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Existence of solutions for hybrid differential equation with fractional order

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Abstract

In this paper, we study the existence of solutions for the following fractional hybrid differential equations

$$D^{\alpha}\left(\frac{x(t)}{f(t,x(t))}\right)=g(t,x(t)) \quad , \quad 0\le t\langle 1 \ ; \quad x(1)=x'(1)=0 \quad ;$$

involving Riemann-Liouville differential operators of order $1 \prec \alpha \le 2$. An existence theorem for fractional hybrid differential equations is proved under mixed Lipschitz and Carathéodory conditions and using the Dhage point fixe theorem.

Keywords: Quadratic perturbations; Riemann-Liouville derivative; Hybrid differential equations

1. Introduction

During the past decades, fractional differential equations have attracted many authors (see[1, 9, 10, 11, 12, 13,14, 16]). The differential equations involving fractional derivatives in time, compared with those of integer order in time, are more realistic to describe many phenomena in nature (for instance, to describe the memory and hereditary properties of various materials and processes), the study of such equations has become an object of extensive study during recent years.

The quadratic perturbations of nonlinear differential equations have attracted much attention. We call such fractional hybrid differential equations. There have been many works on the theory of hybrid differential equations, and we refer the readers to the articles [2, 3, 4, 5, 6, 7, 17].

Dhage and Lakshmikantham [5] discussed the following first order hybrid differential equation

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{x(t)}{f(t,x(t))}\right)=g(t,x(t)) \quad a.e \ \ t\in J=[0,1] \\ x(t_0)=x_0 \ \in \ IR \end{cases}$$

Where $f\in C(J\times\mathbb{R},\mathbb{R}\setminus\{0\})$ and $g\in C_*(J\times\mathbb{R},\mathbb{R})$, ($C_*(J\times\mathbb{R},\mathbb{R})$ is called the Carathéodory class of functions). They established the existence, uniqueness results and some fundamental differential inequalities for hybrid differential equations initiating the study of theory of such systems and proved utilizing the theory of inequalities, its existence of extremal solutions and a comparaison results.

Zhao, Sun, Han and Li [15] have discussed the following fractional hybrid differential equations involving Riemann-Liouville differential operators

$$\begin{cases} D^{q}\left[\dfrac{x(t)}{f(t,x(t))}\right]=g(t,x(t)) \quad a.e \quad t\in J=[0,T] \\ x(0)=0 \in \mathbb{R} \end{cases}$$

Where $f\in C(J\times\mathbb{R},\mathbb{R}\setminus\{0\})$ and $g\in C_{*}(J\times\mathbb{R},\mathbb{R})$. The authors of [15] established the existence theorem for fractional hybrid differential equation and some fundamental differential inequalities. They also established the existence of extremal solutions.

Hilal and Kajouni [7] studied boundary fractional hybrid differential equations involving Caputo differential operators of order $0\prec\alpha\prec1$,

$$\begin{cases} D^{\alpha}\left[\dfrac{x(t)}{f(t,x(t))}\right]=g(t,x(t)) \quad a.e \quad t\in J=[0,T] \\ a\dfrac{x(0)}{f(0,x(0))}+b\dfrac{x(T)}{f(T,x(T))}=c \end{cases}$$

Where $f\in C(J\times\mathbb{R},\mathbb{R}\setminus\{0\})$ and $g\in C_{*}(J\times\mathbb{R},\mathbb{R})$ and a,b,c are real constants with $a+b\neq0$. They proved the existence result for boundary fractional hybrid differential equations under mixed Lipschitz and Carathéodory conditions. Some fundamental fractional differential inequalities are also established which are utilized to prove the existence of extremal solutions. Necessary tools are considered and the comparaison principle is proved which will be useful for further study of qualitative behavior of solutions.

In this paper we consider the fractional hybrid differential equations with involving Riemann-Liouville differential operators of order $1\prec\alpha\leq2$

$$\begin{cases} D^{\alpha}\left[\dfrac{x(t)}{f(t,x(t))}\right]=g(t,x(t)) \quad a.e \quad 0\leq \mathrm{t}\prec1 \\ x(1)=x'(1)=0 \end{cases} \tag{1.1}$$

Where $f\in C(J\times\mathbb{R},\mathbb{R}\setminus\{0\})$ and $g\in C_{*}(J\times\mathbb{R},\mathbb{R})$. Using the fixed point theorem, we give an existence theorem of solutions for the boundary value problem of the above nonlinear fractional differential equation under both Lipschitz and Carathéodory conditions. We present two examples to illustrate our results.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminaries facts which are used throughout this paper.

By $X=C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from $J=[0,1]$ into $\mathbb{R}$ with the norm

$$\|y\| = \sup\{|y(t)| \ , \ t \in J\}$$

We denote by $C_*(J \times \mathbb{R}, \mathbb{R})$ the class of functions $g : J \times \mathbb{R} \to \mathbb{R}$ such that

(i) the map $t \mapsto g(t,x)$ is mesurable for each $x \in \mathbb{R}$, and

(ii) the map $t \mapsto g(t,x)$ is continuous for each $t \in J$.

The class $C_*(J \times \mathbb{R}, \mathbb{R})$ is called the Carathéodory class of functions on $J \times \mathbb{R}$ which are Lebesgue integrable when bounded by a Lebesgue integrable function on J.

By $L^1(J, \mathbb{R})$ denote the space of Lebesgue integrable real-valued functions on J endowed with the norm $\| \cdot \|_{L^1}$ defined by

$$\| \cdot \|_{L^1} = \int_0^T |y(s)| ds$$

Definition 2.1 [8] The Riemann-Liouville fractional integral of the continuous function $h\ (0,+\infty) \to \mathbb{R}$ of order $\alpha \succ 0$ is defined by

$$I^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds$$

Provided that the right side is pointwise defined on $(0,+\infty)$

Definition 2.2 [8] The Riemann-Liouville fractional derivative of order $\alpha \succ 0$ of the continuous function $h : (0,+\infty) \to \mathbb{R}$ is given by

$$D_{0^+}^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{(n)} \int_0^t (t-s)^{n-\alpha-1} h(s) ds$$

Where $n = [\alpha]+1$, $[\alpha]$ denote the integer part of number α, Provided that the right side is pointwise defined on $(0,+\infty)$.

From the definition of the Riemann-Liouville derivative, we can obtain the following statement

Lemma 2.1. [8] Let $\alpha \succ 0$. If we assume $x \in C(0,1) \cap L^1(0,1)$ then the fractional differential equation $D_{0^+}{}^\alpha x(t) = 0$ has $x(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_n t^{\alpha-n}$; $ci \in IR$, $i = 1,\ldots,n$, as unique solutions, where n is the smallest integer greater than or equal to α.

Lemma 2.2. [8] Assume $x \in C(0,1) \cap L^1(0,1)$ with a fractional derivative of $\alpha \succ 0$ that belongs to $C(0,1) \cap L^1(0,1)$. Then

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} x(t) = x(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_n t^{\alpha-n}$$

for some $ci \in \mathbb{R}$, $i = 1, 2, \ldots, n$, where n is the smallest integer greater than or equal to α.

Lemma 2.3. [8] Let $h \in C([0,1], \mathbb{R})$ and $1 \prec \alpha \leq 2$. The unique solution of the problem

$$D^{a}\left(\frac{x(t)}{f(t, x(t))}\right) = h(t) \quad a.e. \quad t \in [0,1] \qquad (2.1)$$

$$x(1) = x'(1) = 0 \qquad (2.2)$$

is

$$x(t) = f(t, x(t)) \int_0^1 H(t,s) h(s) ds$$

where

$$H = \begin{cases} \dfrac{(t-s)^{\alpha-1} - t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \dfrac{s(1-t)t^{\alpha-2}(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}, & 0 \leq s \leq t \leq 1 \\ \dfrac{-t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \dfrac{s(1-t)t^{\alpha-2}(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}, & 0 \leq t \leq s \leq 1 \end{cases} \qquad (2.3)$$

Proof Applying the Riemann-Liouville fractional integral of the order α for the equation (3), we obtain

$$\frac{x(t)}{f(t, x(t))} = I^{\alpha} h(t) + c_1 t^{\alpha-1} + c_2 t^{a-2}$$

for some $c_1, c_2 \in \mathbb{R}$. Consequently, the general solution of (2.1) is

$$x(t) = f(t, x(t)) \left(\frac{1}{\Gamma(\alpha)} \int_0^1 (t-s)^{\alpha-1} h(s) ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} \right) \qquad (2.4)$$

By $x(1) = 0$, we have

$$c_1 + c_2 = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s) ds.$$

From (2.4), we get

$$\frac{x'(t)f(t,x(t))-x(t)f_t(t,x(t))}{f^2(t,x(t))}=\frac{1}{\Gamma(\alpha-1)}\int_0^t(1-s)^{\alpha-2}h(s)ds+(\alpha-1)c_1t^{\alpha-2}+(\alpha-2)c_2t^{\alpha-3}$$

By

$$x'(1)=0$$

we have

$$(\alpha-1)c_1+(\alpha-2)c_2=\frac{1}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}h(s)ds$$

Then

$$\begin{cases} c_1=\frac{1}{\Gamma(\alpha-1)}\int_0^1\left((1-s)^{\alpha-2}-(1-s)^{\alpha-s}\right)h(s)ds \\ c_2=\frac{1}{\Gamma(\alpha-1)}\int_0^1\left((1-s)^{\alpha-1}-(1-s)^{\alpha-2}\right)h(s)ds-\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}h(s)ds \end{cases}$$

Therefore

$$\begin{aligned} x(t) &= f(t,x(t))\left(\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}h(s)ds+\int_0^1\left(\frac{t^{\alpha-1}}{\Gamma(\alpha-1)}\left((1-s)^{\alpha-1}\right.\right.\right. \\ &\quad \left.-(1-s)^{\alpha-2}\right)-\frac{t^{\alpha-1}}{\Gamma(\alpha)}\left((1-s)^{\alpha-1}+\frac{t^{\alpha-2}}{\Gamma(\alpha)}\left((1-s)^{\alpha-2}-\left(1-s^{\alpha-1}\right)\right)h(s)ds\right) \\ &= f(t,x(t))\left(\int_0^t\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}+\frac{t^{\alpha-1}}{\Gamma(\alpha-1)}\left((1-s)^{\alpha-1}-(1-s)^{\alpha-2}\right)-\frac{t^{\alpha-1}}{\Gamma(\alpha)}(1-s)^{\alpha-1}\right. \\ &\quad +\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\left((1-s)^{\alpha-2}-(1-s)^{\alpha-2})\right)h(s)ds+\int_t^1\left(\frac{t^{\alpha-1}}{\Gamma(\alpha-1)}\left((1-s)^{\alpha-1}-(1-s)^{\alpha-2}\right)\right. \\ &\quad \left.-\frac{t^{\alpha-1}}{\Gamma(\alpha)}(1-s)^{\alpha-1}+\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\left((1-s)^{\alpha-2}-(1-s)^{\alpha-1})\right)h(s)ds\right) \\ &= f(t,x(t))\left(\int_0^t\left(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{s(1-t)t^{\alpha-2}(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}-\frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}\right)h(s)ds\right. \\ &\quad \left.+\int_t^1\left(\frac{s(1-t)t^{\alpha-2}(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}-\frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}\right)h(s)ds\right) \\ &= f(t,x(t))\int_0^1H(t,s)h(s)ds \end{aligned}$$

The proof is complete

Lemma 2.4. The function $H(t,s)$ defined by (5) satisfies the following conditions:

$$\Gamma(\alpha-1)H(t,s)\le q(t)k(s) \qquad (2.5)$$

Where $q(t)=(1-t)t^{\alpha-2}$ and $k(s)=s(1-s)^{\alpha-2}$

3. Existence result

In this section, we prove the existence results for the hybrid differential equations with fractional order (2.1) on the closed and bounded interval $J=[0,1]$ under mixed Lipschitz and Carathéodory conditions on the nonlinearities involved in it.

We defined the multiplication in X by

$(xy)(t)=x(t)y(t)$, for $x,y\in X$

Clearly $X=C(J;\mathbb{R})$ is a Banach algebra with respect to above norm and multiplication in it.

Lemma 3.1 [4] Let S be a non-empty, closed convex and bounded subset of the Banach algebra X and let $A:X\to X$ and $B:S\to X$ be two operators such that

(a) A is Lipschitzian with a Lipschitz constant α

(b) B is completely continuous,

(c) $x=AxBy \Rightarrow x\in S$ for all $y\in S$, and

(d) $\alpha M \prec 1$, where $M=\|B(S)\|=\sup\{\|B(x)\|:x\in S\}$

then the operator equation $AxBx=x$ has a solution in S

We make the following assumptions:

(H0) The function is increasing in $\mathbb{R}$ almost every where for $t\in J$

(H1) There exists a constant $L\succ 0$ such that

$$|f(t,x)-f(t,y)|\le L|x-y| \quad \text{for all } t\in J \text{ and } x,y\in\mathbb{R}.$$

(H2) There exists a function $h\in L^1(J,\mathbb{R}^+)$ such that $|g(t,x)|\le h(t)$ a.e $t\in J$ for all $x\in\mathbb{R}$

For convenience, we denote: $$T=\frac{1}{\Gamma(\alpha-1)}\int_0^1 k(\ )s \quad ($$

Theorem 3.1 Assume that hypotheses (H_1) and (H_2) hold. Further, if

$$LT\,\|h\|_{L^1}\prec 1$$

then the boundary value problem (1.1) has a solution define J

Proof.

We define a subset S of X *by* $S=\{x\in X \mid\mid x \mid\mid\leq N\}$

where $N=\dfrac{F_0 T \mid\mid h \mid\mid_{L^1}}{1-LT \mid\mid h \mid\mid_{L^1}}$ and $F=\sup_{\{t\in J\}}\left|f(t,0)\right|$

It is clear that S satisfies hypothesis of lemma 3.1 . By application of Lemma 2.3 , the equation (1.1) is equivalent to the nonlinear hybrid integral equation

$$x(t)=f(t,x(t))\int_0^1 H(t,s)g(s,x(s))ds \tag{3.3}$$

Define two operators $A: X\rightarrow X$ and $B: S\rightarrow X$ by

$$Ax(t)=f(t,x(t)) \quad , \quad t\in J \tag{3.4}$$

and

$$Bx(t)=\int_0^1 H(t,s)\, g(s,x(s))ds \tag{3.5}$$

Then the hybrid integral equation (3.3) is transformed into the operator equation as

$$x(t)=Ax(t)Bx(t) \quad , \quad t\in J \tag{3.6}$$

We shall show that the operators A and B satisfy all the conditions of Lemma 3.1.

Claim 1, let $x,y\in X$ then by hypothesis (H1),

$$\left|Ax(t)-Ay(t)\right|=\left|f(t,x(t))-f(t,y(t))\right|\leq L\left|x(t)-y(t)\right|\leq L \mid\mid x-y \mid\mid$$

for all $t\in J$. Taking supremum over t, we obtain

$$\mid\mid Ax-Ay \mid\mid\leq L \mid\mid x-y \mid\mid$$

for all $x,y\in X$

Claim 2, B is a continuous in S .

Let (x_n) be a sequence in S converging to a point $x\in S$. By (H_2) and Lebesgue dominated convergence theorem, we have

$$\begin{aligned}\lim_{x\to\infty} Bx_n(t) &= \lim_{n\to\infty}\int_0^1 H(t,s)g(s,x_n(s))ds\\ &=\int_0^1 H(t,s)\lim_{n\to\infty} g(s,x_n(s))ds\\ &=\int_0^1 H(t,s)g(s,x(s))ds=Bx(t)\end{aligned}$$

for all $t\in J$. This shows that B is a continuous operator on S .

Claim 3, B is compact operator on S .

First, we show that $B(S)$ is a uniformly bounded set in X .

Let $x\in S$ be arbitrary. By Lemma 2.4 , we have

$$
\begin{aligned}
|Bx(t)| &= \left|\int_0^1 H(t,s)g(s,x(s))ds\right| \\
&\le q(t)\frac{1}{\Gamma(\alpha-1)}\int_0^1 k(s)h(s)ds \\
&\le T\,\|h\|_{L^1}
\end{aligned}
$$

for all $t\in J$. This implies that B is uniformly bounded on S.

Next, we prove that $B(S)$ is an equi-continuous set on X.

Given $\varepsilon \succ 0$, let

$$
\delta \prec \min\left\{\frac{1}{2},\frac{\Gamma(\alpha+1)\varepsilon}{12\,\|h\|_{L^1}}\right\}
$$

Let $x\in S$, $t_1,t_2\in[0,1]$ with $t_1\prec t_2, 0\prec t_2-t_1\prec\delta$, we have

$$
\begin{aligned}
|Bx(t_2)-Bx(t_1)| &= \left|\int_0^1 H(t_2,s)g(s,x(s))ds-\int_0^1 H(t_1,s)g(s,x(s))ds\right| \\
&\le \|h\|_{L^1}\left|\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}-t_2^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds+\int_0^{t_2}\frac{s(1-t_2)t_2^{\alpha-2}(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds\right. \\
&- \int_{t_2}^1\frac{t_2^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds+\int_{t_2}^1\frac{s(1-t_2)t_2^{\alpha-2}(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds \\
&- \int_0^{t_1}\frac{(t_1-s)^{\alpha-1}-t_1^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds-\int_0^{t_1}\frac{s(1-t_1)t_1^{\alpha-2}(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds \\
&+ \left.\int_{t_1}^1\frac{t_1^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds-\int_{t_1}^1\frac{s(1-t_1)t_1^{\alpha-2}(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds\right|
\end{aligned}
$$

$$
\begin{aligned}
&\le \|h\|_{L^1}\left(\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds-\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}ds+\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right. \\
&\left.+\left(t_2^{\alpha-2}-t_1^{\alpha-2}\right)\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds+\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds\right) \\
&\le \|h\|_{L^1}\left(\frac{t_2^{\alpha}-t_1^{\alpha}+t_2^{\alpha-1}-t_1^{\alpha-1}}{\Gamma(\alpha+1)}+\frac{t_2^{\alpha-2}-t_1^{\alpha-2}}{\Gamma(\alpha)}+\frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{\Gamma(\alpha)}\right) \\
&\le \frac{\|h\|_{L^1}}{\Gamma(\alpha+1)}\left(t_2^{\alpha}-t_1^{\alpha}+(1+\alpha)\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)+\alpha\left(t_2^{\alpha-2}-t_1^{\alpha-2}\right)\right) \\
&\le \frac{\|h\|_{L^1}}{\Gamma(\alpha-1)}\left(t_2^{\alpha}-t_1^{\alpha}+3\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)+2\left(t_2^{\alpha-2}-t_1^{\alpha-2}\right)\right)
\end{aligned}
$$

In order to estimate $t_2^\alpha - t_1^\alpha, t_2^{\alpha-1} - t_1^{\alpha-1}$ and $t_2^{\alpha-2} - t_1^{\alpha-2}$, we consider the following cases

Case1: $0 \le t_1 \prec \delta$, $t_2 \prec 2\delta$

$$t_2^\alpha - t_1^\alpha \le t_2^\alpha \prec (2\delta)^\alpha \le 2^\alpha \delta \le 4\delta \quad , \quad t_2^{\alpha-1} - t_1^{\alpha-1} \le t_2^{\alpha-1} \prec (2\delta)^{\alpha-1} \le 2^{\alpha-1}\delta \le 2\delta$$

$$t_2^{\alpha-2} - t_1^{\alpha-2} \le t_2^{\alpha-2} \prec (2\delta)^{\alpha-2} \le 2^{\alpha}\delta \le 2\delta$$

Case2: $0 \prec t_1 \prec t_2 \le \delta$

$$t_2^\alpha - t_1^\alpha \le t_2^\alpha \prec \delta^\alpha \le \alpha\delta \le 4\delta, \quad t_2^{\alpha-1} - t_1^{\alpha-1} \le t_2^{\alpha-1} \prec \delta^{\alpha-1} \le (\alpha-1)\delta \le$$

$$t_2^{\alpha-2} - t_1^{\alpha-2} \le t_2^{\alpha-2} \prec \delta^{\alpha-2} \le (\alpha-2)\delta \le \delta$$

Case3: $\delta \le t_1 \prec t_2 \le 1$

$$t_2^\alpha - t_1^\alpha \le \alpha\delta \le 4\delta \ , \quad t_2^{\alpha-1} - t_1^{\alpha-1} \le (\alpha-1)\delta \le 2\delta$$

$$t_2^{\alpha-2} - t_1^{\alpha-2} \le (\alpha-2)\delta \le \delta$$

Thus, we obtain

$$\left|Bx(t_2) - B(t_1)\right| \prec \varepsilon$$

For all $t_1, t_2 \in J$ and for all $x \in X$.

This implies that $B(S)$ is an equi-continuous set in X .

Then by Arzelá-Ascoli theorem, B is a continuous and compact operator on S .

Claim 4, The hypothesis (c) of theorem 3.1 is satisfied

Let $x \in X$ and $y \in Y$ be arbitrary such that $x = AxBy$. Then,

$$|x(t)| = |Ax(t)By(t)|$$

$$|f(t, x(t)) - f(t,0)| + |f(t,0)| \left|\int_0^1 H(t,s)\, g(s, x(s))\, ds\right|$$

$$\le \left[L|x(t)| + F_0\right] \left(q(t) \frac{1}{\Gamma(\alpha-1)} \int_0^1 k(s)\, h(s)\, ds \right)$$

$$\le \left[L|x(t)| + F_0\right] T \|h\|_{L^1}$$

Thus, $$|x(t)| \le \frac{F_0 T \|h\|_{L^1}}{1 - LT \|h\|_{L^1}}$$

Taking supremum over t,

$$\|x\| \leq \frac{F_0 T \|h\|_{L^1}}{1 - LT\|h\|_{L^1}}$$

Then $x \in S$ and the hypothesis (c) of Lemma 3.1 is satisfied.

Finally, we have

$$M = \|B(S)\| = \sup\{\|Bx\| : x \in S\} \leq T\|h\|_{L^1}$$

and so, $LM \leq LT\|h\|_{L^1} \prec 1$

Thus, all the conditions of Lemma 3.1 are satisfied and hence the operator equation $AxBx = x$ has a solution in S. As a result, the boundary value problem (2) has a solution defined on J. This completes the proof.

4.Examples

In this section, we will present two examples to illustrate the main results.

Example 4.1. we consider the fractional hybrid differential equation

$$\begin{cases} D^{\frac{3}{2}} x(t) = \sin x & \text{a.e. } 0 \leq t \prec 1 \\ x(1) = x'(1) = 0 \end{cases} \qquad (4.1)$$

Where $f(t,x) \equiv 1$, $g(t,x) = \sin x$ and $h(t) \equiv 1$. Then hypothesis (H_1) and (H_2) hold.

Since

$$T = \frac{1}{\Gamma(\alpha - 1)} \int_0^1 k(s)ds = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^1 s(1-s)^{\frac{1}{2}} ds = \frac{4}{35\sqrt{\pi}}$$

choosing $L = 1$, then $LT\|h\|_{L^1} \prec 1$. Therefore, the fractional hybrid differential equation (4.1) has a solution.

Example 4.2. we consider the fractional hybrid differential equation

$$\begin{cases} D^{\frac{3}{2}} \left[\dfrac{x(t)}{\sin x + 2} \right] = \cos x \quad \text{a.e.} \quad 0 \leq t \prec 1 \\ x(1) = x'(1) = 0 \end{cases} \qquad (4.2)$$

Where $f(t,x) \equiv \sin x + 2$, $g(t,x) = \cos x$ and $h(t) \equiv 1$. Then hypothesis (H_1) and (H_2) hold.

Since $T = \frac{4}{35\sqrt{\pi}}$, choosing $L = 1$, then $LT\|h\|_{L^1} \prec 1$. Therefore, the fractional hybrid differential equation (4.2) has a solution .

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