

Kumaraswamy Exponentiated Inverse Rayleigh Distribution

Muhammad Ahsan ul Haq

College of Statistical & Actuarial Sciences, University of the Punjab, Lahore, Pakistan

Abstract:

A four parameter Kumaraswamy Exponentiated Inverse Rayleigh distribution (KEIR) introduced in this study. Studied some properties of this model, including survival and hazard function, mean, moment generating function, incomplete moments, quintiles and renyi entropy. Fit the proposed distribution to a real data set using the maximum likelihood estimation method to illustrate its potential and flexibility.

Keywords: *Kumaraswamy; Inverse Rayleigh; Moments; Entropy; Maximum Likelihood, Order statistics*

1. Introduction:

An appropriate comprehensive lifetime model is often of concentration in the analysis of data. Trayer (1964) introduced a distribution in order to model reliability and survival data sets, named Inverse Rayleigh distribution. After that Inverse Rayleigh distribution was championed by Voda [1]. He discussed its properties and ML estimation of the scale parameter. Further, Gharraph [2] provided closed-form expressions for the mean, harmonic mean, geometric mean, mode and the median of this distribution. Moreover, [2, 3] estimated the parameters using different classical and Bayesian estimation methods.

A particular prominent class of generalization is Kumaraswamy generalized distributions. In this significant work, authors introduced a number of distributions. To cite a few model, we identify the Kumaraswamy Gumbel by [4], Kumaraswamy Inverse Weibull [5], Kumaraswamy Weibull distribution [6], Kumaraswamy GP distribution by [7], Kumaraswamy Pareto distribution by [8], Kumaraswamy Birnbaum–Saunders Distribution [9], Haq, Usman [10] studied Kumaraswamy Power Function distribution, Haq, Usman [11] also studied Transmuted Power Function distribution and many other distributions were existed in literature which were developed using this generalized class.

The extensions of Inverse Rayleigh distribution is exist in literature and these models are applied in many areas, including reliability, life tests and survival analysis. Exponentiated Inverse Rayleigh distribution developed by [12], similarly Transmuted Inverse Rayleigh by [13] and Beta Inverse Rayleigh [14].

In this paper, we are studying the Kumaraswamy Exponentiated Inverse Rayleigh distribution. We have discussed its mathematical properties, and applied it to a real data for illustration of the flexibility of the model. The rest of the paper unfolds as follows. Section two is based on the derivation of probability density function (pdf), cumulative distribution function (cdf), hazard, survival function and plots for some selected values of parameters. Section three is based on moments, quintile function, random number generator, renyi entropy and order statistics densities of KEIR. Application of the model is given in last section.

The KEIR was developed using the cumulative distribution and probability density function of Kumaraswamy distribution. The following functions of cdf and pdf are respectively given below,

$$F(x) = 1 - (1 - G(x)^{a})^{b} \tag{1}$$

Where a>0 and b > 0 are additional shape parameters and manage skewness. Correspondingly, the density function of distribution is

$$f(x) = abg(x)G(x)^{a-1}[1 - G(x)^{a}]^{b-1} \tag{2}$$

Where $g(x) = {dG(x)}/{dx}$.

The random variable X is said to be Exponentiated Inverse Rayleigh (EIR) distribution with parameters α, θ if its probability density function is by,

$$f(x) = \frac{2\alpha\theta}{x^{3}}\left(e^{-\frac{\theta}{x^{2}}}\right)^{\alpha}$$

and the corresponding cdf is,

$$F(x) = \left(e^{-\frac{\theta}{x^{2}}}\right)^{\alpha}$$

2. The KEIR distribution

By placing the cdf and pdf of EIR distribution in the expressions given in equation 1 & 2. The desire probability density and cumulative distribution function of KEIR distribution are given below correspondingly;

$$f(x) = \frac{2ab\alpha\theta}{x^{3}}\left(e^{-\frac{\theta}{x^{2}}}\right)^{a\alpha}\left[1 - \left(e^{-\frac{\theta}{x^{2}}}\right)^{a\alpha}\right]^{(b-1)} \quad x \geq 0; a, b, \alpha, \theta > 0 \tag{3}$$

The cumulative distribution function of KEIR distribution is

$$F(x) = 1 - \left[1 - \left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\right]^b \quad (4)$$

Expansion for the probability density function and cumulative distribution function in terms of KEIR density function. However, expanding the binomial expansion on equation (3) the probability density function can be written as,

$$f(x) = \frac{2ab\alpha\theta}{x^3}\sum_{i=0}^{\infty}(-1)^i\binom{b-1}{i}\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha(i+1)} \quad x \geq 0; a, b, \alpha, \theta > 0$$

The term $\binom{b-1}{i}$ is written as $\frac{\Gamma(b)}{i!\,\Gamma(b-i)}$ so final expression of probability density function is as follows

$$f(x) = \frac{2ab\alpha\theta}{x^3}\sum_{i=0}^{\infty}\frac{(-1)^i\Gamma(b)}{i!\,\Gamma(b-i)}\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha(i+1)} \quad (5)$$

2.1. Survival and Hazard function

Survival function is the probability that an item does not fail before some time t, is defined as $S(x) = 1 - F(x)$. Survival function of KEIR distribution is denoted by $S_x(a, b, \alpha, \theta)$ can be characterization of life time data analysis. The function is given below,

$$S_x(a, b, \alpha, \theta) = \left[1 - \left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\right]^b$$

Using the series expansion the survival function can be expressed as

$$S_x(a, b, \alpha, \theta) = \sum_{i=0}^{\infty}(-1)^i\binom{b}{i}\left(e^{-\frac{\theta a\alpha}{x^2}}\right)^i$$

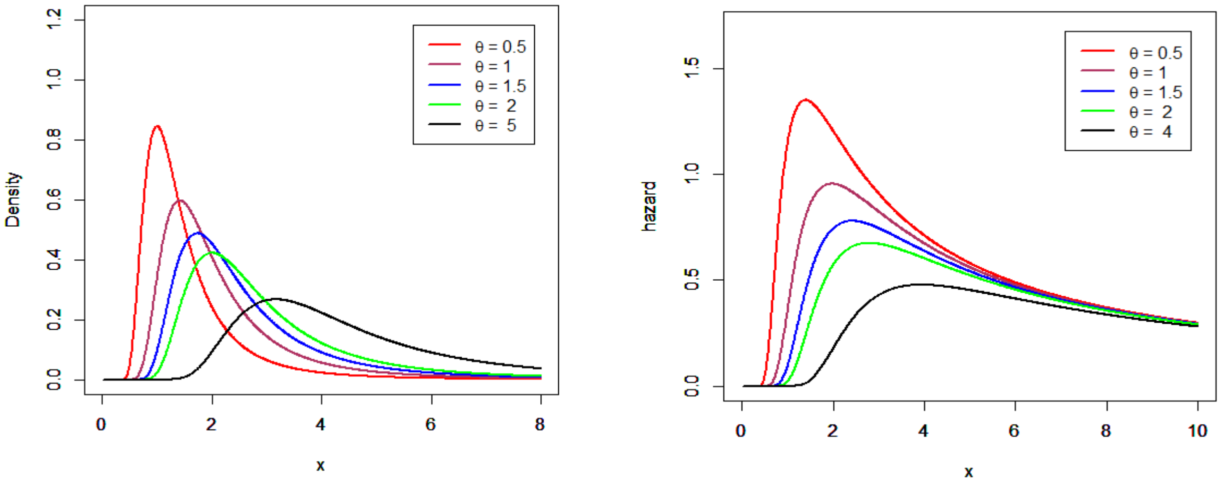
Hazard rate function is the other characteristics of interest of random variable. The hazard rate function is also known failure rate and is obtained using the relation.

$$H(x) = \frac{f(x)}{S(x)}$$

$$h(x) = \frac{\frac{2ab\alpha\theta}{x^3}\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\left[1 - \left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\right]^{(b-1)}}{\left[1 - \left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\right]^b}$$

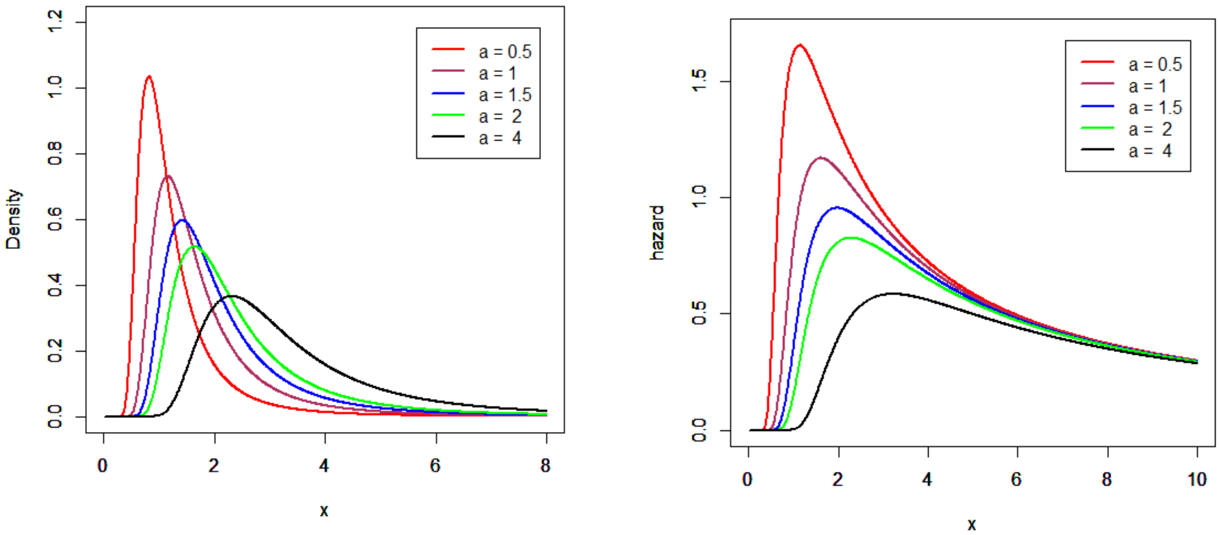
$$h(x) = \frac{2ab\alpha\theta\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}}{x^3\left[1-\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\right]} \tag{6}$$

The plots of probability density function and failure rate function for selected values of parameters are shown in Figure 1 and Figure 2.



(a) (b)

Figure 1: Plots of probability density (a) and hazard (b) functions for parameters (a=2.25, b=1.5, α=1.5)



(c) (d)

Figure 2: Plots of probability density (c) and hazard (d) functions for parameters (b= α= θ=1.5)

3. Mathematical Properties

3.1. Random number generator

The random variates of KEIR distribution can be obtained by using the following relation $\mathrm{F(x)} = \mathrm{R}$ where $\mathrm{R} \sim \mathrm{U}(0,1)$. The expression of random number generator is given below

$$x = \frac{\sqrt{\theta}}{\sqrt{\ln[\left(1-(1-R)^{\frac{1}{b}}\right)^{-\frac{1}{a\alpha}}]}} \qquad (7)$$

The above expression also be used for the computation of quartiles, if we take R values 0.25, 0.5 and 0.75 and obtain first, second and third quartiles.

3.2. Moments

The *rth* moment of the Kumaraswamy Exponentiated Inverse Rayleigh distribution, says μ_r, is given the following form

$$\mu_r = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(b+1)}{i!\,\Gamma(b-i)} \frac{\left(a\alpha\theta(i+1)\right)^{\frac{r}{2}}}{(i+1)} \Gamma\left(1-\frac{r}{2}\right), \qquad r < 2$$

Proof: We have the *rth* moment of the KEIR distribution as follows,

By definition,

$$\mu_r = \int_0^{\infty} x^r f(x; a, b, \alpha, \theta)\, dx$$

$$= 2ab\alpha\theta \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(b)}{i!\,\Gamma(b-i)} \int_0^{\infty} x^{r-3} \left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha(i+1)} dx$$

Using the transformation

$$y = \frac{a\alpha\theta(i+1)}{x^2}; \quad x = \sqrt{\frac{a\alpha\theta(i+1)}{y}} \quad \& \quad dx = -\frac{\left(\frac{a\alpha\theta(i+1)}{y}\right)^{\frac{3}{2}}}{a\alpha\theta(i+1)} dy$$

$$\mu_r = -b \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(b)}{i!\,\Gamma(b-i)} \int_{\infty}^{0} e^{-y} \left(\frac{a\alpha\theta(i+1)}{y}\right)^{\frac{r}{2}-\frac{3}{2}} \frac{\left(\frac{a\alpha\theta(i+1)}{y}\right)^{\frac{3}{2}}}{y^{\frac{3}{2}}(i+1)} dy$$

$$= b\sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(b)}{i!\,\Gamma(b-i)} \frac{\left(a\alpha\theta(i+1)\right)^{\frac{r}{2}}}{(i+1)} \int_0^{\infty} e^{-y}\, y^{-\frac{r}{2}} dy$$

$$\mu_r = \sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(b+1)}{i!\,\Gamma(b-i)} \frac{\left(a\alpha\theta(i+1)\right)^{\frac{r}{2}}}{(i+1)} \Gamma\left(1-\frac{r}{2}\right), \qquad r<2 \tag{8}$$

Mean of KEIR distribution is

$$E(x) = \sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(b+1)}{i!\,\Gamma(b-i)} \frac{\sqrt{\pi}\sqrt{a\alpha\theta(i+1)}}{(i+1)}$$

3.3. Moment generating function

Moment generating of the KEIR is given by the following form;

$$M_x(t) = \sum_{r=0}^{\infty}\sum_{i=0}^{\infty} \frac{t^r}{r!} \frac{(-1)^i\Gamma(b+1)}{i!\,\Gamma(b-i)} \frac{\left(a\alpha\theta(i+1)\right)^{\frac{r}{2}}}{(i+1)} \Gamma\left(1-\frac{r}{2}\right), \quad r<2$$

Proof: We compute the moment generating function using the relation

$$M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x;a,b,\alpha,\theta)\,dx$$

$$= \int_0^{\infty}\left(1+tx+\frac{(tx)^2}{2!}----\right) f(x;a,b,\alpha,\theta)\,dx$$

$$= \int_0^{\infty}\sum_{r=0}^{\infty}\frac{t^r}{r!} x^r f(x;a,b,\alpha,\theta)\,dx$$

$$= \sum_{r=0}^{\infty}\frac{t^r}{r!}E(x^r)$$

$$= \sum_{r=0}^{\infty}\frac{t^r}{r!}\sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(b+1)}{i!\,\Gamma(b-i)} \frac{\left(a\alpha\theta(i+1)\right)^{\frac{r}{2}}}{(i+1)} \Gamma\left(1-\frac{r}{2}\right)$$

$$M_x(t) = \sum_{r=0}^{\infty}\sum_{i=0}^{\infty} \frac{t^r}{r!} \frac{(-1)^i\Gamma(b+1)}{i!\,\Gamma(b-i)} \frac{\left(a\alpha\theta(i+1)\right)^{\frac{r}{2}}}{(i+1)} \Gamma\left(1-\frac{r}{2}\right), \quad r<2 \tag{9}$$

3.4. Incomplete moments of

By definition incomplete moments are

$$M_r(z) = \int_0^z x^r f(x; a, b, \alpha, \theta) dx$$

$$M_r(z) = 2ab\alpha\theta \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(b)}{i!\,\Gamma(b-i)} \int_0^z x^{r-3} \left(e^{-\frac{\theta}{x^2}}\right)^{\alpha(i+1)} dx$$

The final expression of incomplete moments is

$$M_r(z) = 2ab\alpha\theta \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(b)}{i!\,\Gamma(b-i)} \left[\frac{z^r \left(\frac{z^2}{\alpha\theta(i+1)}\right)^{-r/2} \Gamma\left(1-\frac{r}{2}, \frac{\alpha\theta(i+1)}{z^2}\right)}{2\alpha\theta(i+1)} \right] \quad (10)$$

Since the second and higher moments are nonexistent, so the usual expressions of kurtosis and skewness are not defined. Due to the shortcomings of classical methods of skewness and kurtosis the Bowley's method is preferable. These measures exists even if moments does not exist. The expressions of Bowley's measure of skewness and Moor's measure of kurtosis are expressed according to,

$$B = \frac{Q\left(\frac{3}{4}\right) - 2Q\left(\frac{1}{2}\right) + Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)}$$

$$M = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) - Q\left(\frac{3}{8}\right) + Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)},$$

3.5. Renyi' Entropy

The entropy of a random variable refers to the amount of uncertainty. Let x be the Kumaraswamy Exponentiated Inverse Rayleigh random variable then the Rènyi entropy can be obtained using the expression.

$$I(\delta) = \frac{1}{1-\delta} log\left[\int_0^{\infty} f^{\delta}(x) dx\right]$$

$$f^{\delta}(x) = \frac{(2ab\alpha\theta)^{\delta}}{x^{3\delta}} \left(e^{-\frac{\theta}{x^2}}\right)^{\alpha\delta} \left[1 - \left(e^{-\frac{\theta}{x^2}}\right)^{\alpha}\right]^{\delta(b-1)}$$

$$= \frac{(2ab\alpha\theta)^{\delta}}{x^{3\delta}} \sum_{i=0}^{\infty} (-1)^i \binom{\delta(b-1)}{i} \left(e^{-\frac{\theta\alpha}{x^2}}\right)^{(\delta+i)}$$

Now find $\int_0^{\infty} f^{\delta}(x) dx$,

Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.6, No.3, 2016

$$\int_0^\infty f^\delta(x)dx = (2ab\alpha\theta)^\delta \sum_{i=0}^{\infty}(-1)^i \binom{\delta(b-1)}{i} \int_0^\infty x^{-3\delta}\left(e^{-\frac{\theta a\alpha}{x^2}}\right)^{(\delta+i)} dx$$

Making transformation

$$y = \frac{a\alpha\theta(\delta+i)}{x^2}; \quad x = \sqrt{\frac{a\alpha\theta(\delta+i)}{y}} \quad \& \quad dx = -\frac{\left(\frac{a\alpha\theta(\delta+i)}{y}\right)^{\frac{3}{2}}}{a\alpha\theta(\delta+i)}dy$$

and getting the expression

$$\int_0^\infty f^\delta(x)dx = \frac{(2ab\alpha\theta)^\delta}{2} \sum_{i=0}^{\infty}(-1)^i \binom{\delta(b-1)}{i} \left[\frac{1}{a\alpha\theta(\delta+i)}\right]^{\frac{(3\delta-1)}{2}} \Gamma\left(\frac{3\delta}{2}-\frac{1}{2}\right)$$

So the final expression of renyi entropy is,

$$I(\delta) = \frac{1}{1-\delta} log\left[\frac{(2ab\alpha\theta)^\delta}{2} \sum_{i=0}^{\infty}(-1)^i \binom{\delta(b-1)}{i} \left[\frac{1}{a\alpha\theta(\delta+i)}\right]^{\frac{(3\delta-1)}{2}} \Gamma\left(\frac{3\delta}{2}-\frac{1}{2}\right)\right] \tag{11}$$

4. Order Statistics

The order statistics have great importance in life testing and reliability analysis. Let $X_1, X_2, X_3, \ldots\ldots\ldots, X_n$ be random variables and its ordered values is denoted as $X_{(1)}, X_{(2)}, X_{(3)}, \ldots\ldots\ldots, X_{(n)}$. The pdf of order statistics is obtained using the below function

$$f_{s:n}(x) = \frac{n!}{(s-1)!\,(n-s)!} f(x)[F(x)]^{s-1}[1-F(x)]^{n-s}$$

The density of the ordered statistics follows the Kumaraswamy Exponentiated Inverse Rayleigh distribution is derived as follow

$$f_{s:n}(x) = \frac{n!\ 2ab\alpha\theta}{(s-1)!\,(n-s)!\ x^3}\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\left[1-\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\right]^{b(n+1-s)-1}\left(1-\left[1-\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\right]^b\right)^{s-1}$$

The density of the smallest order statistic, is obtained as

$$f_{1:n}(x) = \frac{2ab\alpha\theta n}{x^3}\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\left[1-\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\right]^{bn-1}$$

Using the series expansion the smallest order statistics density can be written as

$$f_{1:n}(x) = \frac{2ab\alpha\theta n}{x^3}\sum_{i=0}^{\infty}(-1)^i\binom{bn-1}{i}\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha(i+1)} \tag{12}$$

The density of the largest order statistic, is obtained as

$$f_{n:n}(x) = \frac{2ab\alpha\theta n}{x^3}\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\left[1-\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\right]^{b-1}$$

The expression also written as

$$f_{n:n}(x) = \frac{2ab\alpha\theta n}{x^3}\sum_{i=0}^{\infty}(-1)^i\binom{b-1}{i}\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha(i+1)} \qquad (13)$$

5. Maximum Likelihood Estimation

The probability density function is

$$f(x) = \frac{2ab\alpha\theta}{x^3}\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\left[1-\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\right]^{(b-1)}$$

The Log-likelihood function is given by

$$l = n\log(2) + n\log(a) + n\log(b) + n\log(\alpha) + n\log(\theta) - \sum_{i=1}^{n}\ln x_i{}^3 - \sum_{i=1}^{n}\frac{a\alpha\theta}{x_i{}^2} + (b-1)\sum_{i=1}^{n}ln\left[1-e^{-\frac{\theta a\alpha}{x_i{}^2}}\right]$$

Therefore the Maximum likelihood estimates of a, b, α and θ which maximizes the above log-likelihood equation must satisfies the normal equations.

$$\frac{\partial L}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n}\frac{\alpha\theta}{x_i{}^2} + (b-1)\sum_{i=1}^{n}\left[\frac{\alpha\theta}{x_i{}^2\left(e^{\frac{\theta a\alpha}{x_i{}^2}}-1\right)}\right]$$

$$\frac{\partial L}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\frac{a\alpha}{x_i{}^2} + (b-1)\sum_{i=1}^{n}\left[\frac{a\alpha}{x_i{}^2\left(e^{\frac{\theta a\alpha}{x_i{}^2}}-1\right)}\right]$$

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\frac{a\theta}{x_i{}^2} + (b-1)\sum_{i=1}^{n}\left[\frac{a\theta}{x_i{}^2\left(e^{\frac{\theta a\alpha}{x_i{}^2}}-1\right)}\right]$$

$$\frac{\partial L}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n}\left[1-e^{-\frac{\theta a\alpha}{x_i{}^2}}\right]$$

The maximum likelihood estimates can be obtained by using the above non-linear system of equations. The above system of equations do not have exact solution, so we adapt iterative based methods such Newton-Raphson to solve them analytically. After application of large sample property of ML Estimates, MLE $\hat{\theta}$ can be treated as being approximately normal with mean θ and variancecovariance matrix equal to the inverse of the expected information matrix, i.e. $\sqrt{n}(\hat{\theta}-\theta) \rightarrow N(0, nI^{-1}(\theta))$. $I(\theta)$ is the information matrix then its inverse of matrix is $I^{-1}(\theta)$ provides the variances and covariance's. Approximate two sided $100(1-\alpha)\%$ confidence intervals for a, b, α, and θ are, respectively, given by

$$\hat{a} \pm Z_{\alpha/2}\sqrt{I_{aa}^{-1}(\hat{\theta})}\,, \qquad \hat{\alpha} \pm Z_{\alpha/2}\sqrt{I_{\alpha\alpha}^{-1}(\hat{\theta})}\,, \qquad \hat{b} \pm Z_{\alpha/2}\sqrt{I_{bb}^{-1}(\hat{\theta})} \quad \text{and} \quad \hat{\theta} \pm Z_{\alpha/2}\sqrt{I_{\theta\theta}^{-1}(\hat{\theta})}$$

Using R we can easily compute the Hessian matrix and its inverse and hence the standard errors and asymptotic confidence intervals.

5.1. Application

In this study we applied the proposed distribution on real data set of strength measured in GPA for single carbon fibers and impregnated 1000-carbon fiber tows which were originally reported by Bader and Priest [15]. The data set consists of, 63 observations: 1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020.

We compare the fitting of the KEIR model with 2 models. The pdf of the other fitted models are:

- Inverse Rayleigh distribution introduced by Voda [1]. The pdf of IR is

$$f(x) = \frac{2\theta}{x^3}\left(e^{-\frac{\theta}{x^2}}\right)$$

- Exponentiated Inverse Rayleigh distribution introduced by Rehman and Dar [12]. The pdf of EIR is

$$f(x) = \frac{2\alpha\theta}{x^3}\left(e^{-\frac{\theta}{x^2}}\right)^{\alpha}$$

In order to compare the models, we have consider criteria like maximized likelihood $-2\hat{\ell}$, Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), and Kolmogorov Smirnov test (KS). With minimum values of AIC, BIC, CAIC and HQIC is considered the best model to fit. These statistics are given by $\text{AIC} = -2\hat{\ell} + 2K, BIC = -2\hat{\ell} + KLog(n), CAIC = -2\hat{\ell} + {}^{2Kn}/_{(n-k-1)}$, where n is sample size, $\hat{\ell}$ is log-likelihood and k is number of parameters.

Table 1: ML Estimates of real data set

Model	MLE estimates			
	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\theta}$
KEIR	0.78406	11.3549	5.6023	5.78081
EIR			0.73505	11.3438
IR				8.3383

Table 2. The measures $-2\hat{\ell}$, AIC, CAIC, BIC and HQIC

Model	$-2\hat{\ell}$	AIC	AIC_C	BIC	HQIC	KS
KEIR	**112.839**	**120.839**	**121.529**	**129.412**	**124.211**	**0.0833**
EIR	180.677	188.576	188.877	194.963	189.363	0.2546
IR	182.677	190.677	190.742	194.819	192.519	0.3544

The table 1 and 2 consist of estimated values and goodness of fit test statistics. The Kumaraswamy Exponentiated Inverse Rayleigh distribution with the IR and EIR distributions. The KEIR distribution gives the smallest values of all goodness of fit measures; $-2\hat{\ell}$, AIC, AICc, BIC, HQIC and KS. So it could be chosen as good distribution among fitted distributions. Figure 2 shows the fitted densities of the estimated values of KEIR, EIR and IR. The graph of fitted densities displays that the KEIR model is a better model than EIR and IR.

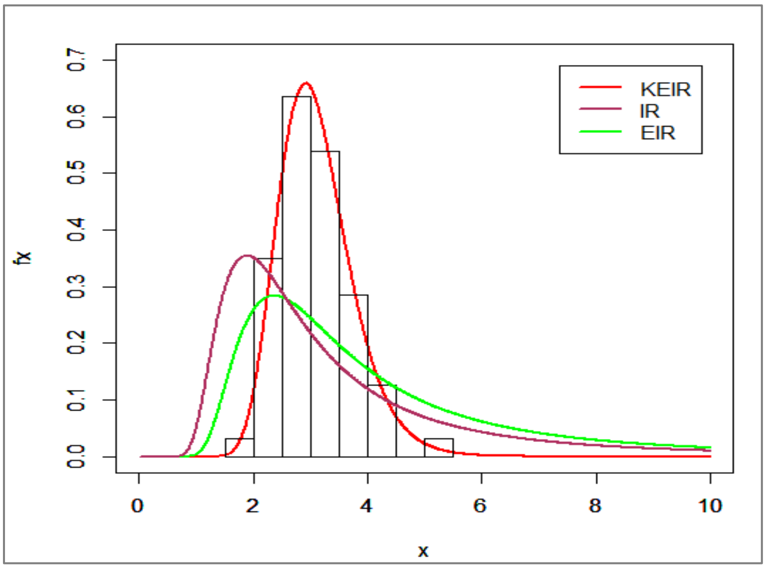


Figure 3. Estimated densities of the KEIR, EIR and IR distributions for the data.

Conclusion

In this study, we derived a generalized form of Inverse Rayleigh distribution known Kumaraswamy Exponentiated Inverse Rayleigh (KEIR) distribution. The generalization of probability theory takes great intention in recent years, due its more flexibility to analyze real life data sets. We studied some mathematical properties of derived distribution. We derive the explicit expressions for the incomplete and ordinary moments, generating and quintile function and renyi entropy. Order statistics density function is also obtained. In the end the new distribution applied to real data found better than other models.

References

1. Voda, V.G., *On the inverse Rayleigh distributed random variable.* Rep. Statist. App. Res., JUSE, 1972. **19**: p. 13-21.
2. Gharraph, M., *Comparison of estimators of location measures of an inverse Rayleigh distribution.* The Egyptian Statistical Journal, 1993. **37**: p. 295-309.
3. Soliman, A., E.A. Amin, and A.A. Abd-El Aziz, *Estimation and prediction from inverse Rayleigh distribution based on lower record values.* Applied Mathematical Sciences, 2010. **4**(62): p. 3057-3066.
4. Cordeiro, G.M., S. Nadarajah, and E.M. Ortega, *The Kumaraswamy Gumbel distribution.* Statistical Methods & Applications, 2012. **21**(2): p. 139-168.
5. Shahbaz, M.Q., S. Shahbaz, and N.S. Butt, *The Kumaraswamy–Inverse Weibull Distribution.* Pakistan journal of statistics and operation research, 2012. **8**(3): p. 479-489.
6. Cordeiro, G.M., E.M. Ortega, and S. Nadarajah, *The Kumaraswamy Weibull distribution with application to failure data.* Journal of the Franklin Institute, 2010. **347**(8): p. 1399-1429.
7. Nadarajah, S. and S. Eljabri, *The Kumaraswamy GP Distribution.* Journal of Data Science, 2013. **11**(4): p. 739-766.
8. Bourguignon, M., et al., *The Kumaraswamy Pareto distribution.* Journal of Statistical Theory and Applications, 2013. **12**(2): p. 129-144.
9. Saulo, H., J. Leão, and M. Bourguignon, *The Kumaraswamy Birnbaum–Saunders Distribution.* Journal of Statistical Theory and Practice, 2012. **6**(4): p. 745-759.
10. Haq, M.A.u., R.M. Usman, and A.A. Fateh, *A Study on Kumaraswamy Power Function Distribution.* Kuwait Journal of Science, 2016. **Forthcoming**.
11. Haq, M.A.u., et al., *Transmuted Power Function Distribution* Gazi University Journal of Science, 2016. **Accepted**.
12. Rehman, S. and I. Dar, Sajjad, *Exponentited Inverse Rayleigh distribution.* MPhil Thesis, 2015.
13. Ahmad, A., S. Ahmad, and A. Ahmed, *Transmuted Inverse Rayleigh Distribution: A Generalization of the Inverse Rayleigh Distribution.* Mathematical Theory and Modeling, 2014. **4**(7): p. 90-98.
14. Leao, J., et al., *On some properties of the beta Inverse Rayleigh distribution.* Chilean Journal of Statistics, 2013. **4**(2): p. 111-131.
15. Bader, M. and A. Priest, *Statistical aspects of fibre and bundle strength in hybrid composites.* Progress in science and engineering of composites, 1982: p. 1129-1136.