

On Weighted Composition Operators on Hardy Space $\mathbb{H}^{\mathbf{2}}$.

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Abstract: In this paper we study some properties of weighted composition. In addition that we discuss the finite rank weighted composition on Hardy space $\mathbb{H}^{\mathbf{2}}$and discuss the eigenvalue equation for weighted composition operator inducing by Koenig's maps on Hardy space $\mathbb{H}^{\mathbf{2}}$.

Keywords: Weighted composition operators, Hardy spaces.

1. Introduction.

Let U denote the open unite disc in the complex plan ,let $\mathbb{H}^{\infty}$ denote the collection of all holomorphic function on U and let $\mathbb{H}^{\mathbf{2}}$ is consisting of all holomorphic self-map on U such that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ whose Maclaurin coefficients are square summable (i.e)

$f(z) = \sum_{n=0}^{\infty} |a_n|^2 < \infty$. More precisely $f(z) = \sum_{n=0}^{\infty} a_n z^n$ if and only if $\|f\| = \sum_{n=0}^{\infty} |a_n|^2 < \infty$.The inner product inducing the$\mathbb{H}^{\mathbf{2}}$ norm is given by $\langle f,g \rangle = \sum_{n=0}^{\infty} a_n \, \overline{b_n}$.

Given any holomorphic self-map φ on U, recall that the composition operator

$$C_{\varphi}(h) = ho\varphi \quad (h \in \mathbb{H}^2)$$

Is called the composition operator with symbol φ, is necessarily bounded. Let

$f \in \mathbb{H}^{\infty}$, the operator $T_f: \mathbb{H}^{\mathbf{2}} \to \mathbb{H}^{\mathbf{2}}$ defined by

$$T_f(h(z)) = f(z)h(z), \quad for\ all\ z \in \text{ U}, h \in \mathbb{H}^2$$

is called the Toeplitz operator with symbol f . Since $f \in \mathbb{H}^{\infty}$, then we call T_f a holomorphic Toeplitz operator. If T_f is a holomorphic Toeplitz operator, then the operator $T_f C_{\varphi}$ is bounded and has the form

$$T_f C_{\varphi} g = f(go\varphi) \qquad (g \in \mathbb{H}^2).$$

We call it the weighted composition operator with symbols f and φ [1] and [3], the linear operator

$$\mathcal{W}_{f,\varphi}\, g = f(go\varphi) \qquad (\, g \in \mathbb{H}^2).$$

We distinguish between the two symbols of weighted composition operator $\mathcal{W}_{f,\varphi}$, by calling f the multiplication symbol and φ composition symbol.

For given holomorphic self-maps f and φ of U, $\mathcal{W}_{f,\varphi}$ is bounded operator even if $f \notin \mathbb{H}^{\infty}$ .To see a trivial example, consider $\varphi(z) = p$ where $p \in$ U and $f \in \mathbb{H}^{\mathbf{2}}$, then for all $g \in \mathbb{H}^{\mathbf{2}}$, we have

$$\left\| \mathcal{W}_{f,\varphi}\, g \right\|_2 = \|g(p)\| \|f\|_2 = \|f\|_2 \left| \langle g, K_p \rangle \right| \le \|f\|_2 \|g\|_2 \left\| K_p \right\|_2.$$

In fact, if $f \in \mathbb{H}^{\infty}$,then $\mathcal{W}_{f,\varphi}$ is bounded operator on $\mathbb{H}^{\mathbf{2}}$ with norm

$$\| \mathcal{W}_{f,\varphi} \| = \|T_f C_\varphi\| \leq \|f\|_\infty \|C_\varphi\| = \|f\|_\infty \sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}}.$$

2. Basic concepts on weighted composition operators.

We start this section, by giving the following results which are collect some properties of Toeplitz and composition operators.

Lemma (2.1):[4, 6] Let φ be a holomorphic self-map of U, then

(a) $C_\varphi T_f = T_{fo\varphi} C_{\varphi.}$
(b) $T_g T_f = T_{gf}$.
(c) $T_{f+\gamma g} = T_f + \gamma T_g$.
(d) $T_f^* = T_{\bar{f}}$.

Proposition (2.2):[1] Let φ and ψ be two holomorphic self-map of U, then

1. $C_\varphi^n = C_{\varphi_n}$ for all positive integer n.
2. C_φ is the identity operator if and only if φ is the identity map.
3. $C_\varphi = C_\psi$ if and only if $\varphi = \psi$.
4. The composition operator cannot be zero operator.

For each $\alpha \in U$,the reproducing kernel at α ,defined by $K_\alpha(z) = \frac{1}{1-\bar{\alpha}z}$

It is easily seen for each $\alpha \in U$ and $f \in H^2$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ that

$$\langle f, K_\alpha \rangle = \sum_{n=0}^{\infty} a_n \, \alpha^n = f(\alpha).$$

When $\varphi(z) = (az+b)/cz+d)$ is linear-fractional self-map of U, Cowen in [2] establishes $C_\varphi^* = T_g C_\sigma T_h^*$, where the Cowen auxiliary functions g, σ and h are defined as follows: $g(z) = \frac{1}{-\bar{b}z+\bar{d}}$, $\sigma(z) = \frac{\bar{a}z-\bar{c}}{-\bar{b}z+\bar{d}}$ $\quad and \quad h(z) = cz + d$.

If φ is linear fractional self-map U, then $W_{f,\varphi}^* = (T_f C_\varphi)^* = C_\varphi^* T_f^* = T_g C_\sigma T_h^*$.

Now the following result contains some simple properties of the weighted composition operator on Hardy space $\mathbb{H}^2$.

Proposition (2.3): Let φ *and* ψ be two holomorphic self-maps of U and $f, h \in \mathbb{H}^\infty$ .Then,

1. If f is non-zero function in $\mathbb{H}^\infty$, then $\mathcal{W}_{f,\varphi}$ cannot be zero operator.
2. $\mathcal{W}_{f,\varphi}$ is the identity operator if and only if φ is the identity self-map of U such that $f(z) = 1$ for all $z \in U$.
3. If f is non-zero function in $\mathbb{H}^\infty$, then$\mathcal{W}_{f,\varphi} = \mathcal{W}_{h,\psi}$ if and only if $\varphi = \psi$.

Proof :

(1) Suppose $\mathcal{W}_{f,\varphi}$ is zero operator for some holomorphic self-map φ of U and $f \in \mathbb{H}^\infty$,then $\mathcal{W}_{f,\varphi}g(z) = 0 \quad (g \in \mathbb{H}^2 \ ,z \in U)$. Hence $f(z).C_\varphi g(z) = 0$. Then, either $f(z) = 0$ or $C_\varphi g(z) = 0$. But $f \in \mathbb{H}^\infty/\{0\}$,then $C_\varphi g(z) = 0 \ (z \in U)$, which is a contradiction with proposition (2.2). Hence $\mathcal{W}_{f,\varphi}$ cannot be zero operator**.**

(2) Assume that $\varphi(z) = z$ and $f(z) = 1$ for all $z \in$ U. Note that, for all $g \in \mathbb{H}^2$ and $z \in$ U ,we have

$$\begin{aligned}\mathcal{W}_{f,\varphi}g(z) &= f(z)C_\varphi g(z)\\ &= f(z)g(\varphi(z))\\ &= g(z).\end{aligned}$$

Hence $\mathcal{W}_{f,\varphi}$ is the identity operator .

Conversely if $\mathcal{W}_{f,\varphi}$ is the identity operator ,then for all $g \in \mathbb{H}^2$and $z \in U$,we have

$$\begin{aligned}\mathcal{W}_{f,\varphi}g(z) &= f(z)C_\varphi g(z)\\ &= f(z)g(\varphi(z))\\ &= g(z).\end{aligned}$$

By taking g is a constant self-map of U in $\mathbb{H}^\infty$, we can see that $f(z) = 1$ for all $z \in$ U. Hence, for all $g \in \mathbb{H}^2$ and $z \in$ U, we have $\mathcal{W}_{f,\varphi}g(z) = C_\varphi g(z) = g(z)$.

Therefore , C_φ is the identity operator on $\mathbb{H}^2$. This implies by proposition (2.2) that φ is the identity self-map of U.

(3) Assume for all $g \in \mathbb{H}^2$ and $z \in$ U that $\mathcal{W}_{f,\varphi}g(z) = \mathcal{W}_{f,\psi}g(z)$. Thus,

$f(z)g\big(\varphi(z)\big) = f(z)g\big(\psi(z)\big)$. Hence, by taking $g(z) = z$ we have $f(z)\varphi(z) = f(z)\psi(z)$.Thus, $f(z)\big(\varphi(z) - \psi(z)\big) = 0$, as desired **.**

The convers is clear .

Following proposition shows that the product of finite number of weighted composition operator is also weighted composition operator.

Proposition (2.4):[5] Let each of $\varphi_1, \varphi_2, \dots \varphi_n$ be holomorphic self-maps of U and $f_1, f_2, \dots f_n \in \mathbb{H}^\infty$**,** then

$$\mathcal{W}_{f_1,\varphi_1}.\mathcal{W}_{f_2,\varphi_2} \dots .\mathcal{W}_{f_n,\varphi_n} = T_h C_\phi$$

Where $T_h = f_1.(f_2 o\varphi_1).(f_3 o\varphi_2 o\varphi_1). \dots (f_2 o\varphi_{n-1} o\varphi_{n-2} o \dots . o\varphi_1)$ and

$C_\phi = \varphi_n o\varphi_{n-1} o \dots . o\varphi_1$.

Corollary (2.5): Let φ be a holomorphic self-map of U and $f \in \mathbb{H}^\infty$ then

$$\mathcal{W}^n_{f,\varphi} = T_{f \;\; (f \;\; o\varphi \;\;)(f \;\; o\varphi_2)\ldots(f \;\; o\varphi_{n-1})} C_{\varphi_n}$$

The following lemma discuss the adjoint of weighted composition operator .

Lemma (2.6):[3] If the operator $\mathcal{W}_{f,\varphi}: \mathbb{H}^2 \to \mathbb{H}^2$ is bounded, then for each $\alpha \in U$

$$\mathcal{W}^*_{f,\varphi} K_\alpha = \overline{f(\alpha)} K_{\varphi(\alpha)}.$$

Now we raise the following question:

Is the adjoint of a weighted composition operator on $\mathbb{H}^2$,a weighted composition operator on $\mathbb{H}^2$?.

Abood E. H. [1] answered the same question for composition operators on Hardy space $\mathbb{H}^2$ as follows:

Theorem (2.7):[1] Let φ be a holomorphic self-map of U . Then C^*_φ is a composition operator on $\mathbb{H}^2$ if and only if $\varphi(z) = \gamma z$ for some γ , $|\gamma| \le 1$.

Now we are ready to give the main result in this section concerning our above question .

Theorem (2.8): Let φ be a holomorphic self-map of U and $f \in \mathbb{H}^\infty$, then $\mathcal{W}^*{}_{f,\varphi}$ is a weighted composition operator on $\mathbb{H}^2$ with symbols $h \; and \; \psi$ if and only if $h(z) = \frac{c}{1-\overline{b_0}z}$ and $\psi(z) = a_0 + \frac{a_1}{1-\overline{b_0}z}$ where $a_0 = \psi(0)$, $a_1 = \acute{\psi}(0)$, $b_0 = \varphi(0)$ and

$c = \overline{f(0)}$.

Proof : Assume that for some holomorphic self-map ψ of U we have for all $g \in \mathbb{H}^2$

$$\mathcal{W}^*{}_{f,\varphi}(g) = \mathcal{W}_{h,\psi}(g).$$

Thus, $\mathcal{W}^*_{f,\varphi} K_\beta(z) = \mathcal{W}_{h,\psi} K_\beta(z) \qquad (z \in U, \beta \in U).$

Then, $\overline{f(\beta)} K_{\varphi(\beta)}(z) = h(z) C_\psi K_\beta(z).$

Hence, $\overline{f(\beta)} K_{\varphi(\beta)}(z) = h(z) K_\beta\big(\psi(z)\big)$. This implies that,

$$\frac{\overline{f(\beta)}}{1 - \overline{\varphi(\beta)}z} = \frac{h(z)}{1 - \bar{\beta}\psi(z)} \tag{2.1}$$

In particular, letting $\beta = 0$ in (2.1) we get

$$h(z) = \frac{\overline{f(0)}}{1 - \overline{\varphi(0)}z} \tag{2.2}$$

Put, $b_0 = \varphi(0) \quad and \; c = \overline{f(0)}$,then we can write h as

$$h(z) = \frac{c}{1 - \overline{b_0}z} \tag{2.3}$$

Combining equation (2.1) and (2.2) we get

$$\frac{c(1-\overline{\varphi(\beta)}z)}{\overline{f(\beta)}} = \left(1-\bar{\beta}\psi(z)\right)\left(1-\overline{b_0}z\right) \qquad (2.4)$$

Clearly that the expression on the left of (2.4) is a polynomial of degree one in variable z. This means that the expression on the right of (2.4) must be a polynomial of degree one in z **,**

$$\left(1-\bar{\beta}\psi(z)\right)\left(1-\overline{b_0}z\right) = 1-\bar{\beta}\psi(z)-\overline{b_0}z+\overline{\beta b_0}z\psi(z) \qquad (2.5)$$

Now suppose $\psi(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ is the Taylor expression for ψ.

Hence by substituting ψ in (1.5) we get ,

$\left(1-\bar{\beta}\psi(z)\right)\left(1-\overline{b_0}z\right)$

$= 1-\bar{\beta}(a_0 + a_1 z + a_2 z^2 + \cdots \ ) - \overline{b_0}z + \overline{\beta b_0}z(a_0 + a_1 z + a_2 z^2 + \cdots \ )$

$= \left(1-\bar{\beta}a_0\right) + \left(\overline{\beta b_0}a_0 - \overline{b_0} - \bar{\beta}a_0\right)z + \left(\overline{\beta b_0}a_1 - \bar{\beta}a_2\right)z^2 + \cdots$

We see that $\overline{\beta b_0}a_{j-1} - \bar{\beta}a_j = 0$ for each integer j for which $j \geq 2$. On the other word $a_j = \overline{b_0}a_{j-1}$ $for\ j \geq 2$.

In particular , $a_2 = \overline{b_0}a_1$ which means that $a_3 = (\overline{b_0})^2 a_1$, and by continuing ,we get that $a_j = (\overline{b_0})^{j-1}a_1$ $,for\ j \geq 2$ .Substituting in Taylor series ,we see that

$$\psi(z) = a_0 + a_1 z + \overline{b_0}a_1 z^2 + \overline{b_0}^2 a_1 z^3 + \cdots = a_0 + \frac{a_1}{1-\overline{b_0}z}$$

where $a_0 = \psi(0)$, $a_1 = \acute{\psi}(0)\ and\ b_0 = \varphi(0)$.

Conversely , if $f(z) = \frac{c}{1-\overline{b_0}z}$ and $\psi(z) = a_0 + \frac{a_1}{1-\overline{b_0}z}$,then a straightforward computation shows that equation (2.1) holds for all $\beta\ and\ z\ in\ U$, as desired .

3. Finite –Rank Weighted Composition Operators.

Recall that an operator T on a Hilbert space $\mathcal{H}$ is called of finite –rank if the rang of T, $\mathcal{R}(T)$ is finite dimensional [6] . An operator T is called an algebraic operator on $\mathcal{H}$ if there exists a nonzero complex polynomial *p* such that *p(T)=0* [6] and T is called an idempotent operator on $\mathcal{H}$ if $T^2 = T$.

In this section we shall determine the self-map φ of U and $f \in \mathbb{H}^\infty$that induce a finite –rank weighted composition operator on Hardy space $\mathbb{H}^{\mathbf{2}}$**.**

Proposition (3.1): Let φ be a holomorphic self-map of U and $f \in \mathbb{H}^\infty$.If φ is a constant map $\varphi(z) = b$ where $b, z \in$ U such that $f(b) = 1$,then $\mathcal{W}_{f,\varphi}$is algebraic operator.

Proof : Take $p(z) = z^2 - z$, $z \in U$ .Let $g \in \mathbb{H}^2$, then

$p(\mathcal{W}_{f,\varphi})g(z) = (\mathcal{W}_{f,\varphi}{}^2 - \mathcal{W}_{f,\varphi})g(z)$

$$= \mathcal{W}_{f,\varphi}(T_f C_\varphi g(z)) - T_f C_\varphi\big(g(z)\big)$$

$$= \mathcal{W}_{f,\varphi}(f(z).g(\varphi(z)) - f(z).g(\varphi(z))$$

$$= T_f C_\varphi(f(z).g(b)) - f(z).g(b)$$

$$= g(b)T_f C_\varphi f(z) - g(b)f(z)$$

$$= g(b)f(z)f(\varphi(z)) - g(b)f(z)$$

$$= g(b)f(z)f(b) - g(b)f(z)$$

$$= g(b)f(z) - g(b)f(z)$$

$$=0$$

Hence $p\big(\mathcal{W}_{f,\varphi}\big)g(z) = \big(\mathcal{W}_{f,\varphi}{}^2 - \mathcal{W}_{f,\varphi}\big)g(z) = 0$ (3.1)

This implies that $\mathcal{W}_{f,\varphi}$ is algebraic operator**.**

From equation (3.1) we have the following corollary

Corollary (3.2): Let φ be a holomorphic self-map of U and $f \in \mathbb{H}^\infty$. If φ is a constant map of U $\varphi(z) = b$ where $b \in$ U such that $f(b) = 1$, then $\mathcal{W}_{f,\varphi}$is an idempotent operator on $\mathbb{H}^{\mathbf{2}}$.

Proposition (3.3): Let φ be a holomorphic self-map of U and $f \in \mathbb{H}^\infty$. If φ and f are constant maps, then $rank\ \mathcal{W}_{f,\varphi} = 1$

Proof: Suppose that φ and f are constant maps, then there exist $\beta, \gamma \in U$ such that

$\varphi(z) = \beta, f(z) = \gamma$.Therefore for all $g \in \mathbb{H}^2$ and $z \in U$

$$\mathcal{W}_{f,\varphi} g(z) = T_f C_\varphi g(z)$$

$$= f(z)(g\big(\varphi(z)\big)$$

$$= \gamma g(z).$$

Hence, $\mathcal{W}_{f,\varphi}(\mathbb{H}^2) = 1$. This implies that $rank\ \mathcal{W}_{f,\varphi} = 1$.

Proposition (3.4): Let φ be a holomorphic self-map of U and $f \in \mathbb{H}^\infty$. If $rank\ \mathcal{W}_{f,\varphi} = 1$, then φ is a constant map**.**

Proof*:* Suppose that $rank\ \mathcal{W}_{f,\varphi} = 1$, then there exist $\beta \in \mathbb{C}, h \in \mathbb{H}^2$,such that for all $g \in \mathbb{H}^2$ and $z \in U$, we have

$(\mathcal{W}_{f,\varphi})g(z) = \beta h(z)$. Hence $T_f C_\varphi g(z) = \beta h(z)$**,** implies that $f(z)(g\big(\varphi(z)\big) = \beta h(z)$. If we take $g(z) = z$, we obtain

$f(z)\varphi(z) = \beta_1 h(z)$, for some $\beta_1 \in \mathbb{C}$ (3.2)

Also if $g(z) = z^2$, then

$f(z)\varphi^2(z) = \beta_2 h(z)$, for some $\beta_2 \in \mathbb{C}$ (3.3)

From equations (3.2) and (3.3) we have the following equation $h(z)(\beta_1\varphi(z) - \beta_2) = 0$

Therefore, either $h(z) \equiv 0$ *or* $(\beta_1\varphi(z) - \beta_2) \equiv 0$. If $h(z) \equiv 0$,then $\mathcal{W}_{f,\varphi}$ is a zero operator which is a contradiction with proposition (2.3).

Hence $(\beta_1\varphi(z) - \beta_2) \equiv 0$,this implies that φ is a constant map, as desired .

Now we are ready to give the necessary condition of the finite-rank weighed composition operator.

Theorem (3.5): Let φ be a holomorphic self-map of U and $f \in \mathbb{H}^\infty$. If $\mathcal{W}_{f,\varphi}$ is of finite-rank operator, then φ is a constant map **.**

Proof: Suppose that $\mathcal{W}_{f,\varphi}$ is of finite- rank . Put $rank\ \mathcal{W}_{f,\varphi} = n$.To show that φ is a constant map, the proof will be by induction on n.

If $rank\ \mathcal{W}_{f,\varphi} = 1$ then by proposition (3.4) φ is a constant map. On the other hand , if the theorem is true for $(n-1)$, we shall prove it for n.

Assume that for each $g \in \mathbb{H}^2, z \in U$

$$(\mathcal{W}_{f,\varphi})g(z) = f(z)(g(\varphi(z))$$

$$= \alpha_1 h_1(z) + \alpha_2 h_2(z) + \cdots + \alpha_n h_n(z)$$

for some $h_i \in \mathbb{H}^2$ and $\alpha_i \in \mathbb{C}$, $i = 1,2,\ldots,n$.

If $g(z) = z$, then

$$f(z)\varphi(z) = \beta_1 h_1(z) + \beta_2 h_2(z) + \cdots + \beta_n h_n(z) \qquad (3.4)$$

for some $\beta_i \in \mathbb{C}$, $i = 1,2,\ldots,n$.

If $g(z) = z^2$, then

$$f(z)\varphi^2(z) = \gamma_1 h_1(z) + \gamma_1 h_2(z) + \cdots + \gamma_1 h_n(z) \qquad (3.5)$$

for some $\gamma_i \in \mathbb{C}$, $i = 1,2,\ldots,n$.

From equations (3.4) and (3.5) we get the following equation

$$\varphi(z)[\beta_1 h_1(z) + \cdots + \beta_n h_n(z)] = \gamma_1 h_1(z) + \cdots + \gamma_1 h_n(z)$$

$$h_n(z)(\beta_n\varphi(z) - \gamma_n) = h_1(z)(\gamma_1 - \beta_1\varphi(z)) + \cdots + h_{n-1}(z)(\gamma_{n-1} - \beta_{n-1}\varphi(z))$$

Note that, if $(\beta_n\varphi(z) - \gamma_n) = 0$, then φ is a constant map. But if

$(\beta_n\varphi(z) - \gamma_n) \neq 0$,then for all $z \in U$, we have

$$h_n(z) = \frac{(\gamma_1 - \beta_1\varphi(z))}{(\beta_n\varphi(z) - \gamma_n)} h_1(z) + \cdots + \frac{(\gamma_{n-1} - \beta_{n-1}\varphi(z))}{(\beta_n\varphi(z) - \gamma_n)} h_{n-1}(z)$$

This implies that, h_n is a linear combination of $h_1, h_2, \dots, h_{n-1}$.

Hence, $rank\ \mathcal{W}_{f,\varphi}$ reduces to $n-1$. But we assumed by the induction that the theorem is true for $rank\ \mathcal{W}_{f,\varphi} = n-1$. Thus φ must be constant map.

Recall that an operator T on Hilbert space $\mathcal{H}$ is called compact if T maps each bounded subset of $\mathcal{H}$ into relatively compact one [6].Before we give the next result we need the following theorem [7].

Theorem (3.6):(Finite Rank Approximation)

Suppose that T is an operator on a Hilbert space $\mathcal{H}$ .Then T is compact if and only if there is a sequence $\{F_n\}$ of finite-rank operator such that $\|T - F_n\| \to 0$.

***Corollary (3.7)*:** Let φ be a non-constant holomorphic self-map of U and $f \in \mathbb{H}^\infty$. If $\mathcal{W}_{f,\varphi}$ is a compact operator then there exists a no sequence of finite-rank weighted composition operator on $\mathbb{H}^2$ that converges to $\mathcal{W}_{f,\varphi}$.

Proof: Suppose, that there exists a sequence $\{T_n\}$ of finite-rank weighted composition operator on $\mathbb{H}^2$ that converges to $\mathcal{W}_{f,\varphi}$, i.e. $\lim_{n\to\infty} T_n = \mathcal{W}_{f,\varphi}$ .We note that if

$h \in \mathcal{R}(\mathcal{W}_{f,\varphi})$ then $\mathcal{W}_{f,\varphi} g = h$ for some $g \in \mathbb{H}^2$. Hence $\lim_{n\to\infty} T_n g = h$,

then we have $h \in \bar{\mathcal{R}}(T_n)$. But $\{T_n\}$ is a sequence of finite-rank operators , then $\{\mathcal{R}(T_n)\}$ is a finite-dimensional subspace of $\mathbb{H}^2$ for all n . Thus $\bar{\mathcal{R}}(T_n) = \mathcal{R}(T_n)$ for all n . Then $h \in \mathcal{R}(T_n)$ for all n .This implies that $\mathcal{R}(\mathcal{W}_{f,\varphi}) \subseteq \mathcal{R}(T_n)$ for all n.

Since $\mathcal{R}(T_n)$ is a finite-dimensional subspace of $\mathbb{H}^2$, then $\mathcal{R}(\mathcal{W}_{f,\varphi})$ is a finite-dimensional subspace of $\mathbb{H}^2$, thus by theorem (3.5) φ is constant map , which a contradiction .

4- The Eigenvalue Equation for Weighted Composition Operator Inducing by Koenig's Maps.

If φ is a holomorphic self-map of U, the eigenvalue equation for the composition operator is

$$C_\varphi g = \lambda g \quad or \quad go\varphi = \lambda g \tag{4.1}$$

this is called Schröder's equation [8].

In 1870, Ernst Schröder's pioneering work on iteration of analytic functions. In trying to understand Newton's method in the complex plane, Schröder's arrived at the idea of using iteration to find solutions of equations involving analytic functions.

In 1884, Gabriel Koenigs published his work on solution of Schröder's equation for the class of holomorphic self-maps φ of U that are not conformal automorphisms, that fix a point p $\in$ U and for which $\varphi(p)\acute{} \neq 0$ (so that φ is noncontact . Shapiero in [7] called these maps Koenigs maps.

Remark (4.1): If φ is a Koenigs map,then by its definition , φ is a holomorphic self-map of U with fixed point $0 \in$ U for which $\varphi(p)\acute{} \neq 0$ and not conformal automorphism . Gabriel

Koenigs published basic existence-uniqueness theory for a unique solutions of Schröder's equation (4.1) near a fixed point. Hence by (1.2.13) $0 < |\acute{\varphi}(p)| < 1$.

Theorem (4.2)(Necessary and Uniqueness): If φ is a Koenigs map ,then the eigenvalue of C_φ are $\acute{\varphi}(p)^n$ for some positive integers n, i.e. suppose $g \in \mathbb{H}^2$ is a nonconstant and there is a complex number λ such that g *and* λ satisfy Schröder's equation, then $\lambda = \acute{\varphi}(p)^n$ for some positive integer n .

Moreover ,these are the only eigenvalues and they all have multiplicity one , i.e. up to multiplicity constant , g is the unique solution for $go\varphi = \lambda g$

In what follows we shall try prove the necessary and uniqueness theorem for analytic solution of the eigenvalue equation of weighted composition operator

$$\mathcal{W}_{f,\varphi} g = \lambda g \quad or \;\; f(\,go\varphi) = \lambda g \qquad (4.2)$$

We shall introduce the main theorem in some stage .We need some preliminaries .

Recall that [6] if T be bounded linear operator on Hilbert space, then the spectrum of *T*, denoted by σ(T), is defined as follows*:*

$$\sigma(T) = \{\lambda \in C : T - \lambda I \text{ is not invertible}\}.$$

Moreover ,the point spectrum of T denoted by $\sigma_p(T)$ is the set of all eigenvalue of T.

Remark (4.3): In fact if φ is Koenigs map whose fixed point p is not 0, we apply the case $p = 0$ result to $\psi = \alpha_p o\varphi o\alpha_p^{-1}$ which fixed the origin. Hence,

$$\begin{aligned} \mathcal{W}_{fo\alpha_p,\psi} &= T_{fo\alpha_p} C_\psi \\ &= T_{fo\alpha_p} C_{\alpha_p} C_\varphi C_{\alpha_p}^{-1} \\ &= C_{\alpha_p} T_f C_\varphi C_{\alpha_p}^{-1} \qquad = C_{\alpha_p} \mathcal{W}_{f,\varphi} C_{\alpha_p}^{-1} \end{aligned}$$

Thus $\mathcal{W}_{fo\alpha_p,\psi}$ and $\mathcal{W}_{f,\varphi}$ are similar . But it well-know that two similar operator have same point spectrum and spectrum . Thus by the special automorphism α_p , we translate the result for ψ and $fo\alpha_p$ into results for φ *and* f, respectively .

Lemma (4.4): Suppose φ be a holomorphic self-map of U and $f \in \mathbb{H}^\infty/\{0\}$. If φ is a non-constant map, then $\mathcal{W}_{f,\varphi}$is one-to-one .

Proof : Suppose that φ is non-constant . Let $h_1, h_2 \in \mathbb{H}^2$ such that

$\mathcal{W}_{f,\varphi}(h_1) = \mathcal{W}_{f,\varphi}(h_2)$, then $T_f C_\varphi(h_1) = T_f C_\varphi(h_2)$. Thus, for all $z \in U$

$f(z)h_1(\varphi(z)) = f(z)h_2(\varphi(z))$, then $h_1(\varphi(z)) = h_2(\varphi(z))$. Hence,

$h_1 \equiv h_2$ *on* $\varphi(U)$. Thus by open mapping theorem, we have $h_1 \equiv h_2$ on a non-empty open subset of U. Therefore ,by Taylor theorem $h_1 = h_2$ .Thus $\mathcal{W}_{f,\varphi}$is one- to-one operator .

Lemma (4.5): Let φ be a holomorphic self-map of U such that φ fixes $p \in U$ and $f \in \mathbb{H}^{\infty}$. If φ is neither a constant nor a conformal automorphism of U and $\mathcal{W}_{f,\varphi}(g) = \lambda g$ for some $g \in \mathbb{H}^{\mathbf{2}}, g \neq 0$ and $\lambda \in \mathbb{C}$, then

1. $\lambda \neq 0$.

2. If g is non-constant function, then $\lambda \neq f(p)$ and $g(p) = 0$.

Proof : 1. Since φ is non-constant, then by Lemma (4.4) we obtain $\mathcal{W}_{f,\varphi}$ is one-to-one. If we assume that $\lambda = 0$, then for some $g \neq 0$

$$\mathcal{W}_{f,\varphi}(g) = f(z).g(\varphi(z)) = \lambda g(z) = 0.$$

But $\mathcal{W}_{f,\varphi}$ is one-to-one, then $g = 0$ which is contradiction. Hence, $\lambda \neq 0$.

2. Assume that $\lambda = f(p)$, then

$$\mathcal{W}_{f,\varphi}(g) = f(z).g(\varphi(z)) = \lambda g(z) = f(p)g(z).\text{ Hence,}$$

$$\left(\mathcal{W}_{f,\varphi}(g(z))\right)^{\boldsymbol{n}} = (f(p))^{\boldsymbol{n}}(g(z))^{\boldsymbol{n}}$$

$$= T_{f\ (f\ \circ\varphi\ )(f\ \circ\varphi_2)\ldots(f\ \circ\varphi_{n-1})}C_{\varphi_n}(g(z)).$$

Since, φ is non-automorphism of U, then by Grand Iteration theorem [] we obtain $\varphi_n \xrightarrow{\kappa} p$.

Therefore for each $z \in U$, we have

$$(f(p))^{\boldsymbol{n}}(g(z))^{\boldsymbol{n}} = \left(\mathcal{W}_{f,\varphi}(g(z))\right)^{\boldsymbol{n}}$$

$$= f(z)f(\varphi(z)) \ldots f(\varphi_{n-1}(z))g(\varphi_n(z))$$

$$\xrightarrow{k} (f(p))^{\boldsymbol{n}}g(p).$$

Hence, $\left(\mathcal{W}_{f,\varphi}(g(z))\right)^{\boldsymbol{n}}$ converges to $(f(p))^{\boldsymbol{n}}(g(z))^{\boldsymbol{n}}$ and $(f(p))^{\boldsymbol{n}}g(p)$ in U.Therefore since the limit is unique and $f(p) \neq 0$ (by part (1)), then g is a constant function, which contradicts our hypothesis .

Now to see that $g(p) = 0$, note that from eigenvalue equation

$f(z).g(\varphi(z)) = \lambda g(z)$, set $z = p$ to get $f(p).g(\varphi(p)) = f(p).g(p) = \lambda g(p)$.
which since $\lambda \neq f(p)$, gives the desired result.

In following theorem proven the existence for analytic solution of the eigenvalue equation (4.1) of weighted composition operator .

Theorem (4.6): Let φ Koenigs map of U that fixed $p \in U$ and $f \in \mathbb{H}^{\infty}$. Suppose that $\mathcal{W}_{f,\varphi}(g) = \lambda g$ for some $g \in \mathbb{H}^{\mathbf{2}}$ such that g is not constant function and $\lambda \in \mathbb{C}$. Then , $\lambda = f(p)\left(\acute{\varphi}(p)\right)^{n}$ for some $n = 0,1,2, \ldots$.

Proof Without loss of generality we give the proof for the special case when $p = 0$ (by remark (4.3)). Since g is not a constant function, then by lemma(4.5) (2) we have

$g(0) = 0$. Therefore, the Taylor expansion of g is

$g(z) = a_n z^n + a_{n+1} z^{n+1} + \cdots$ $\quad where\ a_n \neq 0$. By solving eigenvalue equation $\mathcal{W}_{f,\varphi}(g) = f.(g \circ \varphi) = \lambda g$ we obtain for each $z \in U$ that

$$\lambda = \frac{f(z).(g(\varphi(z))}{g(z)}$$

$$= f(z)\left(\frac{\varphi(z)}{z}\right)^n \frac{a_n + a_{n+1}\varphi(z) + a_{n+2}\varphi(z)^2 + \cdots}{a_n + a_{n+1}z \quad + a_{n+2}z^2 + \cdots}$$

$$\lambda = \lim_{z\to 0} \lambda = \lim_{z\to 0} f(z). \lim_{z\to 0}\left(\frac{\varphi(z)}{z}\right)^n \frac{a_n + a_{n+1}\varphi(z) + a_{n+2}\varphi(z)^2 + \cdots}{a_n + a_{n+1}z \quad + a_{n+2}z^2 + \cdots}$$

$$\lambda = f(p)\big(\acute{\varphi}(p)\big)^n$$

Lemma (4.7): Suppose that φ is a Koenigs of U that fixed $p \in U$ and $f \in \mathbb{H}^\infty$**.** If $\acute{\varphi}(p) = 0$, then $f(p)$ is the only eigenvalue of $\mathcal{W}_{f,\varphi}$.

Proof: Assume that $\lambda \neq f(p)$, λ is the eigenvalue for $\mathcal{W}_{f,\varphi}$. But from the proof of lemma(4.5)(2) the only eigenfunction corresponding to $f(p)$ is the constant multiple of f, therefore the eigenfunction of λ is the constant multiple of f.

Hence, by theorem(3.6) $\lambda = f(p)\big(\acute{\varphi}(p)\big)^n$ for some nonnegative integer n.

But $\acute{\varphi}(p) = 0$, then $\lambda = 0$, which is contradiction lemma(4.5)(1), hence $\lambda = f(p)$.

It is well-known [7] that the non-zero spectral points of compact operator are precisely the non-zero eigenvalues .Also every compact operator on infinite dimensional Hilbert space is not invertible, then zero lies in its spectrum . Therefore from theorem (4.6) and lemma (4.7) we can provide the following description of the spectrum of a compact weighted composition operator .

Corollary (4.8): Let φ be a holomorphic self-map of U and $f \in \mathbb{H}^\infty$**.** If $\mathcal{W}_{f,\varphi}$ is compact operator on $\mathbb{H}^{\mathbf{2}}$ and $p \in U$ is the fixed point of φ, then

$\sigma(\mathcal{W}_{f,\varphi}) = \{0, f(p)\} \cup \{f(p)\big(\acute{\varphi}(p)\big)^n : n = 1,2, \ldots\ \}$.

Remark (4.9): If $\mathcal{W}_{f,\varphi}$ has an eigenfunction which is constant function, then by lemma (4.5)(1) and theorem (4.6) $\lambda = f(p)\acute{\varphi}(p) \neq 0$.

The next theorem discuss the uniqueness of eigenvalue equation (4.1) .

Theorem (4.10): Suppose that φ is a Koenigs of U that fixed $p \in U$ and $f \in \mathbb{H}^\infty$ and λ is an eigenvalue for $\mathcal{W}_{f,\varphi}$, then the multiplicity of λ is one .

Proof : To prove the multiplicity of λ is 1, we show that if g is an eigenfunction for $\mathcal{W}_{f,\varphi}$ corresponding to λ and $\mathcal{W}_{f,\varphi}(h) = \lambda h$,for some $h \in \mathbb{H}^{\mathbf{2}}$, then h is constant multiple of g. Without loss of generality we assume may that $\varphi(0) = 0$ and

$f(0) \neq 0$. If $\lambda = f(0)$ then by lemma (4.5) (2) we see that the only eigenfunctions for the eigenvalue $f(0)$ are the constant function , hence the theorem is true when $\lambda = f(0)$.

Now, if $\lambda \neq f(0)$, by differentiating both sides of eigenvalue equation

$f(z).\, g\big(\varphi(z)\big) = \lambda g(z) \qquad (z \in U)$, we get

$$f(z).\, \acute{g}(\varphi(z)).\, \acute{\varphi}(z) + \acute{f}(z).\, g\big(\varphi(z)\big) = \lambda \acute{g}(z)$$

and evaluate the result at $z = 0$ and $\lambda = f(0)\acute{\varphi}(0)$, awe have

$$(f(0))^2.\, \acute{g}(\varphi(0)).\, \acute{\varphi}(0) + \acute{f}(0).\, g\big(\varphi(0)\big) = \lambda \acute{g}(0)\ .$$

Moreover ,

$$f(z).\, [\acute{g}(\varphi(z)).\, \varphi''(z) + g''\big(\varphi(z)\big).\, (\acute{\varphi}(z))^2] + \acute{f}(z).\, g'\big(\varphi(z)\big)\acute{\varphi}(z) + \acute{f}(z).\, g'\big(\varphi(z)\big)\acute{\varphi}(z) + f''(z).\, g\big(\varphi(z)\big) = \lambda g''(z)$$

and evaluate the result at z=0 and $\lambda = f(0)\acute{\varphi}(0)$ we get

$$f(0).\, \acute{g}(\varphi(0)).\, \varphi''(0) + \lambda.\, \varphi'(0).\, g''\big(\varphi(0)\big) + 2.\, \acute{f}(0).\, g'\big(\varphi(0)\big)\varphi'(0) + f''(0).\, g\big(\varphi(0)\big) = \lambda g''(0)\ . \qquad (2.13)$$

But by lemma (4.5)(2) $g(0) = 0$, then we obtain from equation (2.13) that

$$\lambda[\varphi'(0) - 1]g''(0) = -\big[f(0)\varphi''(0) + 2\acute{f}(0)\varphi'(0)\big]g'(0)\ .$$

Thus the calculation shows that for every $n \geq 2$, the quantity

$\lambda[\varphi^{(n-1)}(0) - 1]g^{(n)}(0)$ is given by an expression that involves the derivatives $\varphi^{(m)}(0)$ for $1 \leq m \leq n$, $g^{(k)}(0)$ for $1 \leq k \leq n-1$ and $f^{(j)}(0)$ for $1 \leq j \leq n-1$.

But by remark (2.3.8) $\lambda \neq 0$ and moreover $\lambda \neq f(0)$, then the induction argument shows that for $n \geq 2$, $g^{(n)}(0)$ is determined solely by $\varphi, f'\, and\ g'(0)$.

So given φ $and\ f$ the coefficients $g^{(n)}(0)\ of\ g$ in its Taylor expansion about 0 are determined solely by $g'(0)$ (note that $g(0) = 0$).

Similarly, the coefficients for every eigenfunction h in its Taylor expansion about 0 determine solely by $h'(0)$ (note that $h(0) = 0$ by lemma(4.5)), hence h is a constant multiple of g .

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