Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.6, No.3, 2016

Oscillation of First Order Neutral Integro-Differential Equations

Hussain Ali Mohamad[1], and Taghreed Hussein Abed[2]
1-2: University of Baghdad, College of Science for Women, Department of Mathematics, Baghdad, IRAQ

Abstract
In this paper necessary and sufficient conditions were obtained to insure that every solution of neutral integro-differential equations oscillates these results improve and generalized Lemma 2.1, Theorem 2.2, Theorem 2.3 in Olach(2005).
Key words: Integro-differential equation, neutral equation.

1. Introduction

Consider neutral integro-differential equations of the form:

$$\left[x(t)-p(t)x\big(\tau(t)\big)\right]'+\delta\int_0^t x(t-s)dr(t,s)=0 \quad t\geq 0,\ \ \delta=\pm 1 \tag{1.1}$$

where $p(t)$ is continuous real-valued functions on the interval $[0,\infty)$, and the integral is in the sense of Riemann-Stieltjes, under the standing hypotheses:

(H1) $r(t;s)$ is increasing with respect to s for $s\in[0,t]$.

(H2) $r(t;t):[0,\infty)\to R$ is continuous; $g(t)=r(t,t)-r(t,0)$

(H3) $\sigma(t):[0,\infty)\to(0,\infty)$ is continuous such that $0<\sigma(t)<t$,

$t-\sigma(t)$ is nondecreasing, $\lim_{t\to\infty} t-\sigma(t)=\infty$.

A continuous real-valued function $x(t)$ defined on the real line R will be called a solution of the neutral integro-differential equation (1.1) if the function $x(t)-p(t)x\big(\tau(t)\big)$ is a continuously differentiable real-valued function for $t\geq 0$ and $x(t)$ satisfies (1.1) for all $t\geq 0$. The purpose of this paper is to obtain some necessary and sufficient conditions for all solution of equation (1.1) to oscillates. Olach(2005) study the integro-differential equations

$$x'(t)+\int_0^t x(t-s)dr(t,s)=0\ , \qquad t\geq 0 \tag{1.2}$$

and established some necessary and sufficient conditions to insure the oscillation of all solutions of eq.(1.2). When $p(t)\equiv 0$ then eq.(1.1) reduce to eq.(1.2), so the results in this paper generalized Lemma 2.1, Theorem 2.2 and Theorem 2.3 in Olach(2005).

Lemma 1.1 (Ladde et al. (1987) **)**

Assume that $\tau(t)>t$, and

$$\liminf_{t\to\infty}\int_t^{\tau(t)} p(s)ds>\frac{1}{e}$$

where $p(t),\tau(t)\in C\big([t_0,\infty);[0,\infty)\big)$ Then

(i) The differential inequality $y'(t) - p(t)y\big(\tau(t)\big) \geq 0,\ t \geq t_0$
has no eventually positive solution.

(ii) The differential inequality $y'(t) - p(t)y(\tau(t)) \leq 0,\ t \geq t_0$
has no eventually negative solution.

2. Main Results

Before start in establishing the results, we give some of lemmas which are useful for extracting results. Let $\delta = 1$, and

$$z(t) = x(t) - p(t)x\big(\tau(t)\big) \tag{2.1}$$

then eq.(1.1) reduce to

$$z'(t) + \int_0^t x(t-s)dr(t,s) = 0, \quad t \geq 0, \tag{2.2}$$

Lemma 2.1 Suppose that H1 − H3, hold and

$$\liminf_{t\to\infty} \int_{t-\sigma(t)}^{t} \big[r(s,s) - r\big(s,\sigma(s)\big)\big]ds > 0 \tag{2.3}$$

and let $x(t)$ be a positive (or negative) solution of equation (2.2) on $[0,\infty)$ then there exist $T > 0$ such that

$$\frac{z(t-\sigma(t))}{z(t)}$$

is bounded on $[T,\infty)$.

Proof. Assume that $x(t)$ is positive for $t \in [0,\infty)$ then from eq.(2.2) we get

$$z'(t) = -\int_0^t x(t-s)dr(t,s) \leq 0, \qquad t \geq 0$$

hence $z(t)$ must be eventually monotonically decreasing, and we have two cases to consider for $z(t)$:

1. $z(t) < 0,\quad t \geq t_0 \geq 0$; 2. $z(t) > 0,\quad t \geq t_0 \geq 0$

Case 1. $z(t) < 0,\ z'(t) \leq 0, \qquad t \geq t_0 > 0$

$$z\big(t-\sigma(t)\big) > z(t)$$

then

$$0 < \frac{z\big(t-\sigma(t)\big)}{z(t)} < 1$$

Case 2. $z(t) > 0,\ z'(t) \leq 0, \qquad t \geq t_0 \geq 0.$

Since $x(t) \geq z(t)$

$$-z'(t) \geq \int_0^t z(t-s)dr(t,s) \geq \int_{\sigma(t)}^t z(t-s)dr(t,s), \qquad t \geq t_0 \tag{2.4}$$

$$-z'(t) \geq \big[r(t,t) - r\big(t,\sigma(t)\big)\big]z\big(t-\sigma(t)\big) \tag{2.5}$$

from (2.3), there exist $c > 0$ and $T > 0$ such that

$$\int_{t-\sigma(t)}^{t} \big[r(s,s) - r\big(s,\sigma(s)\big)\big]ds \geq c > 0$$

for $t \in [T,\infty)$ we can find $t^* \in [t-\sigma(t), t]$ such that

$$\int_{t-\sigma(t)}^{t^*}[r(s,s)-r(s,\sigma(s))]d\,s \geq \frac{c}{2}$$

and

$$\int_{t^*}^{t}[r(s,s)-r(s,\sigma(s))]ds \geq \frac{c}{2}$$

By integrating (2.5) from $t-\sigma(t)$ to t^* yields

$$-\int_{t-\sigma(t)}^{t^*} z'(s)ds \geq \int_{t-\sigma(t)}^{t^*}[r(s,s)-r(s,\sigma(s))]z(s-\sigma(s))\,ds$$

$$z(t-\sigma(t))-z(t^*) \geq \frac{c}{2}z(t^*-\sigma(t^*))$$

$$z(t-\sigma(t)) \geq \frac{c}{2}z(t^*-\sigma(t^*)) \tag{2.6}$$

Similarly integrating (2.5) from t^* to t we get

$$-\int_{t^*}^{t} z'(s)ds \ \geq \int_{t^*}^{t}[r(s,s)-r(s,\sigma(s))]z(s-\sigma(s))\,ds$$

$$-z(t)+z(t^*) \geq z(t-\sigma(t))\frac{c}{2}$$

$$z(t^*)-z(t) \ \geq \frac{c}{2}z(t-\sigma(t))$$

$$z(t^*) \ \geq \frac{c}{2}z(t-\sigma(t)) \tag{2.7}$$

Combining the inequalities (2.6) and (2.7) we obtain

$$z(t^*) \ \geq \frac{c}{2}z(t-\sigma(t)) \geq \frac{c^2}{4}z(t^*-\sigma(t^*))$$

$$\frac{z(t^*-\sigma(t^*))}{z(t^*)} \leq \frac{4}{c^2}.$$

The proof is complete.

Theorem 2.2 Assume that H1 − H3, hold, $0<p(t)\leq p$, $\tau(t)<t$, and (2.3) holds, if

$$\liminf_{t\to\infty}\int_{t}^{\tau^{-1}(t)}\int_{0}^{v}\frac{dr(v,s)}{p(\tau^{-1}(v-s))}dv > \frac{1}{e} \tag{2.8}$$

$$\liminf_{t\to\infty}\frac{\int_0^t \exp(\lambda\int_{t-\sigma(t)}^{t} g(\rho)d\rho)dr(t,s)}{g(t)} > \lambda \tag{2.9}$$

for all $\lambda>0$. Then every solution of equation (2.2) oscillates on $[0,\infty)$.

Proof. For the sake of contradiction assume that $x(t)$ is an eventually positive solution of eq.(2.2), (the proof of the case when $x(t)$ is an eventually negative is similar and will be omitted). Hence for $t\geq t_0\geq 0$, let $x(\tau(t))>0$, $x(t-s)>0$.

From (2.2) it follows that $z(t)$ is nonincreasing function.

We have two cases to consider for $z(t)$:

1. $z(t)>0$, $t\geq t_1\geq t_0$; 2. $z(t)<0$, $t\geq t_1\geq t_0$

Case 1. $z(t)>0$, $z'(t)\leq 0$, $t\geq t_1\geq t_0$

Since $x(t)\geq z(t)$ then from eq.(2.2) and using the decreasing nature of $z(t)$ we obtain

$$z'(t) + z(t)[r(t,t) - r(t,0)] \le 0, \qquad t \ge t_1 \ge 0$$

and by (H2) we get

$$z'(t) + z(t)g(t) \le 0 \tag{2.10}$$

Set $\Omega = \{\lambda > 0 : z'(t) + \lambda z(t)g(t) \le 0\}$.

It is obvious that $1 \in \Omega$, so Ω is non-empty set. By Lemma 2.1, it follows that

$$\frac{z\big(t-\sigma(t)\big)}{z(t)} \le \alpha \quad \text{for} \quad t \ge T. \tag{2.11}$$

Where $\alpha > 0$ is a constant.

From H1, H2 we get

$$r(t,t) - r\big(t,\sigma(t)\big) \le r(t,t) - r(t,0), \qquad t \ge t_1$$

So (2.3) implies that

$$\liminf_{t\to\infty} \int_{t-\sigma(t)}^{t} g(s)ds > 0.$$

We can choose k large enough such that $e^{k\alpha} > \alpha$,

$$0 < k \le \int_{t-\sigma(t)}^{t} g(s)ds \quad \text{for } t \ge t_2 \ge t_1.$$

Now we claim that $\sup \Omega \le \alpha < \infty$. Otherwise $\sup \Omega > \alpha$ which means that $\alpha \in \Omega$, then it follows from

$$\frac{d}{dt}[z(t)\mathrm{e}^{\alpha\int_{t_1}^{t} g(s)ds}] = [z'(t) + \alpha\, r(t,t)z(t)]\, \mathrm{e}^{\alpha\int_{t_2}^{t} g(s)ds} \le 0$$

which means that the function

$$z(t)\mathrm{e}^{\alpha\int_{t_2}^{t} r(s,s)ds}$$

is nonincreasing on $[t_1, \infty)$. Hence

$$z(t-\sigma(t))\mathrm{e}^{\alpha\int_{t_2}^{t-\sigma(t)} g(s)ds} \ge z(t)\mathrm{e}^{\alpha\int_{t_2}^{t} g(s)ds}$$

$$z\big(t-\sigma(t)\big) \ge z(t)\mathrm{e}^{\alpha\int_{t-\sigma(t)}^{t} g(s)ds} \ge z(t)e^{k\alpha} > \alpha z(t), \qquad t \ge t_2$$

Then

$$\frac{z\big(t-\sigma(t)\big)}{z(t)} > \alpha$$

for $t \ge t_2$, which contradicts (2.11). Thus

$$\sup \Omega \le \alpha.$$

Suppose $\lambda^* = \sup \Omega$ and let $\mu \in (0, \lambda^*)$ then $\mu \in \Omega$, moreover

$$\lambda^* - \mu = \beta \in \Omega.$$

Hence there exists $t_3 \ge t_2$ such that

$$z'(t) + \beta g(t)z(t) \le 0 \quad \text{for} \quad t \ge t_3.$$

Then for any t, s with $t \ge t_3$, $0 \le s \le t$, and by using the last inequality we get

$$\frac{z(t-s)}{z(t)} = \exp(-\ln\frac{z(t)}{z(t-s)}) = \exp(-\int_{t-s}^{t}\frac{z'(\rho)}{z(\rho)}d\rho)$$

$$\ge \exp\left(\beta\int_{t-s}^{t} g(\rho)d\rho\right) \ge \exp\left(\beta\int_{t-\sigma(s)}^{t} g(\rho)d\rho\right)$$

Where $\sigma(t) < t$, that is

$$z(t-s) \geq z(t)\exp\left(\beta \int_{t-\sigma(s)}^{t} g(\rho)d\rho\right)$$

Substituting the last inequality in (2.4) yields

$$0 \geq z'(t) + \int_0^t \exp\left(\beta \int_{t-\sigma(s)}^{t} g(\rho)d\rho\right) dr(t,s)z(t)$$

$$= z'(t) + \frac{\int_0^t \exp\left(\beta \int_{t-\sigma(s)}^{t} d\rho\, g(\rho)\right) dr(t,s)}{g(t)} g(t)z(t). \qquad (2.12)$$

According to (2.12) we claim that

$$\liminf_{t\to\infty} \frac{\int_0^t \exp(\beta \int_{t-\sigma(s)}^{t} g(\rho)d\rho)dr(t,s)}{g(t)} \leq \lambda^* \qquad (2.13)$$

otherwise there exist $\lambda_1^* > \lambda^*$ and $T \geq t_3$ such that

$$\frac{\int_0^t \exp(\beta \int_{t-\sigma(s)}^{t} g(\rho)d\rho)dr(t,s)}{g(t)} \geq \lambda_1^*$$

for all $t \geq T$ and therefore (2.12) reduce to

$$0 \geq z'(t) + \lambda_1^* g(t)z(t) \quad \text{for} \quad t \geq T$$

hence $\lambda_1^* \in \Omega$, which contradicts the hypothesis that $\lambda_1^* > \lambda^*$. Thus (2.13) has been established. Finally (2.13) implies that

$$-\lambda^* + \liminf_{t\to\infty} \frac{\int_0^t \exp[(\lambda^* - \mu) \int_{t-\sigma(s)}^{t} g(\rho)d\rho]dr(t,s)}{g(t)} \leq 0$$

as $\mu \in (0, \lambda^*)$ is arbitrary we obtain

$$-\lambda^* + \liminf_{t\to\infty} \frac{\int_0^t \exp\left(\lambda^* \int_{t-\sigma(s)}^{t} g(\rho)d\rho\right) dr(t,s)}{g(t)} \leq 0$$

which contradicts (2.9).

Case 2. $z(t) < 0, \quad z'(t) \leq 0, \quad t \geq t_0 > 0$

From (2.1) we get

$$x(\tau(t)) \geq \frac{-z(t)}{p(t)}, \qquad \text{then} \quad x(t) \geq \frac{-z(\tau^{-1}(t))}{p(\tau^{-1}(t))}$$

from eq.(2.2) we obtain

$$z'(t) - \int_0^t \frac{z(\tau^{-1}(t-s))}{p(\tau^{-1}(t-s))} dr(t,s) \leq 0$$

$$z'(t) - z(\tau^{-1}(t)) \int_0^t \frac{dr(t,s)}{p(\tau^{-1}(t-s))} \leq 0$$

By Lemma 2.1-ii and condition (2.11) it follows that the last inequality cannot have eventually negative solution which a contradiction.

Example 2.3 Consider neutral integro-differential equation

$$\left[x(t)-\frac{1}{2}x(t-2\pi)\right]'+\int_0^t x(t-s)dr(t,s)=0 \tag{2.14}$$

Where $p(t)=\frac{1}{2},\ r(t,s)=\frac{1}{2}(t+s),\ \tau(t)=t-2\pi,\ \sigma(t)=\frac{t}{3},\ \lambda=2.$

One can see that all conditions of Theorem 2.2 met as follows

$$\liminf_{t\to\infty}\int_{t-\sigma(t)}^{t}\left[r(s,s)-r\big(s,\sigma(s)\big)\right]ds=\lim_{t\to\infty}\int_{\frac{2t}{3}}^{t}\frac{s}{3}ds=\frac{5}{54}\lim_{t\to\infty}(t^2)=\infty$$

$$\liminf_{t\to\infty}\int_{t}^{\tau^{-1}(t)}\int_0^v\frac{dr(t,s)}{p\big(\tau^{-1}(t-s)\big)}dv=\lim_{t\to\infty}\int_t^{t+2\pi}\int_0^v ds\,dv=\infty$$

$$\liminf_{t\to\infty}\frac{\int_0^t\exp(\lambda\int_{s-\sigma(s)}^{s}g(\rho)d\rho)dr(t,s)}{g(t)}=\lim_{t\to\infty}\frac{\int_0^t e^{\frac{\lambda}{2}\int_{\frac{2s}{3}}^{s}\rho d\rho}ds}{t}=\infty$$

So according to Theorem 2.2 every solution of (2.14) oscillates. For instance $x(t)=\cos t$ is such a solution.

Now, Set $\delta=-1$ then eq.(1.1) reduce to

$$z'(t)-\int_0^t x(t-s)dr(t,s)=0,\qquad t\geq 0 \tag{2.15}$$

Lemma 2.4 Suppose that $\mathrm{H1}-\mathrm{H3}$ hold, $0<p(t)\leq p$, and

$$\limsup_{t\to\infty}\int_{t-\sigma(t)}^{t}(r(s,s)-r(s,\sigma(s)))ds<\infty \tag{2.16}$$

Let $x(t)$ be an eventually positive (or eventually negative) solution of equation (2.15) on $[0,\infty)$. Then there exist $T>0$ such that

$$\frac{z(t-\sigma(t))}{z(t)}$$

is bounded on $[T,\infty)$.

Proof. Assume that $x(t)$ is positive for $t\in[0,\infty)$, then from eq.(2.15) we get

$$z'(t)=\int_0^t x(t-s)\,dr(t,s)\geq 0,\qquad t\geq 0$$

hence $z(t)$ is an eventually monotonically increasing function. we have two cases to consider for $z(t)$:

1. $z(t)>0,\quad t\geq t_0\geq 0$ 2. $z(t)<0,\quad t\geq t_0\geq 0$

Case 1. $z(t)>0,\ z'(t)\geq 0,\quad t\geq t_0\geq 0$

$$z\big(t-\sigma(t)\big)\leq z(t)$$

then

$$0<\frac{z\big(t-\sigma(t)\big)}{z(t)}<1$$

Case 2. $z(t)<0,\ z'(t)\geq 0,\quad t\geq t_0\geq 0$

then

$$z(t)\geq -p(t)x\big(\tau(t)\big)$$

$$x(t-s) \geq -\frac{z(\tau^{-1}(t-s))}{p(\tau^{-1}(t-s))}$$

from eq.(2.15) we get

$$z'(t) = \int_0^t x(t-s)dr(t,s) \geq -\int_{\sigma(t)}^t \frac{z(\tau^{-1}(t-s))}{p(\tau^{-1}(t-s))}dr(t,s)$$

$$-z'(t) \leq \int_{\sigma(t)}^t \frac{z(\tau^{-1}(t-s))}{p(\tau^{-1}(t-s))}dr(t,s)$$

then

$$-z'(t) \leq \frac{z(\tau^{-1}(0))}{p}\left[r(t,t) - r(t,\sigma(t))\right]$$

then there exists t_1 such that $t-\sigma(t) \geq t_1,\ t_1 = \max\{t_0, \tau^{-1}(0)\}$ the last inequality leads to

$$-z'(t) \leq \frac{z(t-\sigma(t))}{p}\left[r(t,t) - r(t,\sigma(t))\right], \qquad t \geq t_1 \tag{2.17}$$

from (2.16) there exist $c > 0$ and $T > 0$ such that

$$\int_{t-\sigma(t)}^t \left[r(s,s) - r(s,\sigma(s))\right]ds \leq c$$

for $t \in [T,\infty)$ we can find $t^* \in [t-\sigma(t),t]$ such that

$$\int_{t-\sigma(t)}^{t^*} \left[r(s,s) - r(s,\sigma(s))\right]ds \leq \frac{c}{2}$$

and

$$\int_{t^*}^{t} \left[r(s,s) - r(s,\sigma(s))\right]ds \leq \frac{c}{2}$$

By integrating (2.17) from $t-\sigma(t)$ to t^* we find

$$-\int_{t-\sigma(t)}^{t^*} z'(s)ds \leq \frac{1}{p}\int_{t-\sigma(t)}^{t^*} z(s-\sigma(s))\left[r(s,s) - r(s,\sigma(s))\right]ds$$

$$-z(t^*) + z(t-\sigma(t)) \leq \frac{z(t^*-\sigma(t^*))}{p}\int_{t-\sigma(t)}^{t^*} \left[r(s,s) - r(s,\sigma(s))\right]ds$$

$$-z(t^*) + z(t-\sigma(t)) \leq \frac{c}{2p} z(t^*-\sigma(t^*))$$

$$z(t-\sigma(t)) \leq \frac{c}{2p} z(t^*-\sigma(t^*)) \tag{2.18}$$

Similarly by integrating (2.17) from t^* to t we find

$$-\int_{t^*}^{t} z'(s)ds \leq \frac{1}{p}\int_{t^*}^{t} z(s-\sigma(s))\left[r(s,s) - r(s,\sigma(s))\right]ds$$

$$-z(t) + z(t^*) \leq \frac{c}{2p} z(t-\sigma(t))$$

$$z(t^*) \leq \frac{c}{2p} z(t-\sigma(t)) \tag{2.19}$$

Combining the inequalities (2.18) and (2.19) we obtain

$$z(t^*) \leq \frac{c^2}{2p^2} z(t^*-\sigma(t^*))$$

$$\frac{z\big(t^* - \sigma(t^*)\big)}{z(t^*)} \leq \frac{2p^2}{c^2}.$$

The proof is complete.

Theorem 2.5 Assume that $\text{H1} - \text{H3}$ hold and $0 < p(t) \leq p$, $\tau(t) < t$, and (2.16) holds, if

$$\liminf_{t\to\infty} \int_t^{\tau^{-1}(t)} \int_0^v \frac{dr(t,s)}{p\big(\tau^{-1}(t-s)\big)} dv > \frac{1}{e} \tag{2.20}$$

$$\limsup_{t\to\infty} \int_0^t g(s)ds = \infty \tag{2.21}$$

$$\liminf_{t\to\infty} \frac{1}{g(t)} \int_0^t \frac{\exp\left(\frac{\lambda}{p}\int_{t-\sigma(s)}^t g(\rho)d\rho\right)}{p\big(\tau^{-1}(t-s)\big)} dr(t,s) > \frac{\lambda}{p} \tag{2.22}$$

for all $\lambda > 0$. Then every solution $x(t)$ of equation (2.15) either oscillates or $|x(t)| \to \infty$ as $t \to \infty$.

Proof. For the sake of contradiction assume that $x(t)$ is bounded eventually positive solution of eq.(2.15), (the proof of the case when $x(t)$ is an eventually negative is similar and will be omitted). Hence for $t \geq t_0 \geq 0$, let $x\big(\tau(t)\big) > 0,\ x(t-s) > 0$.

From (2.15) it follows that $z(t)$ is nondecreasing function.

We have two cases to consider for $z(t)$:

1. $z(t) > 0,\ \ t \geq t_1 \geq t_0$; 2. $z(t) < 0,\ \ t \geq t_1 \geq t_0$

Case 1. $z(t) > 0,\ \ z'(t) \geq 0,\ \ t \geq t_1 \geq t_0$

Let $\lim_{t\to\infty} z(t) = L,\ 0 < L \leq \infty$

if $L = \infty$, then $\lim_{t\to\infty} x(t) = \infty$, since $x(t) \geq z(t)$, which is a contradiction.

if $0 < L < \infty$, from (2.5) we get

$$x(t) \geq z(t)$$

substituting in eq.(2.15) we obtain

$$z'(t) - \int_0^t z(t-s)dr(t,s) \geq 0$$

$$z'(t) - z(t)g(t) \geq 0$$

By integrating the last inequality from 0 to t we find

$$z(t) - z(0) \geq z(0)\int_0^t g(s)ds.$$

as $t \to \infty$ then the last inequality implies that

$$\limsup_{t\to\infty} \int_0^t g(s)ds < \infty,$$

which is a contradiction.

Case 2. $z(t) < 0,\ \ z'(t) \geq 0,\ \ t \geq t_1 \geq t_0$

from (2.5) we get $z(t) \geq -p(t)x\big(\tau(t)\big)$ then

$$x\big(\tau(t)\big) \geq -\frac{z(t)}{p(t)}$$

$$x(t-s) \geq -\frac{z(\tau^{-1}(t-s))}{p(\tau^{-1}(t-s))}$$

Substituting in eq.(2.15) we obtain

$$z'(t) + \int_0^t \frac{z(\tau^{-1}(t-s))}{p(\tau^{-1}(t-s))} dr(t,s) \geq 0$$

$$z'(t) + \int_0^t \frac{z(t-s)}{p(\tau^{-1}(t-s))} dr(t,s) \geq 0, \qquad \tau(t) > t, \qquad (2.23)$$

$$z'(t) + z(0)\int_0^t \frac{1}{p(\tau^{-1}(t-s))} dr(t,s) \geq 0.$$

Since $p(t) \leq p$, then by using the increasing nature of $z(t)$ and (H2) we get from the last inequality

$$z'(t) + \frac{1}{p} z(t)g(t) \geq 0, \qquad t \geq t_1 \geq 0,$$

Set $\Omega = \left\{\lambda > 0 : z'(t) + \frac{\lambda}{p} z(t)g(t) \geq 0\right\}$.

It is obvious that $1 \in \Omega$, and so Ω is non-empty set. By Lemma 2.4, it follows that

$$\frac{z(t-\sigma(t))}{z(t)} \leq \alpha \quad \text{for} \quad t \geq T. \qquad (2.24)$$

Where $\alpha > 0$ is a constant, we can choose k large enough such that $e^{\frac{k\alpha}{p}} > \alpha$, and

$$0 < k \leq \int_{t-\sigma(t)}^t g(s)ds \ \text{ for } \ t \geq t_2 \geq t_1.$$

Now we claim that $\sup \Omega \leq \alpha < \infty$. Otherwise $\sup \Omega > \alpha$ which means that $\alpha \in \Omega$ and we obtain

$$\frac{d}{dt}[z(t)e^{\frac{\alpha}{p}\int_{t_2}^t g(s)ds}] = [z'(t) + \frac{\alpha}{p} r(t,t)z(t)] \, e^{\frac{\alpha}{p}\int_{t_2}^t g(s)ds} \geq 0$$

since $\alpha \in \Omega$, hence the function

$$z(t)e^{\frac{\alpha}{p}\int_{t_2}^t g(s)ds}$$

is nondecreasing on $[t_2, \infty)$. Hence

$$z(t-\sigma(t))e^{\frac{\alpha}{p}\int_{t_2}^{t-\sigma(t)} g(s)ds} \leq z(t)e^{\frac{\alpha}{p}\int_{t_2}^t g(s)ds}$$

$$z(t-\sigma(t)) \leq z(t)e^{\frac{\alpha}{p}\int_{t-\sigma(t)}^t g(s)ds} \leq z(t)e^{\frac{k\alpha}{p}} < \alpha z(t), \qquad t \geq t_2$$

Then

$$\frac{z(t-\sigma(t))}{z(t)} > \alpha$$

for $t \geq t_2$, which contradicts with (2.24). Thus

$$\sup \Omega \leq \alpha.$$

Suppose $\lambda^* = \sup \Omega$, then for any $\mu \in (0, \lambda^*)$ imply $\mu \in \Omega$, let

$\lambda^* - \mu = \beta \in \Omega.$

Hence there exists $t_3 \geq t_2$ such that

$$z'(t) + \frac{\beta}{p} g(t)z(t) \geq 0 \quad \text{for} \quad t \geq t_3.$$

Then for any t, s with $t \geq t_3$, $0 \leq s \leq t$, and using the last inequality we get

$$\frac{z(t-s)}{z(t)} = \exp(-\ln\frac{z(t)}{z(t-s)}) = \exp(-\int_{t-s}^{t}\frac{z'(\rho)}{z(\rho)}d\rho)$$
$$\geq \exp\left(\frac{\beta}{p}\int_{t-s}^{t} g(\rho)d\rho\right) \geq \exp\left(\frac{\beta}{p}\int_{t-\sigma(s)}^{t} g(\rho)d\rho\right)$$

where $\sigma(t) < t$, that is

$$z(t-s) \leq z(t)\exp\left(\frac{\beta}{p}\int_{t-\sigma(s)}^{t} g(\rho)d\rho\right)$$

Substituting the last inequality in (2.23) yields

$$0 \leq z'(t) + \int_0^t \frac{1}{p\left(\tau^{-1}(t-s)\right)}\exp\left(\frac{\beta}{p}\int_{t-\sigma(s)}^{t} d\rho\, g(\rho)\right)dr(t,s)z(t)$$

$$z'(t) + \frac{1}{g(t)}\int_0^t \frac{\exp\left(\frac{\beta}{p}\int_{t-\sigma(s)}^{t} g(\rho)d\rho\right)}{p\left(\tau^{-1}(t-s)\right)}dr(t,s)\, g(t)z(t) \geq 0. \quad (2.25)$$

According to (2.25) we claim that

$$\liminf_{t\to\infty}\frac{1}{g(t)}\int_0^t \frac{\exp\left(\frac{\beta}{p}\int_{t-\sigma(s)}^{t} g(\rho)d\rho\right)}{p\left(\tau^{-1}(t-s)\right)}dr(t,s) \leq \frac{\lambda^*}{p} \quad (2.26)$$

otherwise there exist a $\lambda_1^* > \lambda^*$ and a $T \geq t_3$ such that

$$\frac{1}{g(t)}\int_0^t \frac{\exp\left(\frac{\beta}{p}\int_{t-\sigma(s)}^{t} g(\rho)d\rho\right)}{p\left(\tau^{-1}(t-s)\right)}dr(t,s) \geq \frac{\lambda_1^*}{p}$$

for all $t \geq T$ and therefore (2.25) leads to

$$z'(t) + \frac{\lambda_1^*}{p}g(t)z(t) \geq 0 \quad \text{for} \quad t \geq T$$

hence $\lambda_1^* \in \Omega$, which contradicts the hypothesis that $\lambda_1^* > \lambda^*$. Thus (2.26) has been established. Finally (2.26) implies that

$$-\frac{\lambda^*}{p} + \liminf_{t\to\infty}\frac{1}{g(t)}\int_0^t \frac{\exp\left(\frac{\lambda^*-\mu}{p}\int_{t-\sigma(s)}^{t} g(\rho)d\rho\right)}{p\left(\tau^{-1}(t-s)\right)}dr(t,s) \leq 0$$

as $\mu \in (0,\lambda^*)$ is arbitrary, we obtain

$$-\frac{\lambda^*}{p} + \liminf_{t\to\infty}\frac{1}{g(t)}\int_0^t \frac{\exp\left(\frac{\lambda^*}{p}\int_{t-\sigma(s)}^{t} g(\rho)d\rho\right)}{p\left(\tau^{-1}(t-s)\right)}dr(t,s) \leq 0$$

which contradicts (2.22) and complete the proof.

Example 2.6 Consider neutral integro-differential equation

$$\left[x(t) - \frac{5}{2}x(t-2\pi)\right]' - \int_0^t x(t-s)dr(t,s) = 0 \quad (2.27)$$

Where $p(t) = \frac{5}{2}$, $r(t,s) = t - e^{-s}$, $\tau(t) = t - 2\pi$, $\sigma(t) = \frac{t}{2}$, $\lambda = 2$.

One can see that all conditions of Theorem 2.5 met as follows

$$\liminf_{t\to\infty}\int_{t-\sigma(t)}^{t}\left[r(s,s) - r\left(s,\sigma(s)\right)\right]ds = \lim_{t\to\infty}\int_{\frac{t}{2}}^{t} e^{-\frac{s}{2}} - e^{-s}ds = 0$$

$$\liminf_{t\to\infty}\int_t^{\tau^{-1}(t)}\int_0^v \frac{dr(t,s)}{p\left(\tau^{-1}(t-s)\right)}dv = \frac{2}{5}\lim_{t\to\infty}\int_t^{t+2\pi}\int_0^v e^{-s}\,ds\,dv = \frac{4\pi}{5} > \frac{1}{e}$$

$$\limsup_{t\to\infty}\int_0^t g(s)ds = \lim_{t\to\infty}\int_0^t (1-e^{-s})ds = \infty$$

$$\liminf_{t\to\infty}\frac{1}{g(t)}\int_0^t \frac{\exp\left(\frac{\lambda}{p}\int_{t-\sigma(s)}^t g(\rho)d\rho\right)}{p\left(\tau^{-1}(t-s)\right)}dr(t,s) = \frac{2}{5}\lim_{t\to\infty}\frac{\int_0^t e^{\frac{2}{5}\lambda\int_{\frac{s}{2}}^{s}(1-e^{-\rho})d\rho}ds}{1-e^{-t}} = \infty$$

So according to Theorem 2.5 every solution of (2.27) oscillates on $[0,\infty)$.

References

Agarwal R. P., Bohner M. & Wan-Tong Li. (2004): "*Nonoscillation and Oscillation Theory for Functional Differential Equations*". Marcel Dekker, New York..

Bainov, D. D. & Mishev, D. P. (1991): "*Oscillation Theory for Neutral Differential Equation with Delay*". Adam Hilger, Bristol, Philadelphia and New York.

Ergen W. K. (1954): "*Kinetics of the circulating fuel nuclear reactor*" . Journal of Applied Physics , Vol. 25, 702-711.

Gyori I., Ladas G. (1991): "*Oscillation Theory of Delay Differential Equations*". Clarendon Press- Oxford.

Kordonis I. G. E., Philos CH. G. & Purnaras I. K. (2004):" *On the Behavior of Solutions of Linear Neutral Integro-Differential Equations with Unbounded Delay*". Georgian Mathematical Journal, Vol. 11, No. 2, 337–348.

Ladde G. S.; Lakshmikantham V. & Zhang B. G. (1987):"*Oscillation Theorem of Differential Equation with Deviating Arguments*". Marcel Dekker Inc. New York and Basel.

Ladas G., Philos CH. G. & Sficas Y. G. (1991): "*Oscillations of integro-differential equations*"**.** *Differential and Integral Equations*, Vol. 4, 1113-1120.

Olach R. & Samajova H. (2005): "*Oscillations of Linear Integro-Differential Equations* " . Central European Journal of Mathematics, Vol.3, No.1, 98-104.

Tuncay C. (2010): "*Oscillation of solutions for Odd-Order Neutral Functional Differential Equations*" Electronic Journal of Differential Equations, No. 23, 1-10.