. Is astable submodule of$M$

In fact, for any R −homomorphism $f: ann_M(I) \longrightarrow M$

And each $m \in ann_M(I). Im = (0), If(m) = f(Im) = f\big((0)\big) = (0)$

Hence $f(m) \in ann_M(I)$. Thus $ann_M(I)$ is stable submodule.

Recall that , a submodule$N$ of an R −module M is said to be annihilator if $N = ann_M(I)$ for some ideal $I$of R. As in (b) by the a bove an R −module , in which all its submodules are annihilators is fully stable.

Following remark show us ,it issufficies to consider stability over a very restricted class of submodules.

[2] Remark (1.3)

Let M be an$R$ −module. If every cyclic submodule of M is stable then M is fully stable module.

Proposition (1.4) [2]

An R −module M is fully stable if and only if for each x, yin M, $y \notin (x)$ implies $ann_R(x) \nsubseteq ann_R(y)$.

Corollary (1.5) [2]

Let M be a fully stable R −module .Then for each x, yin M, $ann_R(y) = ann_R(x)$ implies $(x) = (y)$.

Definition (1.6) [2]

Let M be an R −module, and Nbe anysubmodule of M. We say that Nsatisfies Baer criterion if for every R −homomorphism $f: N \longrightarrow M$, there exists an element $r \in R$ such that $f(n) = rn$ for each $n \in N$An R −module M is said to be satisfy Baer criterion if each submodule of M satisfies Baer criterion ,that is for every submodule N of M and R −homomorphism $f: N \longrightarrow M$, there exists an element r in R such that $f(n) = rn$ for each $n \in N$.

Note :-From above definition notice that every module which satisfies Baer criterion is fully stable.

In thefollowing proposition and its corollary obtain another characterization of fully stable modules.

Proposition (1.7) [2]

Let M be anR −module .Then Baer criterion holds for cyclic submodules of M if and only if $ann_M(ann_R(x)) = (x)$ $for\ each\ x \in M$.

Corollary (1.8) [2]

AnR −module$M$is fully stable if and onlyif $ann_M(ann_R(x)) = (x)$ $for\ each\ x\ in\ M$.

2. Main results

We see that we need first to define follows

Definition (2.1) [3]

A map from a left Banach$A$ −module X in to a left Banach A −module Y (A is not necessarily commutative) is said a multiplier if it satisfies

$T(a.x) = a.Tx$for all $a \in A, x \in X$.

Definition (2.2) [4]

For a nonempty subset M in a left Banach A −module X, the annihilater $ann_A(M)$ of M is $ann_A(M) = \{a \in A\ ; a.x = 0\ for\ all\ x \in M\}$.

Definition (2.3) [5]

Aleft Banach$A$ −module X is called n −generated for $n \in N$ if there exists$x_1, \ldots, x_n \in X$such thateach $x \in X$ can represented as$x = \sum_{k=1} a_k . x_k$for some $a_1, \ldots, a_n \in A$ .A cyclic module is just a 1-generated one.

Now,we start by introducing the concept of stability forBanach A −module.

Definition (2. 4)

Let Xbe Banach$A$ − module , X is called fully stableBanach$A$ −module if for every submodule N of X and for each multiplier $\theta: N \longrightarrow X$ such that $\theta(N) \subseteq N$.

In the following proposition wediscuss another characterization of fully stable modules .

Notations:-

Let X a Banach A −module

1)$N_x = \{n_x| \ n \in N, \ x \in X\}$

$K_y = \{k_y| \ k \in K, \ y \in X\}$

2) $ann_A N_x = \{a \in A, \ a.n_x = 0, \ \forall \ n_x \in N_x\}$

$ann_A K_y = \{a \in A, \ a.k_y = 0, \ \forall k_y \in K_y\}$

Proposition (2.5)

$X$is fully stable Banach A −module if and only if for each $x, y \in X$

AndN_x, K_ysubsetsof X, $y \notin N_x$ implies $ann_A(N_x) \nsubseteq ann_A(K_y)$.

Proof :-

Suppose that X is fully stable Banach A −module there exists $x, y \in X$ such that $y \notin N_x$ and $ann_A(N_x) \subseteq ann_A(K_y)$

Define $\theta: < N_x > \longrightarrow X$ by $\theta(a.n_x) = a.k_y$,for all $a \in A$

if$a.n_x = 0$ then $a \in ann_A(N_x) \subseteq ann_A(K_y)$

This implies that $a.k_y = 0$,henceθ is well define , clear θ

is a multiplier ,because X is fully stable , there exsits an element $t \in A$such that$\theta(m_x) = tm_x$ $for\ each$ $m_x \in N_x$

In particular,$k_y = \theta(n_x) = tn_x \in N_x$

Which is a contradiction

Conversely, assume that there is a subsetN_x of X and a multiplier $\theta: < N_x > \longrightarrow X$ such that $\theta(N_x) \nsubseteq N_x$ then there exists an element $m_x \in N_x$ such that $\theta(m_x) \notin N_x$. Let $s \in ann_A(N_x)$therefor $sn_x = 0$,$s\theta(m_x) = \theta(sm_x) = \theta(stn_x) = \theta(tsn_x) = \theta(0) = 0$.

Hence $ann_A(N_x) \subseteq ann_A(\theta(m_x))$, which is a contradiction.■

Corollary (2. 6)

Let X be a fully stable Banach A −module .Then for each x, y in X, $ann_A(K_y) = ann_A(N_x)$ implies $N_x = K_y$

Proof:-

Assume that there are two elements x, y in X such that $ann_A(N_x) = ann_A(K_y)$ and $N_x \neq K_y$

Then without loss of generality there is an element z_x in N_x not in K_y .By proposition (2.5)we have $ann_A(K_y) \nsubseteq ann_A(Z_x)$ but $ann_A(N_x) \subseteq ann_A(Z_x)$,hence,$ann_A(K_y) \nsubseteq ann_A(N_x)$ which is a contradiction ■

Definition (2. 7)

A Banach$A$ $-$module X is said to satisfy Baer criterion if eachsubmoduleof X satisfies Baer criterion ,that is for every submodule N of X and $A -$ multiplier$\theta: N \longrightarrow X$,there exists an element a in A such that $\theta(n) = an$for all $n \in N.$

In the following proposition and its corollary another characterization of fully stable Banach$A$ $-$module is given.

Proposition (2.8)

Let X be a Banach A $-$module .Then Baer criterion holds for cyclic submodules of X if and only if $ann_X(ann_A(N_x)) = N_x$ $for\ each\ x \in X.$

Proof :-

Assume that Baer criterion holds. Let $y \in ann_X(ann_A(N_x))$

Define $\theta: < N_x > \longrightarrow X$ by $\theta(a.n_x) = a.k_y$,for all $a \in A$

Let $a_1.n_x = a_2.n_x$

$$(a_1 - a_2)n_x = 0\ , \quad a_1 - a_2 \in ann_A(N_x)$$

$$(a_1 - a_2) \in ann_A\left(K_y\right) \longrightarrow \quad (a_1 - a_2)k_y = 0$$

$a_1k_y = a_2k_y$

hence ,θ is well define.

It is clear that clear θ is an$A -$ multiplier. By the assumption, there exists an element $t \in A$ such that

$$\theta(m_x) = tm_x \quad for\ each\ m_x \in N_x$$

In particular,$k_y = \theta(n_x) = tn_x \in N_x ann_X(ann_A(N_x)) \subseteq N_x\ ;\ hence\ ann_X(ann_A(N_x)) =$.This implies that N_x

Conversely, assume that $ann_X(ann_A(N_x)) = N_x$

For each $N_x \subseteq X$.Then for each $A -$ multiplier$\theta: N_x \rightarrow X$

And s $\in ann_A(N_x)$,we have $s\theta(n_x) = \theta(sn_x) = 0$

Thus $\theta(n_x) \in ann_X(ann_A(N_x)) = N_x$, then $\theta(n_x) = t\,n_x$ for some $t \in A$,thus Baer criterion is holds.

Corollary (2.9)

Xisfully stable Banach$A$ $-$module if and only if

$ann_X(ann_A(N_x)) = N_x$ $for\ each\ x \in X.$

References
[1] Matthew David Peter Daws , Banach algebras of operators
The University of Leeds, School of Mathematics, Department of Pure Mathematics , (2004).
[2]M.S.ABBAS,On Fully Stable Modules , Ph. D ,Thesis , University of Baghdad, Iraq,(1990).
[3]JANKO BRAˇCIˇC,Simple Multipliers onBanachModules .University of Ljubljana,Slovenia,GlasgowMathematical Journal Trust ,(2003).
[4]J. BRAˇCIˇC ,Local Operators onBanach Modules, University of Ljubljana, Slovenia,Mathematical Proceedings of the Royal Irish Academy,(2004).
[5]Antonio M.Cegarra, Projective Covers of Finitely
Generated Banach Modules and the Structure of Some Banach Algebras, O.Yu.Aristov,Russia,(2006).