On a Sub Branch of Element Related to an Element in a (BCC, gBCK)-algebras

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Abstract

In this paper, we introduce a new notion that we call a sub branch of an element (a) related to an element (b), denoted by $S(a)_b$ in a BCC-algebra and gBCK-algebra, and we link this notion with another notions of BCC- algebra and gBCK-algebra. We give some properties of $S(a)_b$ in a (BCC, gBCK)-algebras.

Keywords: Sub branch of element related to an element, BCC-algebra, gBCK-algebra.

INTRODUCTION

In 1966, Y. Imai and K. Iseki introduced two classes of abstract algebras: BCK-algebra and BCI-algebra, the notion of BCI-algebra was a generalization of a BCK-algebra[4]. In 1984, Y. Komori, introduced the notion of BCC-algebra[8]. In 1998, J. Hao. introduced the concept of ideals in a BCC-algebra[3]. In 2003, S. M. Hong, Y. B. Jun and M. A. Ozturk construct a new algebra, called a generalized BCK-algebra (gBCK-algebra for short), which is a generalization of a positive implicative BCK-algebra [5]. The aim of this paper is to construct a new subset of (BCC, gBCK)-algebras, called a sub branch of element related to an element. We study the properties of this subset in (BCC, gBCK)-algebras

1.Preliminaries

In this section, we review some basic definitions and notations of BCC-algebras, gBCK- algebras and some types of ideals, that we need in our work.

Definition (1.1) [8] :

By BCC-algebra we mean a non-empty set X, with a constant 0 and a binary operation"*" satisfying the following axioms: for all $x,y,z\in X$,

1- $((x*y) * (z*y)) * (x*z) = 0$,

2- $0*x = 0$,

3- $x*0 = x$,

4- $x*y = 0$ and $y*x = 0$ imply $x = y$.

in X we can define a binary relation "$\leq$" by $x \leq y$ if and only if $x * y = 0$, is called a BCC-order on X.

A nonempty subset S of X is called a subalgebra of X if $x * y \in S$ for all $x, y \in S$.

Example (1.2) [8] :

Let X = {0,1,2, 3,4} be a set with the following table:

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	1	0	0
4	4	3	4	3	0

Then (X; * , 0) is a BCC-algebra.

proposition (1.3) [8] :

In a BCC-algebra, the following hold, for any x,y,z∈X,

1- x*x = 0,
2- (x*y)*x = 0,
3- x*y = 0 imply (x*z)*(y*z)=0.
4- x*y = 0 imply (z*y)*(z*x)=0.

Definition (1.4) [5] :

By a generalized BCK-algebra (gBCK-algebra, for short) we mean a triplet (X, *, 0), where X is a nonempty set, * is a binary operation on X and 0 ∈ X, such that

1- x * 0 = x,
2- x * x = 0,
3- (x * y) * z = (x * z) * y,
4- (x * y) * z = (x * z) * (y * z).

Example (1.5) [5] :

Let X = {0, 1, 2, 3} be a set with the following table:

*	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	0	0	0
3	3	0	0	0

Then X is a gBCK-algebra

Proposition (1.6) [5] :

Let X be a gBCK-algebra. Then

1- 0 * x = 0,
2- (x * y) * x = 0.
3- x * y = 0 implies (x * z) * (y * z) = 0.

Definition (1.7) [3] :

A nonempty subset I is called an ideal of X if it satisfies:

1- 0∈I.
2- x*y∈I and y∈I imply x∈I.

Definition (1.8) [2] :

An ideal I is called a closed ideal of X if: for every x∈I, we have 0*x∈I.

Definition (1.9) [2] :

An ideal I is called a closed ideal with respect to an element b∈X (denoted b-closed ideal) if b*(0*x)∈I , for all x∈I.

Definition (1.10) [1] :

An ideal I of X is called a completely closed ideal of X if: for every $x,y \in I$, we have $x*y \in I$.

Definition (1.11) [1] :

An ideal I is called a completely closed ideal with respect to an element $b \in X$ (denoted b-completely closed ideal) if $b*(x*y) \in I$, for all $x,y \in I$.

Definition (1.12) [2] :

An ideal I satisfies the condition: $x \in I$ and $a \in X \backslash I$ imply $x*a \in I$, is called a *-ideal of X.

Definition (1.13) [7] :

A nonempty subset I of a BCC-algebra X is called a BCC-ideal of X, if

1- $0 \in I$,
2- $(x* y) * z \in I$ and $y \in I$ imply $x* z \in I$; $\forall x; y; z \in X$.

Definition (1.14) [6] :

Let X be a gBCK-algebra. A nonempty subset I of X is called a generalized BCK-ideal (gBCK-ideal, for short) of X if it satisfies the following conditions:

1- $x \in X$ and $a \in I$ imply $a * x \in I$,
2- $x \in X$ and $a, b \in I$ imply $x * (x * a) * b \in I$.

2.The Main Results:

In this section, we define the notion of a sub branch of element related to an element of (BCC, gBCK)-algebras, and link this notion with another notions in BCC-algebra and gBCK-algebra.

Definition (2.1.1):

Let X be a (BCC, gBCK)-algebra, we define a sub branch of element $a \in X$ related to an element $b \in X$ is the set $S(a)_b=\{x \in X: (x*a)*b=0\}$

Now, we study this notion in BCC-algebra and gBCK-algebra.

2.1 The Main Results in BCC-algebra:

To explain Definition(2.1.1) in BCC-algebra we give the following example.

Example (2.1.1):

Let X be a BCC-algebras in example (1.2). Then

$S(1)_2=\{0,1,2\}$, $S(3)_1=\{0,1,2,3\}$, $S(1)_0=\{0,1\}$.

Proposition (2.1.2):

Let X be a BCC-algebra. Then

1- $0 \in S(a)_b$ $\forall a,b \in X$.
2- $a \in S(a)_b$ $\forall a,b \in X$.
3- $S(0)_0 =\{0\}$.

Proof :

1- Let $a,b \in X$
Then $(0* a)*b=0*b=0$ [definition (1.1)(2)]
$\Rightarrow 0 \in S(a)_b$ $\forall a,b \in X$.

2- Let $a,b \in X$
Then $(a*a)*b=0*b=0$ [definition (1.1)(2)]
$\Rightarrow a \in S(a)_b$ $\forall a,b \in X$.

3- Let $x \in S(0)_0$. Then $(x*0)*0=0$
Now, $(x*0)*0=x*0=x \Rightarrow x=0$
$\Rightarrow S(0)_0 =\{0\}$.

Remark (2.1.4):

Let X be a BCC-algebra. The $\bigcup_{b \in X} S(a)_b = X$.

Proposition (2.1.5):

Let X be a BCC-algebra. Then

1- If $x \in S(a)_0 \Rightarrow x*z \in S(a*z)_0$, $\forall z \in X$.
2- If $x \in S(a)_0 \Rightarrow z*a \in S(z*x)_0$, $\forall z \in X$.
3- $x*y \in S(z*y)_{x*z}$, $\forall y,z \in X$.

Proof :

1- let $z \in X$ and $x \in S(a)_0 \Rightarrow (x*a)*0=0 \Rightarrow x*a=0 \Rightarrow (x*z)*(a*z)=0$ [proposition (1.3)(3)]
$\Rightarrow ((x*z)*(a*z))*0=0$

$\Rightarrow x*z \in S(a*z)_0$

2- let $z \in X$ and $x \in S(a)_0 \Rightarrow (x*a)*0=0 \Rightarrow x*a=0 \Rightarrow (z*a)*(z*x)=0$ [proposition (1.1)(4)]
$\Rightarrow ((z*a)*(z*x))*0=0$

$\Rightarrow z*a \in S(z*x)_0$

3- Since $((x*y)*(z*y))*(x*z)=0$ [definition (1.1)(1)]
$\Rightarrow x*y \in S(z*y)_{x*z}$

Proposition (2.1.6):

Let X,Y be a BCC-algebras, x, a, $b \in X$ such that $x \in S(a)_b$, and f be a homomorphism from X to Y. Then $f(x) \in S(f(a))_{f(b)}$

Proof :

Since $x \in S(a)_b \Rightarrow (x*a)*b=0$

Now,

$(f(x)*f(a))*f(b)= f(x*a)*f(b)= f((x*a)*b)=f(0)=0$

$\Rightarrow f(x) \in S(f(a))_{f(b)}$

2.2 The Main Results in gBCK-algebra:

We now give some properties of S(a)b in generalized BCk-algebra

Example (2.2.1):

Let X be a gBCK-algebras in example (1.5). Then

$S(2)_1=\{0,1,2,3\}$, $S(3)_2=\{0,2,3\}$,

Theorem (2.2.2):

Let X be a gBCK-algebra, Then $S(a)_b$ is a subalgebra $\forall a,b \in X$.

Proof :

Let $a,b \in X$ $x,y \in S(a)_b$

Then $(x*a)*b=0$. and $(y*a)*b=0$

Now, $((x*y)*a)*b=((x*a)*y)*b$ [definition (1.4)(3)]

$=((x*a)*b)*y$ [definition (1.4)(3)]

$=0*y= 0$ [proposition (1.6)(1)]

$\Rightarrow S(a)_b$ is a subalgebra

Theorem (2.2.3):

Let X be a gBCK-algebra. Then $S(a)_b$ is an ideal $\forall a,b \in X$.

Proof :

1- Let $a, b \in X$
Then $(0* a)*b=0*b=0$ [proposition (1.6)(1)]
$\Rightarrow 0 \in S(a)_b$.
2- Let $a,b \in X$, $x*y \in S(a)_b$ and $y \in S(a)_b$
Then $((x*y)*a)*b=0$ and $(y*a)*b=0$
Now, $((x*y)*a)*b=0$

$\Rightarrow((x*a)*(y*a))*b=0$ [definition (1.4)(4)]

$\Rightarrow((x*a)*b)*((y*a)*b)=0$ [definition (1.4)(4)]

$\Rightarrow((x*a)*b)*0=0$ [since $y\in S(a)_b$]

$\Rightarrow(x*a)*b=0$

$\Rightarrow x\in S(a)_b$

$\Rightarrow S(a)_b$ is an ideal.

Proposition (2.2.4):

Let X be a gBCK-algebra. Then $S(a)_b$ is a *-ideal $\forall a,b\in X$.

Proof :

1- $S(a)_b$ is an ideal $\forall a,b\in X$ [theorem(2.2.3)]

2- Let $a,b\in X$, $x\in S(a)_b$ and $y\in X/I$

$\Rightarrow (x*a)*b=0$

Now,

$((x*y)*a)*b=((x*a)*y)*b$ [definition (1.4)(3)]

$=((x*a)*b)*y$ [definition (1.4)(3)]

$=0*y=0$ [propisition (1.6)(1)]

$\Rightarrow x*y\in S(a)_b$

$\Rightarrow S(a)_b$ is a *- ideal $\forall a,b\in X$.

Proposition (2.2.5):

Let X be a gBCK-algebra. Then

1- $b\in S(a)_b$ $\forall a,b\in X$.

2- if $x\in S(a)_b \Rightarrow x\in S(b)_a$

3- if $x\in S(a)_b \Rightarrow x*c\in S(a)_{b*c}$ $\forall a,b,c\in X$.

Proof :

1- $\forall a,b\in X$

Then $(b*a)*b=(b*b)*a$ [definition (1.4)(3)]

$=0*b=0$ [proposition (1.6)(1)]

$\Rightarrow b\in S(a)_b$ $\forall a,b\in X$.

2- Let $a,b\in X$ and $x\in S(a)_b$ $\Rightarrow (x*a)*b=0$

$\Rightarrow (x*b)*a=0$ [definition (1.4)(3)]

$\Rightarrow x\in S(b)_a$.

3- Let $a,b,c\in X$ and $x\in S(a)_b$ $\Rightarrow (x*a)*b=0$

Now,

$((x*a)*c)*(b*c)=0$ [proposition (1.6)(3)]

$\Rightarrow ((x*c)*a)*(b*c)=0$ [definition (1.4)(3)]

$\Rightarrow x*c\in S(a)_{b*c}$.

Proposition (2.2.6):

Let X be a gBCK-algebra, If $x\in S(a)_0$ Then $x\in S(a)_b$ $\forall a,b\in X$.

Proof :

1- Let $a,b\in X$ and $x\in S(a)_0 \Rightarrow (x*a)*0=0 \Rightarrow x*a=0 \Rightarrow (x*b)*(a*b)=0$ [proposition (1.6)(3)]

$\Rightarrow (x*a)*b=0$ [definition (1.4)(4)]

$\Rightarrow x\in S(a)_b$.

Proposition (2.2.7):

Let X be a gBCK-algebra. Then $S(a)_b$ is a closed ideal $\forall a,b\in X$.

Proof :

1- $S(a)_b$ is an ideal [theorem (2.2.3)]
2- Let $x \in S(a)_b$

Now,

$((0*x)*a)*b=(0*a)*b=0*b=0$
$0*x \in S(a)_b$
$\Rightarrow S(a)_b$ is a closed ideal.

Proposition (2.2.8):

Let X be a gBCK-algebra. Then $S(a)_b$ is a x-closed ideal $\forall$ $x \in S(a)_b, \forall a,b \in X$.

Proof :

1- $S(a)_b$ is an ideal [theorem (2.2.3)]
2- Let $x,y \in S(a)_b \Rightarrow (x*a)*b=0$

Now,

$((x*(0*y))*a)*b=((x*0)*a)*b=(x*a)*b= 0$
$\Rightarrow x*(0*y) \in S(a)_b$
$\Rightarrow S(a)_b$ is x-closed ideal .

Proposition (2.2.9):

Let X be a gBCK-algebra. Then $S(a)_b$ is a completely closed ideal $\forall a,b \in X$.

Proof :

1- $S(a)_b$ is an ideal [theorem (2.2.3)]
2- Let $x,y \in S(a)_b \Rightarrow (x*a)*b=0, (y*a)*b=0$

Now, $((x*y)*a)*b=((x*a)*y)*b$ [definition (1.4)(3)]

$=((x*a)*b)*y$ [definition (1.4)(3)]

$=0*y= 0$ [since $x \in S(a)_b$]

$x*y \in S(a)_b$

$\Rightarrow S(a)_b$ is a completely closed ideal $\forall a,b \in X$

Proposition (2.2.10):

Let X be a gBCK-algebra. Then $S(a)_b$ is a c-completely closed ideal $\forall a,b \in X$, $c \in S(a)_b$.

Proof :

1- $S(a)_b$ is an ideal [theorem (2.2.3)]
2- Let $c,x,y \in S(a)_b \Rightarrow (c*a)*b=0, (x*a)*b=0, (y*a)*b=0$

Now, $((c*(x*y))*a)*b = ((c*a)*(x*y))*b$ [definition (1.4)(3)]

$= ((c*a)*b)*(x*y)$ [definition (1.4)(3)]

$=0*(x*y)$ [since $c \in S(a)_b$]

$= 0$

$c*(x*y) \in S(a)_b$

$\Rightarrow S(a)_b$ is an c- completely closed ideal $\forall a,b \in X$, $c \in S(a)_b$.

Theorem (2.2.11):

Let X be a gBCK-algebra. Then $S(a)_b$ is a BCC-ideal $\forall a,b \in X$.

Proof :

1- $0 \in S(a)_b$ [theorem (2.2.3)(1)]

2- Let a, b, x, y, z $\in X$, $(x*y)*z \in S(a)_b$, $y \in S(a)_b$
$\Rightarrow (((x*y)*z)*a)*b=0$, $(y*a)*b=0$

Now, $(((x*y)*z)*a)*b=0$

$\Rightarrow$ (((x*z)*y)*a)*b=0 [definition (1.4)(3)]

$\Rightarrow$ (((x*z)*a)*((y*z)*a))*b=0 [definition (1.4)(4)]

$\Rightarrow$ (((x*z)*a)*b)*(((y*z)*a)*b)=0 [definition (1.4)(4)]

$\Rightarrow$ (((x*z)*a)*b) * (((y*a)*z)*b)=0 [definition (1.4)(3)]

$\Rightarrow$ (((x*z)*a)*b) * (((y*a)*b)*z)=0 [definition (1.4)(3)]

$\Rightarrow$ (((x*z)*a)*b) * (0*z)=0 [since y $\in$ S(a)b]

$\Rightarrow$ (((x*z)*a)*b) * 0 = 0 [proposition (1.6)(1)]

$\Rightarrow$ (((x*z)*a)*b) = 0

$\Rightarrow$ x*z$\in$ $S(a)_b$

$\Rightarrow$ $S(a)_b$ is a BCC- ideal $\forall$a,b$\in$X

Proposition (2.2.12):

Let X be a gBCK-algebra. Then $S(a)_b$ is a gBCK-ideal $\forall$a,b$\in$X.

Proof :

1- Let x$\in$X and y$\in$ $S(a)_b$ $\Rightarrow$ (y*a)*b=0
Now,
((y*x)*a)*b=((y*a)*(x*a))*b=((y*a)*b)*((x*a)*b)= 0*((x*a)*b)=0
$\Rightarrow$ y*x$\in$ $S(a)_b$

2- Let x$\in$X and y,z$\in$ $S(a)_b$ $\Rightarrow$ (y*a)*b=0, (z*a)*b=0
Now,
((x*((x*y)*z))*a)*b=((x*a)*(((x*y)*z)*a))*b
=((x*a)*(((x*y)*a)*(z*a)))*b
=((x*a)*(((x*a)*(y*a))*(z*a)))*b
=[(x*a)*b]*[(((x*a)*(y*a))*(z*a))*b]
=[(x*a)*b]*[(((x*a)*(y*a))*b)*((z*a)*b)]
=[(x*a)*b]*[(((x*a)*b)*(y*a)*b)*((z*a)*b)]
=[(x*a)*b]*((x*a)*b)*0)*0
=((x*a)*b)*((x*a)*b)=0
$\Rightarrow$ x*(x*y)*z $\in$ $S(a)_b$.
$\Rightarrow$$S(a)_b$ is a gBCK-ideal $\forall$a,b$\in$X.

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