

Ultimate Boundedness of Solutions for Certain Third Order Nonlinear Differential Equations

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Abstract

We investigate in this paper, the ultimate boundedness of solutions for certain special class of third order nonlinear differential equations. Using suitable complete Lyapunov function, we obtain the criteria for the ultimate boundedness of solutions for this equation. Our result extends and improves on some well known results on boundedness of solutions of third order differential equations in the literature.

Key words: Nonlinear differential equations; Third order; Ultimate boundedness of solutions; Lyapunov's method.

1 INTRODUCTION

We consider the third-order nonlinear ordinary differential equation,

$$\dddot{x} + a\ddot{x} + \phi(x,\dot{x}) = p(t,x,\dot{x},\ddot{x}) \tag{1}$$

where a is positive constant, $\phi \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$ and $p \in C([0,\infty)\times\mathbb{R}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and $\mathbb{R} = (-\infty,\infty)$. The functions ϕ and p depend only on the argument displayed explicitly and the dots denote differentiation with respect to t. The derivatives ϕ_x and ϕ_y exist and are continuous. Moreover, the existence and uniqueness of solutions of (1) will be assumed.

It is well known that the boundedness of solutions is a very important problem in the theory and application of differential equations, and an effective method for studying the stability and boundedness of solutions is still the Lyapunov's second (direct) method [3]. The major advantage of this method is that the stability of solutions can be obtained without any prior knowledge of solutions. However, the construction of these Lyapunov functionals remains a general problem. So far, in the literature, several authors have investigated the boundedness of solutions of some differential equations of third order [6]. For example, many of the third other differential equations which have been discussed in [6] are special cases of (1). We can mention in this direction, the works of Omeike [4] who considered the case for which $a = \psi(x,\dot{x})$ of (1) where an incomplete Lyapunov function was used to obtain the global asymptotic stability of zero solution $x(t) = 0$ of this equation. Omeike [4] proved under less restrictive conditions the stability result obtained by Qian [5] for the case mentioned above. Tunc [9] further improved the result of Omeike [4] on the boundedness of solutions of nonlinear differential equations. Other articles in this direction include Barbashin [1] and Tunc [7,8] where Lyapunov's second method was used. The Lyapunov function used in the papers mentioned above is not complete (see [2]). Particularly, the boundedness result considered in Tunc [9] is of the type in which the bounding constant depends on the solution in question.

Our aim in this paper is to further study the boundedness of solutions of (1). We obtain the criteria for the ultimate boundedness of solutions of (1), which extends and improves Tunc [9].

2 MAIN RESULT

Our main result is the following theorem.

Theorem 1 *In addition to the basic assumptions imposed on the functions* a, ϕ *and* p *appearing in (1), we assume that there exist positive constants* $\alpha, \Delta_o, \Delta_1, m, a, b$ *and* c $(ab > c)$ *such that the following conditions hold:*

$$\phi_x(x,0) \geq c,\ \ 0 < \phi_x(x,y) < c,$$

$$\phi_y(x,\theta y) \geq b,\quad \phi_z(x,\theta y) \geq m,\quad 0 \leq \theta \leq 1$$

$$y\int_0^y \phi_x(x,\nu)d\nu \leq 0 \ \ and$$

$$|\, p(t,x,y,z)| \leq \Delta_o + \Delta_1(|\,x\,| + |\,y\,| + |\,z\,|).$$

Then, every solution $x(t)$ of (1) satisfies

$$|\,x(t)| \leq D,\ |\,\dot{x}(t)| \leq D,\ |\,\ddot{x}(t)| \leq D \qquad (2)$$

for all sufficiently large t, where D is a constant depending only on $a, b, c, \Delta_o, \Delta_1, \alpha$ *and* m.

2.1 PRELIMINARY RESULTS

Equation (1) may be replaced with the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= -az - \phi(x,y) + p(t,x,y,z). \end{aligned} \qquad (3)$$

Our main tool in the proof of the theorem will be the following function,

$$V(x,y,z) = V_1(x,y,z) + V_2(x,y,z) \qquad (4)$$

where V_1 and V_2 are defined by

$$V_1 = a\int_0^x \phi(\xi,0)d\xi + \phi(x,0)y + \int_0^y \phi(x,\nu)d\nu + \frac{1}{2}a^2y^2 + ayz + \frac{1}{2}z^2,$$

and

$$V_2 = \alpha bx^2 + 2a\int_0^x \phi(\xi,0)d\xi + z^2 + a^2y^2 + \int_0^y \phi(x,\nu)d\nu + 2\alpha axy + 2\alpha xz + 2ayz + 2\phi(x,0)y - \alpha y^2,$$

where

$$ab - c > 0 \quad and$$

$$0 < \alpha < min\{b, \frac{ab-c}{a}, \frac{ab-1}{a}\}, \qquad (5)$$

with c chosen such that $0 < c < 1$.

We can easily verify the following for V.

Lemma 1

V(0,0,0)=0 and

there exist finite constants $D_1 > 0$, $D_2 > 0$ such that

$$D_1(x^2 + y^2 + z^2) \leq V \leq D_2(z^2 + y^2 + z^2) \qquad (6)$$

Proof: It is clear that $V(0,0,0)=0$.

Since $\phi_y(x,y)\geq b$,

V_1 in (4) can be re-arranged as follows:

$$V_1 = a\int_0^x \phi(\xi,0)d\xi + \phi(x,0)y + \frac{b}{2}y^2 + \frac{1}{2}a^2y^2$$
$$+ayz + \frac{1}{2}z^2,$$

$$V_1 \geq \frac{1}{2b}\{by+\phi(x,0)\}^2 + \frac{1}{2}\{ay+z\}^2$$
$$+\frac{1}{2b}\int_0^x [ab-\phi_\xi(\xi,0)]\phi(\xi,0)d\xi - \phi^2(0,0),$$

By the condition (i) of Theorem 1 and $\phi(0,0)=0$, we have that

$$\frac{1}{2b}\int_0^x [ab-\phi_\xi(\xi,0)]\phi(\xi,0)d\xi \geq \frac{1}{2b}(ab-c)cx^2$$

Hence,

$$V_1 \geq \frac{1}{2b}\{by+\phi(x,0)\}^2 + \frac{1}{2}\{ay+z\}^2$$
$$+\frac{1}{2b}(ab-c)cx^2.$$

Similarly, V_2 can be re-arranged as follows:

$$V_2 \geq \{\alpha x+ay+z\}^2 + \{a^{-\frac{1}{2}}y + a^{\frac{1}{2}}\phi(x,0)\}^2$$
$$+\alpha(b-\alpha)x^2 + \int_0^y [\phi_\nu(x,\nu) - a^{-1} - \alpha]\nu d\nu$$
$$+a[\int_0^x \phi(\xi,0)d\xi - \phi^2(x,0)],$$

where

$$\int_0^y [\phi_\nu(x,\nu) - a^{-1} - \alpha]\nu d\nu \geq (b - \frac{1}{a} - \alpha)y^2,$$

and

$$a[\int_0^x \phi(\xi,0)d\xi - \phi^2(x,0)]$$
$$= a[\int_0^x \{1-\phi_\xi(\xi,0)\}\phi(\xi,0)d\xi - \phi^2(0,0)]$$

Using condition (i) of Theorem 1 and $\phi(0,0)=0$, we have,

$$= a[\int_0^x \{1-\phi_\xi(\xi,0)\}\phi(\xi,0)d\xi - \phi^2(0,0)] \geq a(1-c)cx^2,$$

V_2 becomes,

$$V_2 \geq \{\alpha x+ay+z\}^2 + \{a^{-\frac{1}{2}}y + a^{\frac{1}{2}}\phi(x,0)\}^2$$
$$+\{\alpha(b-\alpha)+a(1-c)\}x^2 + (b-\frac{1}{a}-\alpha)y^2.$$

Combining the estimates for V_1 and V_2, and we obtain,

$$V = V_1 + V_2,$$

$$V \geq \{\alpha x + ay + z\}^2 + \frac{1}{2}\{ay + z\}^2$$
$$+\{a^{-\frac{1}{2}}y + a^{\frac{1}{2}}\phi(x,0)\}^2 + \frac{1}{2b}\{by + \phi(x,0)\}^2$$
$$+\alpha(b-\alpha)x^2 + \{\frac{1}{2b}(ab-c)c + a(1-c)\}x^2 + (b - \frac{1}{a} - \alpha)y^2, \quad (7)$$

where α satisfies (5).

Now, it is obvious from (7) that the function V defined in (4) is a positive definite function. Hence, there is a positive constant D_1 such that

$$V(x,y,z) \geq D_1(x^2 + y^2 + z^2). \quad (8)$$

Hence, inequality (6) follows from (8) and (4) if we choose $D_1 > 0$ and $D_2 > 0$.

Lemma 2 *There are finite constants* $D_3 > 0, D_4 > 0$ *depending only on* $a, b, c, \Delta_o, \Delta_1, m$ *and* α *such that for any solution* $(x(t), y(t), z(t))$ *of (3)*

$$\dot{V} \equiv \frac{d}{dt}V(x(t), y(t), z(t)) \leq -D_3,$$

provided that $x^2 + y^2 + z^2 \geq D_4$.

Proof : On using (4), a direct differentiation of $\frac{dv}{dt}$ gives after simplification of V_1, yields

$$\dot{V}_1 = y^2\phi_x(x,0) + y\int_0^y \phi_x(x,\nu)d\nu - U_1 - U_2$$
$$+\{ay + z\}p(t,x,y,z), \quad (9)$$

where

$$U_1 = ay\{\phi(x,y) - \phi(x,0\})$$
$$\geq ay^2\phi_y(x,\theta_1 y) \geq aby^2, 0 \leq \theta_1 \leq 1,$$

and

$$U_2 = z\{\phi(x,y) - \phi(x,0)\}$$
$$= z^2\phi_z(x,\theta_2 y),$$
$$\geq mz^2, 0 \leq \theta_2 \leq 1$$

also under the assumptions of the theorem we have

$$y^2\phi_x(x,0) \geq cy^2,$$

$$y\int_0^y \phi_x(x,\nu)d\nu \leq 0,$$

$$\{ay + z\}p(t,x,y,z) \leq (a|y| + |z|)|p(t,x,y,z)|$$

From the estimates U_1, U_2 above and (9) we obtain

$$\dot{V}_1 \leq -(ab-c)y^2 - mz^2 + (a|y| + |z|)|p(t,x,y,z)|.$$

For $\dot{V}_2$, we have

$$\dot{V}_2 = \alpha a y^2 + y\int_0^y \phi_x(x,\nu)d\nu + y^2\phi_x(x,0)$$
$$-\alpha x^2\phi_x(x,y) - U_3 - U_4 - U_5$$
$$+\{\alpha x + ay + z\}p(t,x,y,z).$$

where

$$U_3 = ay\{\phi(x,y) - \phi(x,0)\}$$
$$= ay^2\phi_y(x,\theta_3 y), 0 \le \theta_3 \le 1,$$
$$\ge aby^2,$$

$$U_4 = z\{\phi(x,y) - \phi(x,0)\}$$
$$= z^2\phi_z(x,\theta_4 y), 0 \le \theta_4 \le 1,$$
$$\ge mz^2,$$

$$U_5 = \{\alpha x\phi(x,y) - \alpha bxy\}$$
$$= \alpha\{\phi_y(x,y) - b\}xy \ge 0$$

and

$$y\int_0^y \phi_x(x,\nu)d\nu \le 0.$$

Hence, we obtain for $\dot{V}_2$ as,

$$\dot{V}_2 \le -\alpha c x^2 - \{(ab-c) - \alpha a\}y^2 - mz^2$$
$$+\{\alpha|x| + a|y| + |z|\}|p(t,x,y,z)|,$$

Thus, for $\dot{V} = \dot{V}_1 + \dot{V}_2$, we have

$$\dot{V}(t) \le -\alpha c x^2 - \{2(ab-c) - \alpha a\}y^2 - 2mz^2$$
$$+\{\alpha|x| + 2a|y| + 2|z|\}|p(t,x,y,z)|,$$

where by (5) there exist positive constants δ_1, δ_2 such that

$$\dot{V}(t) \le -\delta_1(x^2 + y^2 + z^2)$$
$$+\delta_2\{|x| + |y| + |z|\}[\Delta_o + \Delta_1(|x| + |y| + |z|)]$$

$$\dot{V}(t) \le -\delta_1(x^2 + y^2 + z^2) + 3\Delta_1\delta_2(x^2 + y^2 + z^2)$$
$$+3^{\frac{1}{2}}\Delta_o\delta_2(x^2 + y^2 + z^2)^{\frac{1}{2}},$$

where $\delta_1 = \min\{\alpha c; (ab-c) - \alpha a; 2m\}$ and $\delta_2 = \max\{\alpha; 2a; 2\}$.

It follows that

$$\dot{V}(t) \le -\delta_3(x^2 + y^2 + z^2) + \delta_4(x^2 + y^2 + z^2)^{\frac{1}{2}} \tag{10}$$

where $\delta_3 = (\delta_1 - 3\Delta_1\delta_2)$, $\Delta_1 < 3^{-1}\delta_2^{-1}\delta_1$, *and* $\delta_4 = 3^{\frac{1}{2}}\Delta_o\delta_2$.

If we choose

$$(x^2+y^2+z^2)^{\frac{1}{2}} \geq \delta_5 = \delta_4\delta_3^{-1}.$$

Inequality (10) implies that

$$\dot{V}(t) \leq -\delta_3(x^2+y^2+z^2). \tag{11}$$

Then, there exists δ_6 such that

$$\dot{V}(t) \leq -1 \quad if \quad x^2+y^2+z^2 \geq \delta_6^2.$$

Proof of Theorem 1 : Let $(x(t), y(t), z(t))$ be any solution of (1). Then there is evidently a $t_o \geq 0$ such that

$$x^2(t_o)+y^2(t_o)+z^2(t_o) < D_4,$$

where D_4 is the constant in the Lemma 2, for otherwise, that is, if

$$x^2(t)+y^2(t)+z^2(t) < D_4, \quad t \geq 0,$$

then by Lemma 2, i.e (11),

$$\dot{V} \leq -D_3 < 0, \quad t \geq 0,$$

and this in turn implies that $V(t) \to -\infty$ as $t \to \infty$, which contradicts (8). Hence, to prove (2) it will suffice to show that if

$$x^2(t)+y^2(t)+z^2(t) < D_4, \quad for \ t = T,$$

where $D_5 \geq D_4$ is a finite constant, then there is a constant $D_6 > 0$, depending on $a, b, c, \Delta_o, \Delta_1, \alpha$ *and* D_5, such that

$$x^2(t)+y^2(t)+z^2(t) \leq D_6, \quad for \ t \geq T. \tag{12}$$

The proof of (12) is based essentially on an extension of an argument proceeding exactly along the lines just indicated in the proof of [10, Lemma 1]. Hence, we omit the detailed proof.

Thus, (12) holds. This completes the proof of (2) and Theorem 1 now follows.

Remark 2.1 *The proof of Theorem 1 shows clearly that the bounding constant D does not depend on the solution of (1) which is not the case with [9].*

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