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A Comparative Study of Chi-Square Goodness-of-Fit Under Fuzzy Environments

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Abstract
Testing goodness-of-fit plays a vital role in data analysis. This problem seems to be much more complicated in the presence of vague data. In this paper, the chi-square goodness-of-fit under trapezoidal fuzzy numbers (tfns.) is proposed using alpha cut interval method. And the ranking grades of tfns. are also used to compute the chi-square test statistic. The proposed technique is illustrated with two different numerical examples along with different methods of ranking grades for a concrete comparative study.
Keywords: Chi-square Test, Fuzzy Sets, Trapezoidal Fuzzy Numbers, Alpha Cut, Ranking Function, Graded Mean Integration Representation.

1. Introduction:

Most of statistical procedures are based on fairly specific assumptions regarding the underlying population distribution, like normality, exponentiality, etc. Therefore it might be desirable to check whether these assumptions are reasonable. Statistical procedures for testing hypotheses about the underlying distribution are called **goodness-of-fit test**.

Fuzzy set theory [34] has been applied to many areas which need to manage uncertain and vague data. Such areas include approximate reasoning, decision making, optimization, control and so on. In traditional statistical testing [13], the observations of sample are crisp and a statistical test leads to the binary decision. However, in the real life, the data sometimes cannot be recorded or collected precisely. The statistical hypotheses testing under fuzzy environments has been studied by many authors using the fuzzy set theory concepts introduced by Zadeh [34]. Viertl [29] investigated some methods to construct confidence intervals and statistical tests for fuzzy data. Wu [32] proposed some approaches to construct fuzzy confidence intervals for the unknown fuzzy parameter. A new approach to the problem of testing statistical hypotheses is introduced by Chachi et al. [8]. Mikihiko Konishi et al. [19] proposed a method of ANOVA for the fuzzy interval data by using the concept of fuzzy sets. Hypothesis testing of one factor ANOVA model for fuzzy data was proposed by Wu [31, 33] using the h-level set and the notions of pessimistic degree and optimistic degree by solving optimization problems. Gajivaradhan and Parthiban analysed one-way ANOVA test using alpha cut interval method for trapezoidal fuzzy numbers [20] and they presented a comparative study of 2-factor ANOVA test under fuzzy environments using various methods [21].

Wang et al. presented a method for centroid formulae for a generalized fuzzy number and arrived some different approach for ranking tfns. [30]. Salim Rezvani analysed the ranking functions with tfns. [25]. Thorani et al. approached the ranking function of a tfns. with some modifications [26]. Salim Rezvani and Mohammad Molani presented the shape function and Graded Mean Integration Representation for tfns. [24].

In this paper, we propose the chi-square goodness-of-fit under fuzzy data. That is, if the observed large samples are unavoidably in terms of trapezoidal fuzzy numbers (or triangular fuzzy numbers), we suggest here how to modify the classical chi-square test for such data using their alpha cut intervals. And the decision rules of the proposed technique are given. In the proposed approach, the degrees of optimism, pessimism and h-level set are not used but used in Wu [31]. In fact we would like to present a conclusion that α-cut interval method is general enough to deal with chi-square test of goodness-of-fit under fuzzy data (tfns.). Also, in this paper we have analysed what can be the result if the centroid/ranking grades of tfns. are employed in hypotheses testing. The same concept can also be applied for the data which are in terms of triangular fuzzy numbers. For better understanding, the proposed technique is illustrated with two different kinds of numerical examples with different conclusions.

2. Preliminaries

Definition 2.1. Generalized fuzzy number

A generalized fuzzy number A is described as any fuzzy subset of the real line $\mathbb{R}$, whose membership function $\mu_A(x)$ satisfies the following conditions:

i. $\mu_A(x)$ is a continuous mapping from $\mathbb{R}$ to the closed interval $[0, \omega]$, $0 \le \omega \le 1$,

ii. $\mu_A(x) = 0$, for all $x \in (-\infty, a]$,

iii. $\mu_L(x) = L_A(x)$ is strictly increasing on $[a, b]$,

iv. $\mu_A(x) = \omega$, for all $[b, c]$, as ω is a constant and $0 < \omega \le 1$,

v. $\mu_R(x) = R_A(x)$ is strictly decreasing on $[c, d]$,

vi. $\mu_A(x) = 0$, for all $x \in [d, \infty)$. where a, b, c, d are real numbers such that $a < b \le c < d$.

Definition 2.2. A fuzzy set A is called **normal** fuzzy set if there exists an element (member) 'x' such that $\mu_A(x) = 1$. A fuzzy set A is called **convex** fuzzy set if $\mu_A(\alpha x_1 + (1-\alpha)x_2) \ge \min\{\mu_A(x_1), \mu_A(x_2)\}$ where $x_1, x_2 \in X$ and $\alpha \in [0, 1]$. The set $A_\alpha = \{x \in X / \mu_A(x) \ge \alpha\}$ is said to be the α-cut of a fuzzy set A.

Definition 2.3. A fuzzy subset A of the real line $\mathbb{R}$ with **membership function** $\mu_A(x)$ such that $\mu_A(x): \mathbb{R} \to [0, 1]$, is called a fuzzy number if A is normal, A is fuzzy convex, $\mu_A(x)$ is upper semi-continuous and $\mathrm{Supp}(A)$ is bounded, where $\mathrm{Supp}(A) = \mathrm{cl}\{x \in \mathbb{R} : \mu_A(x) > 0\}$ and 'cl' is the closure operator.

Definition 2.4. **α-cut of a fuzzy number:** A useful notion for dealing with a fuzzy number is a set of its α-cuts. The α-cut of a fuzzy number A is a non-fuzzy set defined as $A_\alpha = \{x \in \mathbb{R} : \mu_A(x) \ge \alpha\}$. A family of $\{A_\alpha : \alpha \in (0,1]\}$ is a set representation of the fuzzy number A. According to the definition of a fuzzy number, it is easily seen that every α-cut of a fuzzy number is a closed interval. Hence we have, $A_\alpha = [A_\alpha^L, A_\alpha^U]$ where $A_\alpha^L = \inf\{x \in \mathbb{R} : \mu_A(x) \ge \alpha\}$ and $A_\alpha^U = \sup\{x \in \mathbb{R} : \mu_A(x) \ge \alpha\}$. A space of all fuzzy numbers will be denoted by $F(\mathbb{R})$.

It is known that for a **normalized tfn** $A = (a, b, c, d; 1)$, there exists four numbers $a, b, c, d \in \mathbb{R}$ and two functions $L_A(x), R_A(x): \mathbb{R} \to [0, 1]$, where $L_A(x)$ and $R_A(x)$ are non-decreasing and non-increasing functions respectively. And its membership function is defined as follows:

$\mu_A(x) = L_A(x) = (x-a)/(b-a)$ for $a \le x \le b$; 1 for $b \le x \le c$; $R_A(x) = (x-d)/(c-d)$ for $c \le x \le d$ and 0 otherwise. The functions $L_A(x)$ and $R_A(x)$ are also called the **left** and **right side** of the fuzzy number A respectively [9]. In this paper, we assume that $\int_{-\infty}^{\infty} A(x)\,dx < +\infty$. The left and right sides of the fuzzy number A are strictly monotone, obviously, A_L and A_U are inverse functions of $L_A(x)$ and $R_A(x)$ respectively. Another important type of fuzzy number was introduced in [6] as follows:

Let $a, b, c, d \in \mathbb{R}$ such that $a < b \le c < d$. A fuzzy number A defined as $\mu_A(x): \mathbb{R} \to [0, 1]$,

$\mu_A(x) = \left(\frac{x - a}{b - a}\right)^n$ for $a \le x \le b$; 1 for $b \le x \le c$; $\left(\frac{d - x}{d - c}\right)^n$ for $c \le x \le d$; 0 otherwise where $n > 0$ is denoted by $A = (a, b, c, d)_n$. And $L(x) = \left(\frac{x - a}{b - a}\right)^n$; $R(x) = \left(\frac{d - x}{d - c}\right)^n$ can also be termed as left and right spread of the tfn. [Dubois and Prade in 1981].

If $A = (a, b, c, d)_n$, then[1-4],

$A_\alpha = [A_L(\alpha), A_U(\alpha)] = [a + (b - a)\sqrt[n]{\alpha}, d - (d - c)\sqrt[n]{\alpha}]$; $\alpha \in [0, 1]$.

When $n = 1$ and $b = c$, we get a triangular fuzzy number. The conditions $r = 1$, $a = b$ and $c = d$ imply the closed interval and in the case $r = 1$, $a = b = c = d = t$ (some constant), we can get a crisp number 't'. Since a trapezoidal fuzzy number is completely characterized by $n = 1$ and four real numbers $a \le b \le c \le d$, it is often denoted as $A = (a, b, c, d)$. And the family of trapezoidal fuzzy numbers will be denoted by $F^T(\mathbb{R})$.

Now, for $n = 1$ we have a normal trapezoidal fuzzy number $A = (a, b, c, d)$ and the corresponding α - cut is defined by

$A_\alpha = [a + \alpha(b - a), d - \alpha(d - c)]$; $\alpha \in [0, 1]$ ---(2.5). And we need the following results which can be found in [13, 14].

Result 2.1. Let D = {[a, b], $a \le b$ and a, $b \in \mathbb{R}$ }, the set of all closed, bounded intervals on the real line $\mathbb{R}$.

Result 2.2. Let A = [a, b] and B = [c, d] in D. Then A = B if a = c and b = d.

3. Chi-square distribution

If X_i (i=1, 2, …, n) are n independent normal variates with mean μ_i and variance σ^2 (i=1, 2, …, n) then $\chi^2 = \sum_{i=1}^{n} ((X_i - \mu_i)/\sigma_i)^2$ is a chi-square variate with n degrees of freedom. The probability density function of the chi-square distribution is given by, $f(\chi^2) = (1/(2^{n/2}\sqrt{n/2}))(\chi^2)^{(n/2)-1} e^{-\chi^2/2}$; $0 < \chi^2 < \infty$ where n is the degrees of freedom. And the exact shape of the distribution depends upon the number of degrees of freedom n. In general, when n is small, the shape of the curve is skewed to the right and as υ gets larger, the distribution becomes more and more symmetrical. The mean and variance of the chi-square distribution are n and 2n respectively. As $n \to \infty$, the chi-square distribution approaches a normal distribution. The sum of independent chi-square variates is also a chi-square variate.

Moreover, chi-square distribution is very useful: (i) to test if the hypothetical value of the population variance is $\sigma^2 = \sigma_0^2$ (say). (ii) to test the "goodness of fit". It is used to determine whether an actual sample distribution matches a known theoretical distribution. (iii) to test the independence of attributes i.e. if a population is known to have two attributes, then chi-square distribution is used to test whether the two attributes are associated or independent, based on a sample. (iv) to test the homogeneity of independent estimates of the population correlation coefficient.

3.1. Conditions for the validity of chi-square test:

(i) The experimental data (sample observations) must be independent of each other. (ii) The total frequency (or number of observations in the sample) must be reasonably large, say ≥ 50. (iii) No individual frequencies should be less than 5, if any frequency is less than 5, then it is pooled with the preceding or succeeding frequency so that the pooled frequency is more than 5. Finally adjust for the degrees of freedom lost in pooling. (iv) The number of classes n must be neither too small nor too large i.e. $4 \le n \le 16$.

3.2. Chi-square test of goodness of fit:

Tests of goodness of fit [12, 27] are used when we want to determine whether an actual sample distribution matches a known theoretical distribution. It enables us to find if the deviation of the experiment from theory is just by chance or it is really due to the inadequacy of the theory to fit the observed data. If O_i, (i=1, …, n) is a set of observed frequencies and E_i, (i=1, …, n) is the corresponding set of expected frequencies, then $\chi^2 = \sum_{i=1}^{n}((O_i - E_i)^2/E_i)$ follows chi-square distribution with (n-1) degrees of freedom. Suppose that a random sample $X_1, \ldots, X_n$ is drawn from a population with unknown cumulative distribution function F. We wish to test the null hypothesis $H_0: F(x) = F_0(x) \ \forall x$ that the population cdf is F_0 (which is completely specified), against $H_A: F(x) \neq F_0(x)$ for some x. To apply this test, the data must first be grouped into categories and then the observed frequencies for these categories are compared with the frequencies expected under the null hypothesis. In the case of a discrete distribution these categories appear in a natural way and are relevant to the distribution under study. When the distribution F_0 is continuous we have to arrange classes which are counterparts of above mentioned categories. Let 'k' be the level of significance. If the calculated $\chi^2 \leq \chi^2_k$ with (n-1) degrees of freedom, we will accept the null hypothesis H_0 then the difference between the observed and expected frequencies is not significant at k% level of significance. If $\chi^2 > \chi^2_k$, we reject H_0 and conclude that the difference is significant. It may happen that a sample used for making decision consists of observations that are not necessarily crisp but may be vague as well. In order to describe the vagueness of data we use the notion of a fuzzy number, introduced by Dubois and Prade [9].

3.3. Fuzzy random variables:

A notion of fuzzy random variables was introduced by Kwakernaak [17, 18]. Other definitions of fuzzy random variables are due to Kruse [15] or to Puri and Ralescu [22]. Suppose that a random experiment is described as usual by a probability space $(\Omega, \mathbb{A}, \mathbb{P})$, where Ω is the set of all possible outcomes of the experiment, $\mathbb{A}$ is the σ-algebra of subsets of Ω (the set of all possible events) and $\mathbb{P}$ is a probability measure. Then the mapping $X : \Omega \to F(\mathbb{R})$ is called a fuzzy random variable if $\{X(\alpha, \omega): \alpha \in (0, 1]\}$ is a set representation of $X(\omega)$ for all $\omega \in \Omega$ and for each $\alpha \in (0, 1]$ both $X^L_\alpha = X^L_\alpha(\omega) = \inf X_\alpha(\omega)$ and $X^U_\alpha = X^U_\alpha(\omega) = \sup X_\alpha(\omega)$ are usual real-valued random variables[16] on $(\Omega, \mathbb{A}, \mathbb{P})$.

A fuzzy random variable X is considered as a perception of an unknown usual random variable $V: \Omega \to \mathbb{R}$, called an original of X (if only vague data are available, it is of course impossible to show which of the possible originals is the true one). Similarly n-dimensional fuzzy random sample $X_1, \ldots, X_n$ may be treated as a fuzzy perception [16] of the usual random sample $V_1, \ldots, V_n$ (where $V_1, \ldots, V_n$ are independent and identically distributed crisp random variables). A random variable is completely characterized by its probability distribution P_θ. In statistical reasoning we assume that a probability distribution under study belongs to a family of distributions $\mathcal{P} = \{ P_\theta : \theta \in \Theta\}$ where Θ is the parameter space. Then very often we identify the distribution with its parameter θ and restrict statistical inference to that parameter. However, if we deal with a fuzzy random variable, we cannot observe the parameter θ directly but only its vague image. Using this reasoning together with Zadeh's extension principle Kruse and Meyer [16] introduced the notion of ***fuzzy parameter of fuzzy random variable*** $\tilde{\theta}$ which may be considered as a ***fuzzy perception*** of the unknown parameter θ. It is defined as a fuzzy subset of the parameter space Θ with membership function $\mu_A : \Theta \to [0, 1]$. Of course, if our data are crisp i.e. X = V, we get $\tilde{\theta} = \theta$.

3.4. Chi-square test for vague data:

Suppose $\mu_{X_1}, \ldots, \mu_{X_n}$ denote membership functions of fuzzy numbers which are observations of a fuzzy random sample $X_1, \ldots, X_n$. Suppose our sample comes from the unknown distribution F, and our aim is to test the null hypothesis $H_0 : F = F_{\tilde{\theta}}$ against $H_A : F \neq F_{\tilde{\theta}}$ where the distribution $F_{\tilde{\theta}}$ is completely specified by a fuzzy parameter θ described by its membership function μ_A. And the test statistic for testing H_0 against H_A with fuzzy data is given by [10, 11] $\chi^2 = \sum_{i=1}^{n}((O_i - E_i)^2/E_i) --- (3.5)$ for large samples and which follows approximately chi-square distribution with (n-1) degrees of freedom. Therefore, we reject H_0 in favor of H_A if $\chi^2 \geq \chi^2_{1-\alpha, n-1}$ where $\chi^2_{1-\alpha, n-1}$ is the ***quantile*** of order $1-\alpha$ from the chi-square distribution with k-1 degrees of freedom.

4. Chi-square goodness of fit using alpha cut interval method:

The fuzzy test of hypotheses of chi-square model in which the sample data are trapezoidal fuzzy numbers is given here. The test statistic for fuzzy observations given by (3.5) is formulated according to the α-cut interval of tfns. (def. 2.4; section 2). Using the relation (def. 2.5; section 2), we transform the fuzzy chi-square model to interval chi-square model. Having the upper limit of the alpha cut interval, we construct upper level crisp chi-square model and using the lower limit of the alpha cut interval, we construct the lower level crisp chi-square model. Thus, in this approach, the test statistic (3.5) is split into two parts up namely lower level and upper level α-cut intervals viz. $A_L(\alpha)=[a_i + \alpha(b_i - a_i)]---(4.1)$ and $A_U(\alpha)=[d_i - \alpha(d_i - c_i)]---(4.2)$; $i=1,..., n; \alpha \in [0, 1]$. Accordingly, the test statistics will be

$$\chi^2_L = \sum_{i=1}^{n} \frac{\left(O_i^L - E_i^L\right)^2}{E_i^L} --- (4.3) \quad \text{and} \quad \chi^2_U = \sum_{i=1}^{n} \frac{\left(O_i^U - E_i^U\right)^2}{E_i^U} --- (4.4)$$

where $O_i^L = [a_i + \alpha(b_i - a_i)]$ and $O_i^U = [d_i - \alpha(d_i - c_i)]$; $i=1, ..., n; \alpha \in [0, 1]$.

Decision rules:

The decision rules for the fuzzy hypotheses are given below:

$$H_0 : F = F_{\tilde{\theta}} \text{ against } H_0 : F \neq F_{\tilde{\theta}} \Rightarrow [H_0] : [F] = [F_{\tilde{\theta}}] \text{ against } [H_0] : [F] \neq [F_{\tilde{\theta}}].$$

$$\Rightarrow [H_0^L, H_0^U] : [F^L, F^U] = [F_{\tilde{\theta}}^L, F_{\tilde{\theta}}^U] \text{ against } [H_0^L, H_0^U] : [F^L, F^U] \neq [F_{\tilde{\theta}}^L, F_{\tilde{\theta}}^U].$$

$\Rightarrow$ The null hypothesis for lower level model: $[H_0^L] : [F^L] = [F_{\tilde{\theta}}^L]$ against $[H_0^L] : [F^L] \neq [F_{\tilde{\theta}}^L]$.

$\Rightarrow$ The null hypothesis for upper level model: $[H_0^U] : [F^U] = [F_{\tilde{\theta}}^U]$ against $[H_0^U] : [F^U] \neq [F_{\tilde{\theta}}^U]$.

Example 1. The following table shows defective articles produced by four machines A_i, i=1, 2, 3, 4. Due to some work congestion, the observed data are unavoidably trapezoidal fuzzy numbers.

Machine	A_1	A_2	A_3	A_4
Production time (in hours)	1	1	2	3
Number of defectives	(10, 12, 13, 15)	(26, 29, 30, 32)	(58, 60, 63, 64)	(94, 96, 98, 101)

We now test whether the tfns. indicate a significant difference in the performance of the machines.

Example 2. The demand for a particular spare part in a factory was found to vary from day-to-day. The observed demand of the spare parts are in terms of tfns. due to some unexpected situations in the non-stop work flow. The obtained sample study is tabulated below:

Days	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.
No. of parts demanded	(1119, 1122, 1124, 1126)	(1120, 1122, 1125, 1128)	(1107, 1108, 1110, 1114)	(1116, 1120, 1122, 1123)	(1121, 1124, 1126, 1127)	(1114, 1115, 1117, 1120)

We test the hypotheses whether the number of parts demanded depends on the day of the week.

Example 4.1. Let us consider example 1, using the relation (2.5) in section 2, the alpha cut interval model for the 4 machines are given by,

Machine	A_1	A_2	A_3	A_4
Production time (in hours)	1	1	2	3
Number of defectives	[10+2α, 15-2α]	[26+3α, 32-2α]	[58+2α, 64-α]	[94+2α, 101-3α]

H_0 : Production rates of the 4 machines are same.

The lower level model (l.l.m.)

Machine	A_1	A_2	A_3	A_4
Production time (in hours)	1	1	2	3
Number of defectives	[10+2α]	[26+3α]	[58+2α]	[94+2α]

H_0^L : Production rates of the 4 machines are same.

The total number of defectives at l.l.m. = [188+9α]

The expected number of defectives and observed number of defectives produced by the four machines are given below respectively,

E_i^L	(1/7)[188+9α]	(1/7)[188+9α]	(2/7)[188+9α]	(3/7)[188+9α]
O_i^L	[10+2α]	[26+3α]	[58+2α]	[94+2α]

Now, the test statistic for l.l.m. is $\chi_L^2 = \sum_{i=1}^{n} \frac{\left(O_i^L - E_i^L\right)^2}{E_i^L} = \frac{1400\alpha^2 - 13552\alpha + 104132}{378\alpha + 7896}$.

And since $\sum E_i^L = \sum O_i^L$, v =4-1=3, from the chi-square table, $\chi^2_{T(5\%)}(v=3) = 7.815$

Here, $\chi_L^2 > \chi^2_{T(5\%)}$ $\forall\alpha, 0 \le \alpha \le 1$. $\Rightarrow$ The null hypothesis H_0^L is rejected at 5% level of significance $\forall\alpha$.

$\Rightarrow$ **There is a significant difference in the performance of machines at l.l.m.**

The upper level model (u.l.m.)

Machine	A_1	A_2	A_3	A_4
Production time (in hours)	1	1	2	3
Number of defectives	[15-2α]	[32-2α]	[64-α]	[101-3α]

H_0^U : Production rates of the 4 machines are same.

The total number of defectives at u.l.m. = [212-8α]

The expected number of defectives and observed number of defectives produced by the four machines are given below respectively,

E_i^U	(1/7)[212-8α]	(1/7)[212-8α]	(2/7)[212-8α]	(3/7)[212-8α]
O_i^U	[15-2α]	[32-2α]	[64-α]	[101-3α]

Now, the test statistic for u.l.m. is $\chi_U^2 = \sum_{i=1}^{n} \frac{\left(O_i^U - E_i^U\right)^2}{E_i^U} = \frac{693\alpha^2+8988\alpha+81368}{8904-336\alpha}$

And since $\sum E_i^U = \sum O_i^U$, v =4-1=3, from the chi-square table, $\chi^2_{T(5\%)}(v=3)=7.815$

Here, $\chi_U^2 > \chi^2_{T(5\%)}\ \forall\alpha,\ 0\le\alpha\le1$. $\Rightarrow$ The null hypothesis H_0^U is rejected at 5% level of significance $\forall\alpha$.

$\Rightarrow$ **There is a significant difference in the performance of machines at u.l.m.**

Hence, observing the decisions from both l.l.m. and u.l.m., the null hypothesis H_0 is rejected at 5% level of significance and we conclude that there is a significant difference in the performance of 4 machines.

Example 4.2. Let us consider example 2, using the relation (2.5) in section 2, the alpha cut interval of the given tfns. are tabulated below:

Days	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.
Demand	[1119+3α, 1126-2α]	[1120+2α, 1128-3α]	[1107+α, 1114-4α]	[1116+4α, 1123-α]	[1121+3α, 1127-α]	[1114+α, 1120-3α]

H_0 : The number of spare parts demanded are same over the 6-day.

The lower level model (l.l.m.)

Days	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.
Demand	[1119+3α]	[1120+2α]	[1107+α]	[1116+4α]	[1121+3α]	[1114+α]

H_0^L : The number of spare parts demanded are same over the 6-day period.

The total number of defectives at l.l.m. = [6697+14α]

Under the null hypothesis, the expected frequencies of the spare parts demanded on each of the six days would be [6697+14α]/6

E_i^L	[6697+14α]/6	[6697+14α]/6	[6697+14α]/6	[6697+14α]/6	[6697+14α]/6	[6697+14α]/6
O_i^L	[1119+3α]	[1120+2α]	[1107+α]	[1116+4α]	[1121+3α]	[1114+α]

Now, the test statistic for l.l.m. is $\chi_L^2 = \sum_{i=1}^{n} \frac{\left(O_i^L - E_i^L\right)^2}{E_i^L} = \frac{264\alpha^2+1344\alpha+4854}{84\alpha+40182}$.

And since $\sum E_i^L = \sum O_i^L$, v =6-1=5, from the chi-square table, $\chi^2_{T(5\%)}(v=5)=11.07$

Here, $\chi_L^2 < \chi^2_{T(5\%)}\ \forall\alpha,\ 0\le\alpha\le1$. $\Rightarrow$ The null hypothesis H_0^L is accepted at 5% level of significance $\forall\alpha$.

$\Rightarrow$ **The number of spare parts demanded are same over the 6-day period at l.l.m.**

The upper level model (u.l.m.)

Days	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.
Demand	[1126-2α]	[1128-3α]	[1114-4α]	[1123-α]	[1127-α]	[1120-3α]

H_0^U : The number of spare parts demanded are same over the 6-day period.

The total number of defectives at l.l.m. = [6738-14α]

Under the null hypothesis, the expected frequencies of the spare parts demanded on each of the six days would be [6738–14α]/6

E_i^L	[6738–14α]/6	[6738–14α]/6	[6738–14α]/6	[6738–14α]/6	[6738–14α]/6	[6738–14α]/6
O_i^L	[1126-2α]	[1128-3α]	[1114-4α]	[1123-α]	[1127-α]	[1120-3α]

Now, the test statistic for l.l.m. is $\chi_U^2 = \sum_{i=1}^{n} \frac{\left(O_i^U - E_i^U\right)^2}{E_i^U} = \frac{100\alpha^2+720\alpha+2520}{20214-42\alpha}$.

And since $\sum E_i^U = \sum O_i^U$, v=6-1=5, from the chi-square table, $\chi^2_{T(5\%)}(v=5)=11.07$

Here, $\chi_U^2 < \chi^2_{T(5\%)}$ $\forall\alpha,\ 0 \le \alpha \le 1$. $\Rightarrow$ The null hypothesis H_0^U is accepted at 5% level of significance $\forall\alpha$.

$\Rightarrow$ **The number of spare parts demanded are same over the 6-day period at u.l.m.**

Hence, observing the decisions obtained from both l.l.m. and u.l.m., the null hypothesis H_0 is accepted at 5% level of significance and we conclude that the number of spare parts demanded are same over the 6-day period.

5. Wang's centroid point and ranking method

Wang et al. [30] found that the centroid formulae proposed by Cheng are incorrect and have led to some misapplications such as by Chu and Tsao. They presented the correct method for centroid formulae for a generalized fuzzy number $A=(a, b, c, d; w)$ as

$$\left(\bar{x}_0, \bar{y}_0\right) = \left[\frac{1}{3}\left((a+b+c+d) - \left(\frac{dc - ab}{(d+c)-(a+b)}\right)\right), \left(\frac{w}{3}\right)\left(1+\left(\frac{c-b}{(d+c)-(a+b)}\right)\right)\right] \text{--- (5.1)}$$

And the ranking function associated with A is $R(A) = \sqrt{\bar{x}_0^2 + \bar{y}_0^2}$ --- (5.2)

For a normalized tfn., we put w = 1 in equations (5.1) so we have,

$$\left(\bar{x}_0, \bar{y}_0\right) = \left[\frac{1}{3}\left((a+b+c+d) - \left(\frac{dc - ab}{(d+c)-(a+b)}\right)\right), \left(\frac{1}{3}\right)\left(1+\left(\frac{c-b}{(d+c)-(a+b)}\right)\right)\right] \text{--- (5.3)}$$

And the ranking function associated with A is $R(A) = \sqrt{\bar{x}_0^2 + \bar{y}_0^2}$ --- (5.4)

Let A_i and A_j be two fuzzy numbers (i) $R(A_i) > R(A_j)$ then $A_i > A_j$ (ii) $R(A_i) < R(A_j)$ then $A_i < A_j$ (iii) $R(A_i) = R(A_j)$ then $A_i = A_j$.

Example 5.1. Let us consider example 1, the ranking grades of tfns. are calculated using relations (5.3) and (5.4) which are given below:

Machine	A_1	A_2	A_3	A_4
Production time (in hours)	1	1	2	3
Number of defectives	12.50≈13	29.19≈29	61.22≈61	97.30≈97

And total no. of defectives = 200, E_i : 200/7; 200/7; 2(200)/7; 3(200)/7 and converting E_i to the whole numbers subject to the condition that $\sum E_i$ =200, we get,

E_i	29	29	57	85
O_i	13	29	61	97

Now, $\chi^2 = \sum_{i=1}^{n}((O_i\text{-}E_i)^2/E_i) = 10.80$; since $\sum E_i = \sum O_i$, $v = 4-1 = 3$, from the chi-square table it is seen that $\chi^2_{T(5\%)}(v=3) = 7.815$. Here $\chi^2 > \chi^2_{T(5\%)}$. $\Rightarrow$The null hypothesis H_0 is rejected at 5% level of significance. $\Rightarrow$**The difference between the performances of 4 machines is significant.**

Example 5.2. Let us consider example 2, the ranking grades of tfns. are calculated using relations (5.3) and (5.4) which are given below:

Days	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.
Demand	1122.704	1123.788	1109.889	1120.111	1124.417	1116.583

And total no. of demand = 6717.492, E_i = (total no. of defectives/6) = (6717.492/6) = 1119.582

E_i	1119.582	1119.582	1119.582	1119.582	1119.582	1119.582
O_i	1122.704	1123.788	1109.889	1120.111	1124.417	1116.583

Now, $\chi^2 = \sum_{i=1}^{n}((O_i\text{-}E_i)^2/E_i) = 0.1376$; since $\sum E_i = \sum O_i$, $v = 6-1 = 5$, from the chi-square table it is seen that $\chi^2_{T(5\%)}(v=5) = 11.07$. Here $\chi^2 < \chi^2_{T(5\%)}$. $\Rightarrow$The null hypothesis H_0 is accepted at 5% level of significance. $\Rightarrow$ **The number of spare parts demanded are same over the 6-day period.**

6. Rezvani's ranking function of TFNs

The centroid of a trapezoid is considered as the balancing point of the trapezoid. Divide the trapezoid into three plane figures. These three plane figures are a triangle (APB), a rectangle (BPQC) and a triangle (CQD) respectively. Let the centroids of the three plane figures be G_1, G_2 and G_3 respectively. The incenter of these centroids G_1, G_2 and G_3 is taken as the point of reference to define the ranking of generalized trapezoidal fuzzy numbers. The reason for selecting this point as a point of reference is that each centroid point are **balancing points** of each individual plane figure and the incenter of these centroid points is much more balancing point for a generalized trapezoidal fuzzy number. Therefore, this point would be a better reference point than the centroid point of the trapezoid.

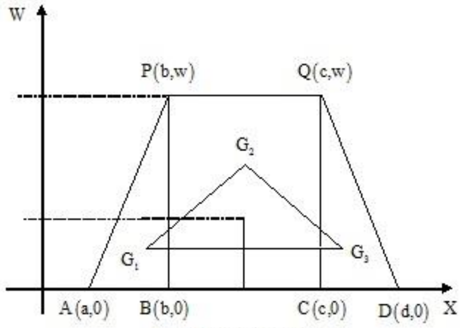

Fig.1 Centroid of centroids

Consider a generalized trapezoidal fuzzy number $A=(a, b, c, d; w)$. The centroids of the three plane figures are:

$$G_1 = \left(\frac{a+2b}{3}, \frac{w}{3}\right),\ G_2 = \left(\frac{b+c}{2}, \frac{w}{2}\right) \text{ and } G_3 = \left(\frac{2c+d}{3}, \frac{w}{3}\right) \text{--- (6.1)}$$

Equation of the line G_1G_3 is $y = \frac{w}{3}$ and G_2 does not lie on the line G_1G_3. Therefore, G_1, G_2 and G_3 are non-collinear and they form a triangle. We define the incenter $I(\bar{x}_0, \bar{y}_0)$ of the triangle with vertices G_1, G_2 and G_3 of the generalized fuzzy number $A=(a, b, c, d; w)$ as [25],

$$I_A(\bar{x}_0, \bar{y}_0) = \left[\frac{\alpha\left(\frac{a+2b}{3}\right)+\beta\left(\frac{b+c}{2}\right)+\gamma\left(\frac{2c+d}{3}\right)}{\alpha+\beta+\gamma}, \frac{\alpha\left(\frac{w}{3}\right)+\beta\left(\frac{w}{2}\right)+\gamma\left(\frac{w}{3}\right)}{\alpha+\beta+\gamma}\right] \text{--- (6.2)}$$

$$\text{where } \alpha = \frac{\sqrt{(c - 3b + 2d)^2 + w^2}}{6}, \beta = \frac{\sqrt{(2c + d - a - 2b)^2}}{3}, \gamma = \frac{\sqrt{(3c - 2a - b)^2 + w^2}}{6} \text{--- (6.3)}$$

And ranking function of the trapezoidal fuzzy number $A=(a, b, c, d; w)$ which maps the set of all fuzzy numbers to a set of all real numbers $[\text{i.e. } R:[A] \rightarrow \mathbb{R}]$ is defined as $R(A) = \sqrt{\bar{x}_0^2 + \bar{y}_0^2}$ --- (6.4) which is the Euclidean distance from the incenter of the centroids. For a normalized tfn., we put w = 1 in equations (6.1), (6.2) and (6.3) so we have,

$$G_1 = \left(\frac{a+2b}{3}, \frac{1}{3}\right),\ G_2 = \left(\frac{b+c}{2}, \frac{1}{2}\right) \text{ and } G_3 = \left(\frac{2c+d}{3}, \frac{1}{3}\right) \text{--- (6.5)}$$

$$I_A(\bar{x}_0, \bar{y}_0) = \left[\frac{\alpha\left(\frac{a+2b}{3}\right)+\beta\left(\frac{b+c}{2}\right)+\gamma\left(\frac{2c+d}{3}\right)}{\alpha+\beta+\gamma}, \frac{\alpha\left(\frac{1}{3}\right)+\beta\left(\frac{1}{2}\right)+\gamma\left(\frac{1}{3}\right)}{\alpha+\beta+\gamma}\right] \text{--- (6.6)}$$

$$\text{where } \alpha = \frac{\sqrt{(c - 3b + 2d)^2 + 1}}{6}, \beta = \frac{\sqrt{(2c + d - a - 2b)^2}}{3} \text{ and } \gamma = \frac{\sqrt{(3c - 2a - b)^2 + 1}}{6} \text{--- (6.7)}$$

And ranking function of the trapezoidal fuzzy number $A=(a, b, c, d; 1)$ is defined as $R(A) = \sqrt{\bar{x}_0^2 + \bar{y}_0^2}$ --- (6.8).

7. Chi-square test using Rezvani's ranking function

We now analyse the chi-square test by assigning rank for each normalized trapezoidal fuzzy numbers and based on the ranking grades the decisions are observed.

Example 7.1. Let us consider example 1, the ranking grades of tfns. are calculated using the relations (6.6), (6.7) and (6.8) which are given below:

Machine	A_1	A_2	A_3	A_4
Production time (in hours)	1	1	2	3
Number of defectives	12.51≈13	29.50≈30	61.50≈62	97≈97

And total no. of defectives = 202, E_i : 202/7; 202/7; 2(202)/7; 3(202)/7 , the expected and observed number of defectives are tabulated below:

E_i	28.86	28.86	57.71	86.57
O_i	13	30	62	97

Now, $\chi^2 = \sum_{i=1}^{n}((O_i - E_i)^2/E_i) = 10.34$; since $\sum E_i = \sum O_i$, $v = 4-1 = 3$, from the chi-square table it is seen that $\chi^2_{T(5\%)}(v=3) = 7.815$. Here $\chi^2 > \chi^2_{T(5\%)}$. $\Rightarrow$ The null hypothesis H_0 is rejected at 5% level of significance. $\Rightarrow$ **The difference between the performances of 4 machines is significant.**

Example 7.2. Let us consider example 2, the ranking grades of tfns. are calculated using relations (6.6), (6.7) and (6.8) which are given below:

Days	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.
Demand	1123	1123.5	1109	1121	1125	1116

And total no. of demand = 6717.5, E_i = (total no. of defectives/6) = (6717.5/6) = 1119.5833

E_i	1119.5833	1119.5833	1119.5833	1119.5833	1119.5833	1119.5833
O_i	1123	1123.5	1109	1121	1125	1116

Now, $\chi^2 = \sum_{i=1}^{n}((O_i - E_i)^2/E_i) = 0.1636$; since $\sum E_i = \sum O_i$, $v = 6-1 = 5$, from the chi-square table it is seen that $\chi^2_{T(5\%)}(v=5) = 11.07$. Here $\chi^2 < \chi^2_{T(5\%)}$. $\Rightarrow$ The null hypothesis H_0 is accepted at 5% level of significance. $\Rightarrow$ **The number of spare parts demanded are same over the 6-day period.**

8. Thorani's centroid point and ranking method

As per the description in Salim Rezvani's ranking method, Y. L. P. Thorani et al. [26] presented a different kind of centroid point and ranking function of tfns. The incenter $I_A(\bar{x}_0, \bar{y}_0)$ of the triangle [Fig. 1] with vertices G_1, G_2 and G_3 of the generalized tfn. $A=(a, b, c, d; w)$ is given by,

$$I_A(\bar{x}_0, \bar{y}_0) = \left[\frac{\alpha\left(\frac{a+2b}{3}\right) + \beta\left(\frac{b+c}{2}\right) + \gamma\left(\frac{2c+d}{3}\right)}{\alpha+\beta+\gamma}, \frac{\alpha\left(\frac{w}{3}\right) + \beta\left(\frac{w}{2}\right) + \gamma\left(\frac{w}{3}\right)}{\alpha+\beta+\gamma} \right] \text{--- (8.1)}$$

$$\text{where } \alpha = \frac{\sqrt{(c-3b+2d)^2 + w^2}}{6}, \beta = \frac{\sqrt{(2c+d-a-2b)^2}}{3}, \gamma = \frac{\sqrt{(3c-2a-b)^2 + w^2}}{6} \text{--- (8.2)}$$

And the ranking function of the generalized tfn. $A=(a, b, c, d; w)$ which maps the set of all fuzzy numbers to a set of real numbers is defined as $R(A)=x_0 \times y_0$ --- (8.3). For a normalized tfn., we put w = 1 in equations (8.1) and (8.2) so we have,

$$I_A(\bar{x}_0, \bar{y}_0)=\left[\frac{\alpha\left(\frac{a+2b}{3}\right)+\beta\left(\frac{b+c}{2}\right)+\gamma\left(\frac{2c+d}{3}\right)}{\alpha+\beta+\gamma}, \frac{\alpha\left(\frac{1}{3}\right)+\beta\left(\frac{1}{2}\right)+\gamma\left(\frac{1}{3}\right)}{\alpha+\beta+\gamma}\right] \text{--- (8.4)}$$

$$\text{where } \alpha=\frac{\sqrt{(c-3b+2d)^2+1}}{6}, \beta=\frac{\sqrt{(2c+d-a-2b)^2}}{3} \text{ and } \gamma=\frac{\sqrt{(3c-2a-b)^2+1}}{6} \text{--- (8.5)}$$

And for $A=(a, b, c, d; 1)$, $R(A)=x_0 \times y_0$ --- (8.6)

9. Chi-square test using Thorani's ranking function

We now analyse the chi-square test by assigning rank for each normalized trapezoidal fuzzy numbers and based on the ranking grades the decisions are observed.

Example 9.1. Let us consider example 1, the ranking grades of tfns. are calculated using the relations (8.4), (8.5) and (8.6) which are given below:

Machine	A_1	A_2	A_3	A_4
Production time (in hours)	1	1	2	3
Number of defectives	5.20≈5	12.28≈12	25.62≈26	40.40≈40

And total no. of defectives = 83, E_i : 83/7; 83/7; 2(83)/7; 3(83)/7 , the expected and observed number of defectives are tabulated below:

E_i	11.857	11.857	23.71	35.57
O_i	5	12	26	40

Now, $\chi^2=\sum_{i=1}^{n}((O_i-E_i)^2/E_i)=4.74$; since $\sum E_i=\sum O_i$, $v=4-1=3$, from the chi-square table it is seen that $\chi^2_{T(5\%)}(v=3)=7.815$. Here $\chi^2<\chi^2_{T(5\%)}$. $\Rightarrow$ The null hypothesis H_0 is accepted at 5% level of significance. $\Rightarrow$ **The difference between the performances of 4 machines is not significant.**

Example 9.2. Let us consider example 2, the ranking grades of tfns. are calculated using relations (8.4), (8.5) and (8.6) which are given below:

Days	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.
Demand	467.722	468.005	461.879	466.875	468.507	464.760

And total no. of demand = 2797.748, E_i = (total no. of defectives/6) = (2797.748/6) = 466.2913

E_i	466.2913	466.2913	466.2913	466.2913	466.2913	466.2913
O_i	467.722	468.005	461.879	466.875	468.507	464.760

Now, $\chi^2=\sum_{i=1}^{n}((O_i-E_i)^2/E_i)=0.0687$; since $\sum E_i=\sum O_i$, $v=6-1=5$, from the chi-square table it is seen that $\chi^2_{T(5\%)}(v=5)=11.07$. Here $\chi^2<\chi^2_{T(5\%)}$. $\Rightarrow$ The null hypothesis H_0 is accepted at 5% level of significance. $\Rightarrow$ **The number of spare parts demanded are same over the 6-day period.**

10. Graded mean integration representation (GMIR)

Let $A=(a, b, c, d; w)$ be a generalized trapezoidal fuzzy number, then the **GMIR** [24] of A is defined by

$$P(A)=\int_0^w h\left[\frac{L^{-1}(h)+R^{-1}(h)}{2}\right]dh \Big/ \int_0^w hdh.$$

Theorem 10.1. Let $A=(a, b, c, d; 1)$ be a tfn. with normal shape function, where a, b, c, d are real numbers such that $a<b\le c<d$. Then the graded mean integration representation (GMIR) of A is $P(A)=\frac{(a+d)}{2}+\frac{n}{2n+1}(b-a-d+c)$.

Proof : For a trapezoidal fuzzy number $A=(a, b, c, d; 1)_n$, we have $L(x)=\left(\frac{x-a}{b-a}\right)^n$ and

$$R(x)=\left(\frac{d-x}{d-c}\right)^n \text{ Then, } h=\left(\frac{x-a}{b-a}\right)^n \Rightarrow L^{-1}(h)=a+(b-a)h^{1/n};$$

$$h=\left(\frac{d-x}{d-c}\right)^n \Rightarrow R^{-1}(h)=d-(d-c)h^{1/n}$$

$$\therefore P(A)=\left(\frac{1}{2}\int_0^1 h\left[\left(a+(b-a)h^{1/n}\right)+\left(d-(d-c)h^{1/n}\right)\right]dh\right)\Big/\int_0^1 hdh$$

$$=\left(\frac{1}{2}\left[\frac{(a+d)}{2}+\frac{n}{2n+1}(b-a-d+c)\right]\right)\Big/\left(1/2\right)$$

$$\text{Thus, } P(A)=\frac{(a+d)}{2}+\frac{n}{2n+1}(b-a-d+c) \text{ Hence the proof.}$$

Result 10.1. If n =1 in the above theorem, we have $P(A)=\frac{a+2b+2c+d}{6}$

11. Chi-square test using GMIR of tfns.

We now analyse the chi-square test by using GMIR of each normalized trapezoidal fuzzy numbers and based on the GMIR of tfns. the decisions are observed.

Example 11.1. Let us consider example 1, the GMIRs of tfns. are calculated using the result (10.1) of theorem 10.1 which are given below:

Machine	A_1	A_2	A_3	A_4
Production time (in hours)	1	1	2	3
Number of defectives	12.5≈13	29.33≈29	61.33≈61	97.17≈97

And total no. of defectives = 200, E_i : 200/7; 200/7; 2(200)/7; 3(200)/7 and converting E_i to the whole numbers subject to the condition that $\sum E_i$ =200, we get,

E_i	29	29	57	85
O_i	13	29	61	97

Now, $\chi^2=\sum_{i=1}^{n}((O_i-E_i)^2/E_i)=10.80$; since $\sum E_i=\sum O_i$, $\nu=4-1=3$, from the chi-square table it is seen that $\chi^2_{T(5\%)}(\nu=3)=7.815$. Here $\chi^2>\chi^2_{T(5\%)}$. $\Rightarrow$ The null hypothesis H_0 is rejected at 5% level of significance. $\Rightarrow$ **The difference between the performances of 4 machines is significant.**

Example 11.2. Let us consider example 2, the GMIRs of tfns. are calculated using the result (10.1) of theorem 10.1 which are given below:

Days	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.
Demand	1122.83	1123.67	1109.50	1120.50	1124.67	1116.33

And total no. of demand = 6717.5, E_i = (total no. of defectives/6) = (6717.5/6) = 1119.5833

E_i	1119.5833	1119.5833	1119.5833	1119.5833	1119.5833	1119.5833
O_i	1122.83	1123.67	1109.50	1120.50	1124.67	1116.33

Now, $\chi^2 = \sum_{i=1}^{n}((O_i\text{-}E_i)^2/E_i) = 0.1485$; since $\sum E_i = \sum O_i$, $v = 6-1 = 5$, from the chi-square table it is seen that $\chi^2_{T(5\%)}(v=5) = 11.07$. Here $\chi^2 < \chi^2_{T(5\%)}$. $\Rightarrow$ The null hypothesis H_0 is accepted at 5% level of significance. $\Rightarrow$ **The number of spare parts demanded are same over the 6-day period.**

12. Conclusion

The decisions obtained from various methods are tabulated below for the null hypothesis.

Acceptance of null hypotheses H_0											
α cut method				**Wang**		**Rezvani**		**Thorani**		**GMIR**	
Eg.1		Eg.2		Eg.1	Eg.2	Eg.1	Eg.2	Eg.1	Eg.2	Eg.1	Eg.2
L	U	L	U								
×	×	✓	✓	×	✓	×	✓	✓	✓	×	✓

Here, the proposed α-cut interval method provides a parallel decision for the acceptance/rejection of null hypothesis in lower level (L) and upper level (U) models for both example 1 and example 2. Wang's ranking method, Rezvani's ranking method and GMIR of tfns. exhibit the same decisions for example 1 and example 2. Thorani's ranking method of tfns. does not provide reliable result as it accepts the null hypothesis in all the cases.

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