Meromorphic p-valent Functions With Fixed Second Coefficients

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Abstract
In this paper we introduce the class $L^*(\alpha, \beta, \gamma, A, B, p)$ of meromorphic p-valent functions with fixed second coefficients estimates. Convex linear combinations, some distortion theorems and radii of starlikeness and convexity for are presented in this paper for $f(z)$ in the class $L^*(\alpha, \beta, \gamma, A, B, p)$.

keywords: Analytic functions, Meromorphic functions, p-valent functions, negative coefficients, growth and distortion theorem.

1 Introduction

Let $\sum_p$ denotes the class of functions of the form:

$$f(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} a_{p+n-1} z^{p+n-1}, \qquad (p \in N), \tag{1.1}$$

which are analytic and p-valent in the punctured disk $E = \{z : 0 < |z| < 1\}$. Further denote $T(\alpha, \beta, \gamma, A, B, P)$ the subclass of $\sum_p$ consisting of function f in the unit disk E of the form

$$f(z) = \frac{1}{z^p} - \sum_{n=1}^{\infty} a_{p+n-1} z^{p+n-1}, \qquad (p \in N). \tag{1.2}$$

Further let $\sum_p^*(\alpha)$ be the class of $\sum_p$ consisting of function f which satisfies the inequality

$$Re\left(\frac{-zf'(z)}{f(z)}\right) > \rho, \qquad (0 \le \rho < p),$$

and let $\sum_p^C(\alpha)$ be the class of $\sum_p$ consisting of function f which satisfies the inequality

$$Re\left(-p - \frac{zf''(z)}{f'(z)}\right) > \rho, \qquad (0 \le \rho < p).$$

Clearly we have

$\sum_p^C(\alpha) \Leftrightarrow zf' \in \sum_p^*(\alpha), \qquad (0 \le \alpha < p, p \in N).$

This condition is obviously analogous to the well-known Alexander equivalent (see for details [2]). Many important properties and characteristics of various interesting subclasses of the class $\sum_p^*$ of meromorphic p-valent functions, including the classes $\sum_p^C(\alpha)$ and $\sum_p^*(\alpha)$ defined above, were studied by Owa [3], Saibah [12], Aouf ([5],[6]), Uralgeddi [4], muad [13], Srivastava ([11], [9]), Morga ([7], [8]), Kulkarni [10], [14], Aabed [18], Gahmin [15], Kamali [16], and Markinde [17]. The function $f \in T(1)$ that are starlike of order α and convex of order $\alpha(0 \le \alpha \le 1)$ have been investigated by Silverman [1].

Now, we define the same class of function, $L(\alpha, \beta, \gamma, A, B, p)$, that have been presented by same author [14] for the meromorphic functions with positive coefficients, and we will extend this definition for the meromorphic p-valent functions with fixed second coefficient. as follows:

Definition 1.1. *A function f given by (1.2) is said to be a member of the class* $L(\alpha, \beta, \gamma, A, B, p)$ *if it satisfies*

$$\left| \frac{z^{p+1}f'(z) + z^p f(z)p}{[(B-A)\gamma - A]z^{p+1}f'(z) - z^p f(z)[(B-A)\gamma\alpha - Ap]} \right| \leq \beta. \qquad (1.3)$$

Where $0 < \beta \leq 1$, $0 \leq \alpha < p$, $-1 < A \leq B \leq 1$, $\frac{A}{B-A} \leq \gamma \leq 1$, for all $z \in E$ [14].

let us write $L^*(\alpha, \beta, \gamma, A, B, p)$=$L^*(\alpha, \beta, \gamma, A, B, p) \cap T(p)$ where $T(P)$ is the class of function of the form (1.2) that are analytic and p-valent in the punctured disk E.

2 Preliminary Results

For the class $L^*(\alpha, \beta, \gamma, A, B, p)$, Osama and Darus [14] showed

Theorem 2.1 *A function f defined by (1.1) is in the class* $L^*(\alpha, \beta, \gamma, A, B, p)$ *if*

$$\sum_{n=1}^{\infty}(2p+n-1)+\beta(B-A)\gamma(p+n-1-\alpha)-\beta A(n-1)a_{p+n-1} \leq \beta(B-A)\gamma(p+\alpha)-2\beta Ap. \quad (2.4)$$

By using definition (1.2) we will show the theorem below is satisfied.

Theorem 2.2 *A function f defined by (1.2) is in the class* $T(p)$ *if and only if*

$$\sum_{n=1}^{\infty}(2p+n-1)+\beta(B-A)\gamma(p+n-1-\alpha)-\beta A(n-1)a_{p+n-1} \leq \beta(B-A)\gamma(p+\alpha)-2\beta Ap \quad (2.5)$$

Where $0 < \beta \leq 1$, $0 \leq \alpha < p$, $-1 < A \leq B \leq 1$, $\frac{A}{B-A} \leq \gamma \leq 1$, for all $z \in E$.

Then $f \in T(p)$ is satisfied.

Proof. Let us suppose that

$$\sum_{n=1}^{\infty}(2p+n-1)+\beta(B-A)\gamma(p+n-1-\alpha)-\beta A(n-1)a_{p+n-1} \leq \beta(B-A)\gamma(p+\alpha)-2\beta Ap$$

for $f \in T(p)$. It suffices to show that

$$\left| \frac{z^{p+1}f'(z) + z^p f(z)p}{[(B-A)\gamma - A]z^{p+1}f'(z) - z^p f(z)[(B-A)\gamma\alpha - Ap]} \right| \leq \beta. \qquad (z \in E).$$

Now:

$$\left| \frac{z^{p+1}f'(z) + z^p f(z)p}{[(B-A)\gamma - A]z^{p+1}f'(z) - z^p f(z)[(B-A)\gamma\alpha - Ap]} \right|$$

$$= \left| \frac{\sum_{n=1}^{\infty} (2p+n-1) a_{p+n-1} z^{2p+n-1}}{-(B-A)\gamma(p+\alpha) + 2Ap + \sum_{n=1}^{\infty} (B-A)\gamma(p+n-1-\alpha) - A(n-1) a_{p+n-1} z^{2p+n-1}} \right|,$$

$$\leq \frac{\sum_{n=1}^{\infty} (2p+n-1) |a_{p+n-1}|}{(B-A)\gamma(p+\alpha) - 2Ap - \sum_{n=1}^{\infty} (B-A)\gamma(p+n-1-\alpha) - A(n-1)|a_{p+n-1}|} < \beta,$$

From (2.5), the last expression satisfies,

$$\sum_{n=1}^{\infty} (2p+n-1) + \beta(B-A)\gamma(p+n-1-\alpha) - \beta A(n-1) a_{p+n-1} \leq \beta(B-A)\gamma(p+\alpha) - 2\beta Ap.$$

which is equivalent to our condition of the theorem, so that $f \in T(p)$.

Conversely, assume that $f \in T(p)$, then

$$\left| \frac{z^{p+1} f'(z) + z^p f(z) p}{[(B-A)\gamma - A] z^{p+1} f'(z) - z^p f(z)[(B-A)\gamma\alpha - Ap]} \right| \tag{2.6}$$

$$= \left| \frac{\sum_{n=1}^{\infty} (2p+n-1) a_{p+n-1} z^{2p+n-1}}{-(B-A)\gamma(p+\alpha) + 2Ap + \sum_{n=1}^{\infty} (B-A)\gamma(p+n-1-\alpha) - A(n-1) a_{p+n-1} z^{2p+n-1}} \right|,$$

$$\leq \frac{\sum_{n=1}^{\infty} (2p+n-1) |a_{p+n-1}|}{(B-A)\gamma(p+\alpha) - 2Ap - \sum_{n=1}^{\infty} (B-A)\gamma(p+n-1-\alpha) - A(n-1)|a_{p+n-1}|} < \beta,$$

that is

$$\sum_{n=1}^{\infty} (2p+n-1) + \beta(B-A)\gamma(p+n-1-\alpha) - \beta A(n-1) a_{p+n-1} \leq \beta(B-A)\gamma(p+\alpha) - 2\beta Ap.$$

.
The result are sharp.

In view of Theorem 2.2, we can see that the function f defined by (1.2) in the class $T(p,c)$ satisfy

$$a_p \leq \frac{c\{\beta(B-A)\gamma(p+\alpha) - 2\beta Ap\}}{2p + \beta(B-A)\gamma(p-\alpha)} \tag{2.7}$$

Let $T(p,c)$ denote the class of function f in the form

$$f(z) = z^{-p} - \frac{c\{\beta(B-A)\gamma(p+\alpha) - 2\beta Ap\}}{2p + \beta(B-A)\gamma(p-\alpha)} z^p - \sum_{n=2}^{\infty} a_{p+n-1} z^{p+n-1}, \tag{2.8}$$

with $0 \leq c \leq 1$.

3 Coefficient Inequalities

In this section, we provide a sufficient condition for a function, analytic in E to be in $T(p,c)$.

Theorem 3.1 *A function f defined by (2.7)is in the class* $L^*(\alpha,\beta,\gamma,A,B,p,c)$*if and only if*

$$\sum_{n=1}^{\infty}(2p+n-1)+\beta(B-A)\gamma(p+n-1-\alpha)-\beta A(n-1)a_{p+n-1}\leq(1-c)\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}. \tag{3.9}$$

Proof. Assume that $f\in L^*(\alpha,\beta,\gamma,A,B,p,c)$,by putting

$$a_p\leq\frac{c\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{2p+\beta(B-A)\gamma(p-\alpha)}\qquad 0\leq c\leq 1 \tag{3.10}$$

in (2.5), we have the result as follows

$$2p+\beta(B-A)\gamma(p+\alpha)a_p-\sum_{n=2}^{\infty}\{(2p+n-1)+\beta(B-A)\gamma(p+n-1-\alpha)-\beta A(n-1)\}a_{p+n-1}$$
$$\leq\beta(B-A)\gamma(p+\alpha)-2\beta Ap$$

and so

$$\sum_{n=1}^{\infty}(2p+n-1)+\beta(B-A)\gamma(p+n-1-\alpha)-\beta A(n-1)a_{p+n-1}\leq(1-c)\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}.$$

Conversely, let us suppose that

$$\sum_{n=1}^{\infty}(2p+n-1)+\beta(B-A)\gamma(p+n-1-\alpha)-\beta A(n-1)a_{p+n-1}\leq(1-c)\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}.$$

it suffices to show that

$$\left|\frac{z^{p+1}f'(z)+z^pf(z)p}{[(B-A)\gamma-A]z^{p+1}f'(z)-z^pf(z)[(B-A)\gamma\alpha-Ap]}\right|\leq\beta.\qquad z\in E$$

Now:

$$\left|\frac{z^{p+1}f'(z)+z^pf(z)p}{[(B-A)\gamma-A]z^{p+1}f'(z)-z^pf(z)[(B-A)\gamma\alpha-Ap]}\right| \tag{3.11}$$

$$=\left|\frac{\sum_{n=1}^{\infty}(2p+n-1)a_{p+n-1}z^{2p+n-1}}{-(B-A)\gamma(p+\alpha)+2Ap+\sum_{n=1}^{\infty}(B-A)\gamma(p+n-1-\alpha)-A(n-1)a_{p+n-1}z^{2p+n-1}}\right|,$$

$$\leq\frac{\sum_{n=1}^{\infty}(2p+n-1)|a_{p+n-1}|}{(B-A)\gamma(p+\alpha)-2Ap-\sum_{n=1}^{\infty}(B-A)\gamma(p+n-1-\alpha)-A(n-1)|a_{p+n-1}|}<\beta,$$

From (1.3), the last expression satisfies,

$$\sum_{j=3}^{\infty}(p+n-1)(1+\alpha\beta)a_{p+n-1}\leq\beta[p(1-\alpha)+(\lambda-\alpha)(p-\alpha)]-\frac{c\beta[p(1-\alpha)+(\lambda-\alpha)(p-\alpha)]}{(1+\alpha\beta)}(1+\alpha\beta),$$

That is

$$\sum_{j=3}^{\infty}(p+j-1)(1+\alpha\beta)a_{p+n-1} \leq (1-c)\beta[p(1-\alpha)+(\lambda-\alpha)(p-\alpha)]$$

which is equivalent to our condition of the theorem, so that $f \in T(p,c)$.

The result is sharp for functions f of the form

$$f_{p+n-1}(z) = z^{-p} - \frac{c\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap}{2p+\beta(B-A)\gamma(p-\alpha)}z^p - \frac{(1-c)\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{(2p+n-1)+\beta(B-A)\gamma(p+n-1-\alpha)-\beta A(n-1)}z^{p+j-1}. \tag{3.12}$$

where $j \geq 3$

Corollary 3.1. *Let the function f defined by (2.7) be in the class $T(\alpha,\beta,\lambda,p,c)$,then*

$$a_{p+n-1} \leq \frac{(1-c)\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{(2p+n-1)+\beta(B-A)\gamma(p+n-1-\alpha)-\beta A(n-1)}. \qquad (n \geq 2) \tag{3.13}$$

4 Distortion Theorem

A distortion property for functions in the class $L^*(\alpha,\beta,\gamma,A,B,p,c)$, is given as follows:

Theorem 4.1. *If the function f defined by (2.7) is in the class $L^*(\alpha,\beta,\gamma,A,B,p,c)$, then for $0<|z|=r<1$, we have*

$$r^{-p} - \frac{c\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{2p+\beta(B-A)\gamma(p-\alpha)}r^p - \frac{(1-c)\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{(2p+1)+\beta(B-A)\gamma(p+1-\alpha)-\beta A}r^{p+1} \leq \left|f(z)\right|$$
$$\leq r^{-p} + \frac{c\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{2p+\beta(B-A)\gamma(p-\alpha)}r^p + \frac{(1-c)\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{(2p+1)+\beta(B-A)\gamma(p+1-\alpha)-\beta A}r^{p+1}$$

The bounds are attained by the function f given by,

$$f_2(z) = z^{-p} - \frac{c\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{2p+\beta(B-A)\gamma(p-\alpha)}z^p - \frac{(1-c)\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{(2p+1)+\beta(B-A)\gamma(p+1-\alpha)-\beta A}z^{p+1}, \qquad (z=ir,r).$$

Proof. *Since $f \in L^*(\alpha,\beta,\gamma,A,B,p,c)$,Theorem 3.1 yields the inequality*

$$\sum_{n=2}^{\infty}a_{p+n-1} \leq \frac{(1-c)\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{(2p+n-1)+\beta(B-A)\gamma(p+n-1-\alpha)-\beta A(n-1)}. \tag{4.14}$$

Thus, for $0<|z|=r<1$, and making use of (4.12) we have

$$|f(z)| \;=\; \left|z^{-p} - \frac{c\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{2p+\beta(B-A)\gamma(p-\alpha)}z^p - \sum_{n=2}^{\infty}a_{p+n-1}z^{p+n-1}\right|$$

$$\begin{aligned}
&\leq \ |z|^{-p}+\frac{c\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{2p+\beta(B-A)\gamma(p-\alpha)}|z|^{p}+\sum_{n=2}^{\infty}a_{p+n-1}|z|^{p+n-1} \qquad (|z|=r),\\
&\leq \ r^{-p}+\frac{c\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{2p+\beta(B-A)\gamma(p-\alpha)}r^{p}+\frac{(1-c)\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{(2p+1)+\beta(B-A)\gamma(p+1-\alpha)-\beta A}r^{p+n-1}
\end{aligned}$$

And

$$\begin{aligned}
|f(z)| &= \left|z^{-p}-\frac{c\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{2p+\beta(B-A)\gamma(p-\alpha)}z^{p}-\sum_{n=2}^{\infty}a_{p+n-1}z^{p+n-1}\right|\\
&\geq |z|^{-p}-\frac{c\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{2p+\beta(B-A)\gamma(p-\alpha)}|z|^{p}-\sum_{n=2}^{\infty}a_{p+n-1}|z|^{p+n-1} \qquad (|z|=r),\\
&\geq r^{-p}-\frac{c\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{2p+\beta(B-A)\gamma(p-\alpha)}r^{p}-\frac{(1-c)\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{(2p+1)+\beta(B-A)\gamma(p+1-\alpha)-\beta A}r^{p+n-1}
\end{aligned}$$

Thus we complete the proof.

Theorem 4.2. *If the function f defined by (2.7) is in the class $L^*(\alpha,\beta,\gamma,A,B,p,c)$, then for $0<|z|=r<1$, we have*

$$\begin{aligned}
&pr^{-p-1}-\frac{cp\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{2p+\beta(B-A)\gamma(p-\alpha)}r^{p-1}-\frac{(p+1)(1-c)\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{(2p+1)+\beta(B-A)\gamma(p+1-\alpha)-\beta A}r^{p}\leq\left|f'(z)\right|\\
&\qquad\leq pr^{-p-1}+\frac{cp\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{2p+\beta(B-A)\gamma(p-\alpha)}r^{p-1}+\frac{(p+1)(1-c)\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{(2p+1)+\beta(B-A)\gamma(p+1-\alpha)-\beta A}r^{p}
\end{aligned}$$

The bounds are attained by the function f given by,

$$f_2(z)=z^{-p}-\frac{c\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{2p+\beta(B-A)\gamma(p-\alpha)}z^{p}-\frac{(1-c)\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{(2p+1)+\beta(B-A)\gamma(p+1-\alpha)-\beta A}z^{p+1}, \qquad (z=ir,r).$$

Proof. *Since $f\in L^*(\alpha,\beta,\gamma,A,B,p,c)$, Theorem 3.1 yields the inequality*

$$\sum_{n=2}^{\infty}(p+n-1)|a_{p+n-1}|\leq\frac{(1-c)\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{(2p+1)+\beta(B-A)\gamma(p+1-\alpha)-\beta A}. \tag{4.15}$$

Thus, for $0<|z|=r<1$, and making use of (3.4.2) we have

$$\begin{aligned}
|f'(z)| &= \left|-pz^{-p-1}-\frac{cp\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{2p+\beta(B-A)\gamma(p-\alpha)}z^{p-1}-\sum_{n=2}^{\infty}(p+n-1)a_{p+n-1}z^{p+n-1}\right|\\
&\leq p|z|^{-p-1}+\frac{cp\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{2p+\beta(B-A)\gamma(p-\alpha)}|z|^{p-1}+\sum_{n=2}^{\infty}(p+n-1)a_{p+n-1}|z|^{p+n-1} \qquad (|z|=r),\\
&\leq pr^{-p-1}+\frac{cp\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{2p+\beta(B-A)\gamma(p-\alpha)}r^{p-1}+\frac{(p+1)(1-c)\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{(2p+1)+\beta(B-A)\gamma(p+1-\alpha)-\beta A}r^{p+n-1}
\end{aligned}$$

And

$$|f'(z)| = \left|-pz^{-p-1}-\frac{cp\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{2p+\beta(B-A)\gamma(p-\alpha)}z^{p-1}-\sum_{n=2}^{\infty}(p+n-1)a_{p+n-1}z^{p+n-1}\right|$$

$$\geq \quad p|z|^{-p-1} - \frac{cp\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{2p+\beta(B-A)\gamma(p-\alpha)}|z|^{p-1} - \sum_{n=2}^{\infty}(p+n-1)a_{p+n-1}|z|^{p+n-1} \qquad (|z|=r),$$

$$\geq \quad pr^{-p-1} - \frac{cp\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{2p+\beta(B-A)\gamma(p-\alpha)}r^{p-1} - \frac{(p+1)(1-c)\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{(2p+1)+\beta(B-A)\gamma(p+1-\alpha)-\beta A}r^{p+n-1}$$

Thus we complete the proof.

5 Radii of Starlikeness and Convexity

The radii of starlikeness and convexity for the class $L^*(\alpha,\beta,\gamma,A,B,p)$,is given by the following theorem:

Theorem 5.1 *If the function f defined by (2.7) is in the class $L^*(\alpha,\beta,\gamma,A,B,c,p,\rho)$, then f is starlike of order $\rho(0\leq\rho<p)$, in the disk $|z|<r_1(\alpha,\beta,\gamma,A,B,c,p,\rho)$, where $r_1(\alpha,\beta,\gamma,A,B,c,p,\rho)$,is the largest value for which*

$$p_0 \quad + \quad 1+\frac{c(p_0-\rho)\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap_0\}}{2p_0+\beta(B-A)\gamma(P_0-\alpha)}r_0^{2p_0}$$
$$+ \quad \frac{(1-c)(p_0+j_0-1-\rho)\{\beta(B-A)\gamma(p_0-\alpha)-2\beta Ap_0\}}{(2p_0+j_0-1)+\beta(B-A)\gamma(p+\alpha)-\beta A(j_0-1)}r_0^{2p_0+j_0-1} \leq 1-\rho.$$

The result is sharp for functionsf_{p+j-1}(z) given by (3.10).

Proof. It suffices to show that

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq (1-\rho),$$

for $|z|\leq r_1$, we have

$$\left|\frac{zf'(z)}{f(z)}-1\right| \tag{5.16}$$

$$= \left|\frac{-pz^{-p}-\frac{cp\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{2p+\beta(B-A)\gamma(p-\alpha)}-\sum_{j=2}^{\infty}(p+j-1)a_{p+j-1}z^{p+j-2}}{z^{-p}-\frac{cp\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{2p+\beta(B-A)\gamma(p-\alpha)}z^p-\sum_{j=2}^{\infty}a_{p+j-1}z^{p+j-1}}\right| \tag{5.17}$$

$$\leq \frac{(p+1)r^{-p}+\frac{c(p-1)\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{2p+\beta(B-A)\gamma(p-\alpha)}r^p+\sum_{j=2}^{\infty}(p+j-2)a_{p+j-1}r^{p+j-1}}{r^{-p}-\frac{cp\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{2p+\beta(B-A)\gamma(p-\alpha)}r^p-\sum_{j=2}^{\infty}a_{p+j-1}r^{p+j-1}} \leq 1-\rho$$

Hence (5.14) holds true if

$$(p+1)+\frac{c(p-\rho)\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{2p+\beta(B-A)\gamma(p-\alpha)}r^{2p}+\sum_{j=2}^{\infty}(p+j-1-\rho)a_{p+j-1}r^{p+j-1} \leq (1-\rho).$$

And it follows that from (3.8), we may take

$$a_{p+j-1} = \frac{(1-c)\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{(2p+j-1)\beta(B-A)\gamma(p+j-1-\alpha)-\beta A(j-1)}\lambda_{p+j-1}, \qquad j \geq 2$$

where $\lambda_{p+j-1} \geq 0,$ and

$$\sum_{j=2}^{\infty}\lambda_{p+j-1} \leq 1.$$

For each fixed r, we choose the positive integer $(p_0+j_0-1) = (p_0+j_0-1)(r)$ for each

$$\frac{(p+j-1-\rho)}{(2p+j-1)+\beta(B-A)\gamma(p+j-1-\alpha)-\beta A(j-1)}$$

is maximal. Then it follows that

$$\sum_{j=2}^{\infty}(p_0+j_0-1-\rho)a_{p_0+j_0-1}r^{2p_0+j_0-1} \leq \frac{(1-c)(p_0+j_0-1-\rho)\{\beta(B-A)\gamma(p_0-\alpha)-2\beta Ap_0\}}{(2p_0+j_0-1)+\beta(B-A)\gamma(p+\alpha)-\beta A(j_0-1)}r_0^{2p_0+j_0-1}$$

Hence f is starlike of order ρ in $|z| < r_1(\alpha,\beta,\gamma,A,B,p,j,\rho)$, provided that

$$\begin{aligned}(p_0+1) &+ \frac{c(p_0-\rho)\{\beta(B-A)\gamma(p_0+\alpha)-2\beta Ap_0\}}{2p_0+\beta(B-A)\gamma(p_0-\alpha)}r^{2p_0} \\ &+ \frac{(1-c)(p_0+j_0-1-\rho)\{\beta(B-A)\gamma(p_0-\alpha)-2\beta Ap_0\}}{(2p_0+j_0-1)+\beta(B-A)\gamma(p+\alpha)-\beta A(j_0-1)}r_0^{2p_0+j_0-1} \leq 1-\rho.\end{aligned}$$

We fined the value $r_0 = r_0(\alpha,\beta,\gamma,A,B,p,j,\rho),$ and the corresponding integer $(p_0+j_0-1)(r_0)$ so that

$$\begin{aligned}(p_0+1) &+ \frac{c(p_0-\rho)\{\beta(B-A)\gamma(p_0+\alpha)-2\beta Ap_0\}}{2p_0+\beta(B-A)\gamma(p_0-\alpha)}r^{2p_0} \\ &+ \frac{(1-c)(p_0+j_0-1-\rho)\{\beta(B-A)\gamma(p_0-\alpha)-2\beta Ap_0\}}{(2p_0+j_0-1)+\beta(B-A)\gamma(p+\alpha)-\beta A(j_0-1)}r_0^{2p_0+j_0-1} \leq 1-\rho.\end{aligned}$$

Then this value r_0 is the radius of starlikeness of order ρ for function f belonging to the class $L^*(\alpha,\beta,\gamma A,B,p,c)$.

Theorem 5.2 *If the function f defined by (2.7) is in the class $L^*(\alpha,\beta,\gamma A,B,p,c)$, then f is convexity of order $\rho(0 \leq \rho < p)$, in the disk $|z| < r_2(\alpha,\beta,\gamma A,B,p,c,\rho)$, where $r_2(\alpha,\beta,\gamma A,B,p,c,\rho)$,is the largest value for which*

$$\begin{aligned}(p_0+1) &+ \frac{c(p_0-\rho)\{\beta(B-A)\gamma(p_0+\alpha)-2\beta Ap_0\}}{2p_0+\beta(B-A)\gamma(p_0-\alpha)}r^{2p_0} \\ &+ \frac{(1-c)(p_0+j_0-1)(p_0+j_0-1-\rho)\{\beta(B-A)\gamma(p_0-\alpha)-2\beta Ap_0\}}{(2p_0+j_0-1)+\beta(B-A)\gamma(p_0+j_0-1-\alpha)-\beta A(j_0-1)}r_0^{2p_0+j_0-1} \leq 1-\rho.\end{aligned}$$

The result is sharp for functions f_{p+j-1}(z) given by (3.10).

Proof. It suffices to show that

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq (1-\rho),$$

for $|z| \leq r_2$, we have

$$\left|\frac{zf''(z)}{f'(z)}\right| \tag{5.18}$$

$$= \left|\frac{(p^2+p)z^{-p-1} - \frac{cp(p-1)\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{2p+\beta(B-A)\gamma(p-\alpha)}z^{p-1} - \sum_{j=2}^{\infty}(p+j-1)(p+j-2)a_{p+j-1}z^{p+j-2}}{-pz^{-p-1} - \frac{cp\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{2p+\beta(B-A)\gamma(p-\alpha)}z^{p-1} - \sum_{j=2}^{\infty}(p+j-1)a_{p+j-1}z^{p+j-1}}\right|$$

$$\leq \frac{(p^2+p)r^{-p-1} + \frac{cp(p-1)\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{2p+\beta(B-A)\gamma(p-\alpha)}r^{p-1} + \sum_{j=2}^{\infty}(p+j-1)(p+j-2)a_{p+j-1}r^{p+j-2}}{pr^{-p-1} - \frac{cp\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{2p+\beta(B-A)\gamma(p-\alpha)}r^{p-1} - \sum_{j=2}^{\infty}(p+j-1)a_{p+j-1}r^{p+j-2}} \leq 1-\rho \tag{5.19}$$

Hence (5.15) holds true if

$$p+1+\frac{c(p-\rho)\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{2p+\beta(B-A)\gamma(p-\alpha)}r^{2p}+\sum_{j=2}^{\infty}(p+j-1)(p+j-1-\rho)a_{p+j-1}r^{2p+j-1} \leq (1-\rho).$$

And it follows that from (3.8), we may take

$$a_{p+j-1} = \frac{(1-c)\{\beta(B-A)\gamma(P+\alpha)-2\beta Ap\}}{(2p+j-1)+\beta(B-A)\gamma(P+j-1-\alpha)-\beta A(j-1)}\lambda_{p+j-1}, \qquad j \geq 2$$

where $\lambda_{p+j-1} \geq 0$, and

$$\sum_{j=2}^{\infty}\lambda_{p+j-1} \leq 1.$$

For each fixed r, we choose the positive integer $(p_0+j_0-1) = (p_0+j_0-1)(r)$ for each

$$\frac{(p+j-1)(p+j-\rho-1)}{(2p+j-1)+\beta(B-A)\gamma(P+j-1-\alpha)-\beta A(j-1)}$$

is maximal. Then it follows that

$$\sum_{j=2}^{\infty}(p_0+j_0-1)(p_0+j_0-1-\rho)a_{p_0+j_0-1}r^{2p_0+j_0-1}$$
$$\leq \frac{(1-c)(p_0+j_0-1)(p_0+j_0-1-\rho)\{\beta(B-A)\gamma(p_0-\alpha)-2\beta Ap_0\}}{(p_0+j_0-1)+\beta(B-A)\gamma(p_0+j_0-1-+\alpha)-\beta A(j_0-1)}r_0^{2p_0+j_0-1}$$

Hence f is convex of order ρ in $|z| < r_2(\alpha,\beta,\gamma,A,B,c,p,j,\rho)$, provided that

$$p_0 \quad + \quad 1+\frac{c(p_0-\rho)\{\beta(B-A)\gamma(p_0+\alpha)-2\beta Ap_0\}}{2p_0+\beta(B-A)\gamma(p_0-\alpha)}r^{2p_0}$$
$$\leq \frac{(1-c)(p_0+j_0-1)(p_0+j_0-1-\rho)\{\beta(B-A)\gamma(p_0-\alpha)-2\beta Ap_0\}}{(p_0+j_0-1)+\beta(B-A)\gamma(p_0+j_0-1-+\alpha)-\beta A(j_0-1)}r_0^{2p_0+j_0-1} \leq 1-\rho.$$

We fined the value $r_0 = r_0(\alpha,\beta,\gamma,A,B,c,p,j,\rho)$, and the corresponding integer $(p_0+j_0-1)(r_0)$ so that

$$(p_0+1) \quad + \quad \frac{c(p_0-\rho)\{\beta(B-A)\gamma(p_0+\alpha)-2\beta Ap_0\}}{2p_0+\beta(B-A)\gamma(p_0-\alpha)}r^{2p_0}$$
$$+ \quad \frac{(1-c)(p_0+j_0-1)(p_0+j_0-1-\rho)\{\beta(B-A)\gamma(p_0-\alpha)-2\beta Ap_0\}}{(2p_0+j_0-1)+\beta(B-A)\gamma(p_0+j_0-1-\alpha)-\beta A(j_0-1)}r_0^{2p_0+j_0-1} \leq 1-\rho.$$

Then this value r_0 is the radius of convexity of order ρ for function f belonging to the class $L^*(\alpha,\beta,\gamma,A,B,p,c)$.

6 Convex linear Combination

Our next result involves a linear combination of function of the type (2.7).

Theorem 6.1 Let

$$f_p(z) = z^{-p} - \frac{c\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{2p++\beta(B-A)\gamma(p-\alpha)}z^p,. \tag{6.20}$$

and

$$f_{p+n-1}(z) = \quad z^{-p} - \frac{c\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{2p++\beta(B-A)\gamma(p-\alpha)}z^p$$
$$-\frac{(1-c)\{\beta(B-A)\gamma(P+\alpha)-2\beta Ap\}}{(2p+n-1)+\beta(B-A)\gamma(P+n-1-\alpha)-\beta A(n-1)}z^{p+n-1}$$
$$, \qquad (n \geq 2).$$

then $f \in L^*(\alpha,\beta,\gamma A,B,p,c)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} \lambda_{p+n-1} f_{p+n-1}(z), \tag{6.21}$$

$$where \qquad \lambda_{p+n-1} \geq 0 \qquad and \qquad \sum_{n=1}^{\infty} \lambda_{p+n-1} \leq 1$$

Proof.$\Leftarrow$ From (6.16),(6.17) and (6.18), we have

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \lambda_{p+n-1} f_{p+n-1}(z), \\ &= z^{-p} - \frac{c\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{2p+\beta(B-A)\gamma(p-\alpha)} z^p \\ &\quad - \sum_{n=2}^{\infty} \frac{(1-c)\{\beta(B-A)\gamma(P+\alpha)-2\beta Ap\}}{(2p+n-1)+\beta(B-A)\gamma(P+n-1-\alpha)-\beta A(n-1)} \lambda_{p+n-1} z^{p+n-1}. \end{aligned}$$

since

$$\sum_{n=2}^{\infty} \frac{(1-c)\{\beta(B-A)\gamma(P+\alpha)-2\beta Ap\}}{(2p+n-1)+\beta(B-A)\gamma(P+n-1-\alpha)-\beta A(n-1)} * \frac{(2p+n-1)+\beta(B-A)\gamma(P+n-1-\alpha)-\beta A(n-1)}{(1-c)\{\beta(B-A)\gamma(P+\alpha)-2\beta Ap\}} \lambda_{p+n-1} = \sum_{n=2}^{\infty} \lambda_{p+n-1} \leq = 1,$$

it follows from Theorem 3.1 that the function $f \in L^*(\alpha, \beta, \gamma A, B, p, c)$.

$\Leftarrow$ Conversely, let us suppose that $f \in L^*(\alpha, \beta, \gamma A, B, p, c)$. Since

$$a_{p+n-1} \leq \frac{(1-c)\{\beta(B-A)\gamma(P+\alpha)-2\beta Ap\}}{(2p+n-1)+\beta(B-A)\gamma(P+n-1-\alpha)-\beta A(n-1)} \qquad \{(n \geq 2)\}$$

Setting

$$\lambda_{p+j-1} = \frac{(2p+n-1)+\beta(B-A)\gamma(P+n-1-\alpha)-\beta A(n-1)}{(1-c)\{\beta(B-A)\gamma(P+\alpha)-2\beta Ap\}} a_{p+n-1} \qquad (n \geq 2)$$

It follows that

$$f(z) = \sum_{j=2}^{\infty} \lambda_{p+j-1} f_{p+j-1}(z)$$

this complete the proof of theorem.

Theorem 6.2 *The class $L^*(\alpha, \beta, \gamma, A, B, p, c)$ is closed under convex linear combinations.*
Proof. Suppose that the functions f_1 and f_2 defined by,

$$f_i(z) = z^{-p} - \frac{c\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{2p+\beta(B-A)\gamma(p-\alpha)} z^p - \sum_{n=2}^{\infty} b_{p+n-1,i} z^{p+n-1}, \qquad i = 1, 2; z \in E \tag{6.22}$$

are in the class $L^*(\alpha, \beta, \gamma A, B, p, c)$.

Setting $\qquad V(z) = \mu f_1(z) + (1-\mu) f_2(z) \qquad (0 \leq \mu \leq 1),$

we want to show that $f \in L^*(\alpha,\beta,\gamma A,B,p,c)$. We find from (6.19) that

$$V(z)=z^{-p}-\frac{c\{\beta(B-A)\gamma(p+\alpha)-2\beta Ap\}}{2p+\beta(B-A)\gamma(p-\alpha)}z^p-\sum_{j=3}^{\infty}\left(\mu a_{p+j-1,1}+(1-\mu)a_{p+j-1,2}\right)z^{p+j-1},$$

In view of theorem 3.1, we have

$$\sum_{n=2}^{\infty}[(2p+n-1)+\beta(B-A)\gamma(p+n-1-\alpha)\left(\mu|a_{p+j-1,1}|+(1-\mu)a_{p+j-1,2}\right), \quad (6.23)$$

$$\begin{aligned}
&= \mu\sum_{n=2}^{\infty}[(2p+n-1)+\beta(B-A)\gamma(p+n-1-\alpha)]|a_{p+n-1,1}| \\
&+ (1-\mu)\sum_{n=2}^{\infty}[(2p+n-1)+\beta(B-A)\gamma(p+n-1-\alpha)]|b_{p+n-1,1}| \\
&\leq \mu\left\{(1-c)\beta(B-A)\gamma(p-\alpha)-2\beta Ap\right\}+(1-\mu)\left\{(1-c)\beta(B-A)\gamma(p-\alpha)-2\beta Ap\right\}, \\
&= (1-c)\beta(B-A)\gamma(p-\alpha)-2\beta Ap.
\end{aligned}$$

which show that $f \in L^*(\alpha,\beta,\gamma,A,B,p,c)$.

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