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Common Fixed Points of Compatible Maps in Fuzzy Metric Spaces

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Abstract

In this Paper a transposition of these notions is being made for 4-tuples *A*, *B*, *C* and *D* of self maps of fuzzy metric space (*X*, *d*). Then under suitable contractive conditions some common fixed point theorems involving such maps are stated and proved. These results open, in our opinion, a wider scope for the study of this topic in the framework of this new fuzzy metric space.

1. Introduction

In this Paper we introduced the concept of compatibility of maps in fuzzy metric space defined by Bandyopadhyay, Samanta and Das [1] and prove some common fixed point theorems illustrating with examples. In 1965 Zadeh [7] developed the concept of fuzzy set as a new way to represent vagueness in everyday life. Subsequently it was developed extensively by many authors and used in various fields.

In 1982 Deng [2] defined fuzzy metric space in a non-empty set by assigning a nonnegative real number for every pair of fuzzy points. On the other hand Kalewa and Seikkala [4] generalized the notion of the metric space by setting the distance between two points to be a non-negative fuzzy number. In the above definitions Bandyopadhyay et al [1] observed that the concept of fuzziness has not been used in the proper context. The distance between two fuzzy points is expected to be fuzzy. Keeping this idea in mind Bandyopadhyay et al [1] introduced a new definition by assigning a fuzzy number corresponding to a pair of a fuzzy point and studied some properties of the metric. They also proved a fixed point theorem in this newly developed fuzzy metric space.

A weaker concept than that of commutativity of maps, namely weak commutativity is introduced by Sessa [6]. Further Jungck [3] proposed a generalization of the concepts of commuting and weakly commuting mappings called compatible mappings. In fact weak commutativity or commutativity implies compatibility but the neither implication is reversible. He established some results regarding common fixed points of such maps in metric spaces.

1.2 Definitions & Preliminaries

Throughout the Paper we use all symbols and basic definitions of Bandyopadhyay et al [3].

For a non-empty set X, I = [0, 1], $\alpha \in (0, 1)$ and $x \in$ X a map x_α : X → I such that $x_\alpha(y) = \alpha$ if $x = y$ and $x_\alpha(y) = 0$ if $x \neq y$, is said to be a fuzzy point in X and the complement $(x_\alpha)^c$ is denoted by $x_{1-\alpha}$. Let $P = \{x_\alpha : x \in X, \alpha \in (0,\ 1)\}$ (Deng use q_x^α instance of x_α with support x and value α).

A fuzzy number is a fuzzy set on the real line i.e. a map z : R → I associating with each real number its grade of membership z(t). Let E be a set of all upper semi-continuous, convex, normal fuzzy numbers and G be the set of all non-negative elements of E. Where a fuzzy number z in E is called

(i) Normal if there exists a t ∈ R such that z(t) = 1,

(ii) non-negative if z(t) = 0 for all t < 0,

(iii) convex if its λ-level set $[z]_\lambda = \{t : z(t) \geq \lambda\}(0 < \lambda \leq 1)$.

The λ-level set of an element z of E is a closed interval $[z]_\lambda = [a^\lambda, b^\lambda]$ where $a^\lambda = -\infty$ and $b^\lambda = +\infty$ is admissible. Addition, multiplication and partially ordering ≤ in E is defined in Mizumoto and Tanaka [5] and Kalewa and Sikkala [4] respectively as:

$[z_1 + z_2]_\lambda = [a_1^\lambda + a_2^\lambda,\ b_1^\lambda + b_2^\lambda]$, $[z_1 \cdot z_2]_\lambda = [a_1^\lambda a_2^\lambda,\ b_1^\lambda b_2^\lambda]$ and $z_1 \leq z_2 \Leftrightarrow a_1^\lambda \leq a_2^\lambda;\ b_1^\lambda \leq b_2^\lambda$, where $z_i \in E$, and $[z_i]_\lambda = [a_i^\lambda,\ b_i^\lambda]$, $(0 < \lambda \leq 1)$, i = 1,2. Since each element m ∈ R can be considered as a fuzzy number $\overline{m}$ such that $\overline{m}$ (t) = 1 if $t = m$ and $\overline{m}$ (t) = 0 if $t \neq m$.

Definition 1.2.1[3] : Let a mapping $d : P \times P \to G$ be such that

FM-1 $d(x_\alpha, x_\beta) = \bar{o}$ where $\alpha \leq \beta$,

FM-2 $d(x_\alpha, x_\beta) = d((y_\beta)^c, (x_\alpha)^c)$,

FM-3 $d(x_\alpha, z_\gamma) \leq d(x_\alpha, y_\beta) + d(y_\beta, z_\gamma)$,

FM-4 $d(x_\alpha, y_\beta) \leq r$ where $r > \bar{o} \Rightarrow$ there exists $\delta > \alpha$ such that $d(x_\delta, y_\beta) \leq r$. Then d is called fuzzy pseudo metric and the pair (X, d) is called fuzzy pseudo metric space. The pair (X, d) is called fuzzy metric (respectively d is fuzzy metric) if

FM-5 $d(x_\alpha, y_\beta) = \bar{o} \Leftrightarrow x = y,\ \alpha \leq \beta$.

Example 1.2.1 [1]: Let X = (-∞,∞) and $d(x_\alpha, y_\beta) = \overline{\max(\alpha - \beta, 0) + |x - y|}$. Then (X, d) is a fuzzy metric space.

Example 1.2.2: Let X = (-∞,∞) and $d(x_\alpha, y_\beta) = \overline{\max\{1 - (\exp^{-1}(\alpha - \beta)), 0\} + |x - y|}$. Then (X, d) is a fuzzy metric space.

Throughout this Paper we use $\{q\}$ for P and q_x^α for x^α .Let $\{q_n\}$ be a sequence of fuzzy points, then

Lemma 1.2.1 : Image of a fuzzy point q_x^α under a map T, $T(q_x^\alpha) = q_{T(x)^\alpha}$.

Definition 1.2 [1]: $q_n \to q \Leftrightarrow d(q_n, q) = \bar{o}$ or for each $\varepsilon > 0$ there exists $N \in I^+$ such that $d(q_n, q) < \varepsilon$, for all $n \geq N$.

Definition 1.3 [1] : $\{q_n\}$ is called m-converges to q if $q_n \to q$ and $q_n^c \to q^c$ (written as $q_n \xrightarrow{m} q$).

Definition 1.4 [1]: {qn} is called Cauchy if d (qn, qm) → ō.

Definition 1.5 [1]: The fuzzy metric space (X, d) is called complete if every Cauchy sequence m-converges to some fuzzy point.

Definition 1.6: A map T: (X, d) → (Y, d') is said to be continuous at a fuzzy point q if for a sequence $\{q_n\}$ of fuzzy points $q_n \to q \Rightarrow T(q_n) \to T(q)$.

Definition 1.7: Two self maps of a fuzzy metric space (X, d) are called weakly commuting if for all q in {q}, $d(STq, TSq) \le d(Sq, Tq)$.

Definition 1.8: Two self maps S and T of a fuzzy metric space (X, d) are said to be compatible if d $(STq_n, TSq_n) \to \bar{O}$.

Whenever $\{q_n\}$ is a sequence in X such that $Sq_n, Tq_n \to q$, for some q in X. Clearly weak commutativity and commutativity implies compatibility but neither implication is reversible, as we can prove in the following lemma:

Lemma 1.2.2: Weak commutativity implies compatibility but the converse is not true in general in fuzzy metric space.

Proof: Let S and T be weakly commuting self maps of a fuzzy metric space (X, d) and $\{q_n\}$ be a sequence of fuzzy points such that $Sq_n, Tq_n \to q$ for some fuzzy point q , then

$d(STq_n, TSq_n) \le d(Sq_n, Tq_n) \to \bar{o}$ as $n \to \infty$, yields S and T are compatible.

For the converse we suppose S and T are defined by $Sx = x^2$ and $Tx = 2 - x$ for all x in X , where X = $(-\infty, +\infty)$ and $d(q_x^{\alpha}, q_x^{\beta}) = \overline{\max(\alpha-\beta,0)+|x-y|}$, consider a sequence $\{q_n\}$ of fuzzy points such that $q_n = q_{x_n}{}^{\alpha} = q_{1+1/n}{}^{\alpha}$, n = 1, 2, 3, … then $q_n \to q_1^{\alpha}$ as $n \to \infty$ and $Sq_n, Tq_n \to q_1^{\alpha}$.

Now $d\left(STq_n,\ TSq_n\right) = \overline{\max(\alpha-\alpha,0)+|(2-x_n)^2-2+x_n{}^2|} \to \bar{o}$, i. e. S and T are compatible maps but, for x = -1 we have

$$d(STq_x{}^{\alpha},\ TSq_x{}^{\alpha}) = \overline{\max(\alpha-\alpha,0)+|(2-x)^2-2+x^2|} = \bar{8},\quad \text{and}$$

$d(Sq_x{}^{\alpha},\ Tq_x{}^{\alpha}) = \overline{\max(\alpha-\alpha,0)+|x^2-2+x|} = \bar{o}$ implies that S and T are not weakly commuting maps.

1.3 Main Results

Theorem 1.3.1 : Let A, B, S and T be self maps of a complete metric space (X, d) satisfying.

(i) $A(X) \subset T(X)$; $B(X) \subset S(X)$,

(ii) S and T are continuous,

(iii) (A, S) and (B, T) are compatible maps,

(iv) $d(Aq_1, Bq_2) \leq k \max \{d(Sq_1, Tq_2), d(Aq_1, Sq_1), d(Bq_2, Tq_2)\}$,

for all $q_1, q_2 \in \{q\}$ and $\bar{o} < k < \bar{\imath}$.

Then A, B, S and T have a unique common fixed point.

Proof: Let q_0 be any arbitrary fuzzy point. Since $A(X) \subset T(X)$ so there exists a point q_1 in X such that $Aq_0 = Tq_1$ and $B(X) \subset S(X)$ there exists a point q_2 such that $Bq_1 = Sq_2$. Inductively construct a sequence $\{p_n\}$ in $\{q\}$ such that

$p_{2n-1} = Tq_{2n-1} = Aq_{2n-2}$ and $p_{2n} = Sq_{2n} = Bq_{2n-1}$; $n = 1, 2, 3 \ldots$.

From (iv), we have

$$d(p_{2n+1}, p_{2n+2}) = d(Aq_{2n}, Bq_{2n+1})$$

$$\leq k \max\{d(Sq_{2n}, Tq_{2n+1}), d(Aq_{2n}, Sq_{2n}), d(Bq_{2n+1}, Tq_{2n+1})\}$$

$$\leq k \max (p_{2n}, p_{2n+1}), d(p_{2n+1}, p_{2n}), d(p_{2n+2}, p_{2n+1})\},$$

yields,

$$d(p_{2n+1}, p_{2n+2}) \leq k\, d(p_{2n}, p_{2n+1}).$$

Similarly,

$$d(p_{2n}, p_{2n+1}) \leq k\, d(p_{2n-1}, p_{2n}).$$

In general

$$d(p_n, p_{n+1}) \leq k\, d(p_{n-1}, p_n), \text{ for all n.}$$

Now,

$$d(p_n, p_{n+1}) \leq k\, d(p_{n-1}, p_n) \leq k2\, d(p_{n-2}, p_{n-1}), \ldots, \leq k_n\, d(p_0, p_1).$$

If $0 < \lambda \leq 1$ and $[k]_\lambda = [k_1^\lambda, k_2^\lambda]$ then $[k_n]_\lambda = [(k_1^\lambda)^n, (k_2^\lambda)^n]$. Also since $\bar{O} < k < \bar{\imath}$ and so $0 < k_1^\lambda < 1$ and $0 < k_2^\lambda < 1$ then $(k_1^\lambda)^n, (k_2^\lambda)^n \to 0$ as $n \to \infty$ implies that $k_n \to \bar{O}$ as $n \to \infty$

Now for n > m we have

$$d(p_m, p_n) \leq d(p_m, p_{m+1}) + d(p_{m+1}, p_{m+2}) + \ldots + d(p_{n-1}, p_n)$$

$$\leq (k^m + k^{m+1} + k^{m+2} + \ldots + k^{n-1})\, d(p_0,p_1).$$

Now for all $p_m, p_n \in \{q\}$, $[d(p_1, p_2)]_\lambda = [\mu_\lambda(p_1, p_2), \rho_\lambda(p_1, p_2)]$, $0 < \lambda \leq 1$.

Then

$$\mu_\lambda(p_m, p_n) \leq [(k_1^\lambda)^m + (k_1^\lambda)^{m+1} + (k_1^\lambda)^{m+2} + \ldots + (k_1^\lambda)^{n-1}]\, \mu_\lambda(p_0,p_1).$$

$$\leq (k_1^\lambda)^m\, [1- (k_1^\lambda)^{n-m} / 1- k_1^\lambda]\, \mu_\lambda(p_0,p_1) \to 0 \text{ as } m, n \to \infty.$$

Similarly, $\rho_\lambda(p_m, p_n) \to 0$ as $m, n \to \infty$ implies that $d(p_m, p_n) \to \bar{O}$ as $m, n \to \infty$.

Hence $\{p_n\}$ is a Cauchy sequence. By completeness of (X, d), $\{p_n\}$ and consequently its subsequences $\{Aq_{2n-2}\}$, $\{Sq_{2n}\}$, $\{Bq_{2n-1}\}$ and $\{Tq_{2n-1})$ m-converges to some fuzzy point say q.

By continuity of S, $SAq_{2n} \to Sq$ and from condition (iii), $d(ASq_{2n}, SAq_{2n}) \to \bar{O}$.

Therefore,

$$d(ASq_{2n}, Sq) \le d(ASq_{2n}, SAq_{2n}) + d(SAq_{2n}, Sq) \to \bar{O} \text{ , yields } \quad ASq_{2n} \to Sq.$$

Similarly, $BTq_{2n-1} \to Tq$.

Using (iv), we have

$$d(ASq_{2n}, BTq_{2n-1}) \le k \max\{d(S^2q_{2n}, T^2q_{2n-1}), d(ASq_{2n}, S^2q_{2n}), d(BTq_{2n-1}, T^2q_{2n-1})\}.$$

Letting $n \to \infty$, we have $d(Sq, Tq) \le kd(Sq, Tq)$, yields $Sq = Tq$.

Further from (iv), we have

$d(Aq, BTq_{2n-1}) \le k \max\{d(Sq, T^2q_{2n-1}), d(Aq, Sq), d(BTq_{2n-1}, T^2q_{2n-1})\}$.

Letting $n \to \infty$, we have $d(Aq, Tq) \le kd(Aq, Tq)$, yields $Aq = Tq$.

Now again using (iv), we have

$d(Aq, Bq) \le k \max\{d(Sq, Tq), d(Aq, Sq), d(Bq, Tq)\}$,

yields $Aq = Bq$.

Consequently, $Aq = Bq = Sq = Tq$.

Now we claim that $B(x) = x$, if it is not then by (iv), we have

$d(Aq_{2n}, Bq) \le k \max\{d(Sq_{2n}, Tq), d(Aq_{2n}, Sq_{2n}), d(Bq, Tq)\}$

Letting $n \to \infty$, $d(q, Bq) \le k \max\{d(q, Tq), d(q, q), d(Bq, Tq)\}$

or $d(q, Bq) < d(q, Bq)$, i.e. $d(q_x^{\alpha}, q_{B(x)}^{\alpha}) < d(q_x^{\alpha}, q_{B(x)}^{\alpha})$, which is a contradiction. Hence $B(x) = x$. Therefore for all $0 < \alpha < 1$, $B(q_x^{\alpha}) = q_x^{\alpha}$. Consequently x is the common fixed point of A, B, S and T.

For the uniqueness of x suppose that x′ be another common fixed point of A, B, S and T. Then from (iv), we have

$$d(q_x^{\alpha}, q_{x'}^{\alpha}) = d(Aq_x^{\alpha}, Bq_{x'}^{\alpha}) \le max\{d(Sq_x^{\alpha}, Tq_{x'}^{\alpha}), d(Aq_x^{\alpha}, Sq_x^{\alpha}), d(Bq_{x'}^{\alpha}, Tq_{x'}^{\alpha})\}$$

$$\le k \ max\{d(q_x^{\alpha}, q_{x'}^{\alpha}), d(q_x^{\alpha}, q_x^{\alpha}), d(q_{x'}^{\alpha}, q_{x'}^{\alpha})\}$$

Which implies that , $d(q_x^{\alpha}, q_{x'}^{\alpha}) < d(q_x^{\alpha}, q_{x'}^{\alpha})$) which is the contradiction. So x = x'.

Theorem 1.3.2: Let A, B, S and T be self maps of a complete fuzzy metric space (X, d) satisfying (ii) and

(v) $A^a(X) \subset T^t(X)$; $B^b(X) \subset S^s(X)$,

(vi) AS = SA ; BT = TB,

(vii) $d(A^a q_1, B^b q_2) \le k \max\{d(S^s q_1, T^t q_2), d(A^a q_1, S^s q_1), d(B^b q_2, T^t q_2)\}$

for all $q_1, q_2 \in \{q\}$ and $\bar{0} < k < \bar{1}$ and a, b, s, t $\in$ N.

Then A, B, S and T have a unique common fixed point.

Proof: Since A and B commute with S and T, so A^a, B^b commute with S^s, T^t respectively. Also commutativity implies compatibility. Hence by theorem 8.2.1 A^a, B^b, S^s and T^t have a unique common fixed point say q,

i.e. $q = A^a q = B^b q = S^s q = T^t q$.

Now $Aq = A(A^a q) = A^a(Aq)$ and $Aq = A(S^s q) = S^s(Aq)$, hence Aq is the common fixed point of A^a and S^s. Similarly Bq is the common fixed point of B^b and T^t.

Using(vii), we have

$$S(Aq, Bq) = d(A^a(Aq), B^b(Bq))$$

$$\le k \max \{d(S^s(Aq), T^t(Bq)), d(A^a(Aq), S^s(Aq)), d(B^b(Bq), T^t(Bq))\},$$

$$\le k \max \{d(Aq, Bq), d(Aq, Aq), d(Bq, Bq)\},$$

$\le k\, d(Aq, Bq)$, yields $Aq = Bq$. Similarly $Sq = Tq$.

Since q is the unique common fixed point of A^a, B^b, S^s *and* T^t and $Aq(= Bq)$; $Sq(= Tq)$ are common fixed points of A^a, S^s *and* B^b, T^t respectively. Hence $q = Aq = Bq = Sq = Tq$. This completes the proof.

Now we illustrate our results by following examples.

Example 1.3.1: Let (X, d) be a fuzzy metric space where X = (-∞, ∞) and $d(q_x^{\alpha}, q_y^{\beta}) = \overline{\max(\alpha-\beta, 0) + |x-y|}$. Define self maps A, B, S and T of X such that Ax = x/16, Bx = x/8, Sx = x/4 and Tx = x/2 for all x in X. Here the conditions (i),(ii) and (iii) are satisfied and

$$d(Aq_x^{\alpha}, Bq_y^{\beta}) = \overline{\max(\alpha-\beta, 0) + |x/16 - y/8|},$$

$$\le \bar{k}\ \overline{\max(\alpha-\beta, 0) + |x/4 - y/2|},$$

(where k = (4a+A)/4(a+A) < 1, a = α-β and A = |x/4-y/2|),

$$\le \bar{k}\, d(Sq_x^{\alpha},\ Tq_y^{\beta}),$$

$$\le \bar{k}\ \max\{d(Sq_x^{\alpha}, Tq_y^{\beta}), d(Aq_x^{\alpha}, Sq_x^{\alpha}), d(Bq_y^{\beta}, Tq_y^{\beta})\},$$

which is (iv) and zero is the unique common fixed point of A, B, S and T.

Example 1.3.2: Let (X, d) be a fuzzy metric space where X = (-∞, ∞) and $d(q_x^{\alpha}, q_y^{\beta}) = \overline{\max\{1-(\exp^{-1}(\alpha-\beta)),0\}+|x-y|}$. Define self maps A, B, S and T of X as in ex. 8.2.1 then the conditions (i),(ii) and (iii) are satisfied and

$$d(Aq_x^{\alpha},\ Bq_y^{\beta}) = \overline{\max\{1-(\exp^{-1}(\alpha-\beta)),0\}+|x/16-y/8|}$$

$$\le \bar{k}\ \overline{\max\{1-(\exp^{-1}(\alpha-\beta)),0\}+|x\ \ y|}$$

(where k = (4a+A)/4(a+A) < 1, a = 1 – exp-1(α-β) and A = |x/4-y/2|),

$$\le \bar{k}\ d(Sq_x^{\alpha},\ Tq_y^{\beta})$$

$$\le \bar{k}\ \max\{d(Sq_x^{\alpha},\ Tq_y^{\beta}), d(Aq_x^{\alpha}, Sq_x^{\alpha}), d(Bq_y^{\beta},\ Tq_y^{\beta})\},$$

which is (iv) and zero is the unique common fixed point of A, B, S and T.

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