Effect of Heat Transfer on Peristaltic Transport of a Johnson Segalman Fluid Through a Porous Medium in an Inclined Asymmetric Channel

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Abstract

In this paper, the influence of heat transfer on peristaltic transport of Johnson Segalman fluid in an inclined asymmetric channel are investigated theoretically and graphically. A regular perturbation method is used to obtain the analytical solutions for the stream functions, temperature fields, axial pressure gradient, and pressure rise. The effects of the physical parameters of the problem on these distributions are discussed and illustrated graphically through a set of figures.

Keywords: Johnson Segalman fluid; Peristaltic transport; Heat transfer; Porous medium; Inclined asymmetric channel

1. Introduction

The Johnson-Segalman model is a viscoelastic fluid model which was developed to allow for non-affine deformation [19]. Recently, this model has been used by a number of researchers [20,24,25] to explain the “spurt” phenomenon. Experimentalists usually associate a spurt with a slip at the wall, and on this issue experiments have been carried out [21,22,26,27,33]. Also Rao and Rajagopal [34] discussed three distinct flows of a Johnson-Segalman fluid. The three flows are cylindrical poiseuill flow. The mechanism of peristaltic transport has been exploited for industrial applications like sanitary fluid transport, blood pumps in heart lung machine and transport of corrosive fluids where the contact of the fluid with the machinery parts is prohibited. Since the first investigation of Latham [23], a number of analytical, numerical and experimental [1,2,3,6-10] studies of peristaltic flows of different fluid have been reported under different conditions with reference to physiological and mechanical situations. The peristaltic flows can be divided to Newtonian and non-Newtonian flows that have been reported analytically, numerically, and experimentally by a number of researchers [29-32]. Although most prior studies of peristaltic transport have focused on Newtonian fluids, there are also studies involving non-Newtonian fluid [11-15]. Srinivas and Pushparaj [36] discussed the non-linear peristaltic transport in an inclined asymmetric channel. Hayat et al. [26] investigated the peristaltic transport of a Johnson-Segalman fluid in an asymmetric channel. Reddy and Roju [35] studied the non-linear peristaltic pumping of Johnson-Segalman fluid in an asymmetric channel under the effect of a magnetic field. Recently, Hayat et. [16] discussed the peristaltic flow in an asymmetric channel for a J-S fluid and
previously the examiner Haroun [17] discussed the peristaltic flow in an inclined asymmetric channel for fourth grad fluid. He observed that with the increase in inclined angle the trapped bolus volume increase.

Considering the importance of heat transfer in peristaltic an attempt is made to study the combined effects of heat transfer and inclined angle on the peristaltic transport at a Johnson-Segalman fluid in an asymmetric channel. Nadeem and Noreen Sher Akbar [28] are studied the effect of heat transfer on peristaltic transport of a J-S fluid in an inclined asymmetric channel. El Shehawey and Husseny[4] and El Shehawey et al. [5] studied the peristaltic mechanism of a Newtonian fluid through a porous medium. Hall effects on peristaltic flow of a Maxwell fluid through a porous medium in a channel was studied by Hayat et. [18].

In the present note, a mathematical model is presented to understand the interaction between peristalsis and heat transfer for the motion of a viscous in compressible fluid in a tow–dimensional asymmetric inclined channel. The momentum and energy equations have been linearized under long–wavelength and low Reynold’s number assumptions and analytical solutions for the flow variables have been obtained.

2. The mathematical model

The constitutive equations for an incompressible Johnson-Segalman fluid are given by

2.1. Governing equations

Consider an incompressible, Johnson-Segalman fluid confined in a two-dimensional infinit inclined asymmetric channel of width \(d_1+d_2\) (see Fig. 1). We consider an infinite wave train travelling with velocity \(c\) along the channel walls. The asymmetry in the channel is induced by assuming the peristaltic wave train on the walls to have different amplitudes and phase. The resulting asymmetric channel walls are defined as

\[
Y = \bar{h}_1(\bar{X}, \bar{T}) = d_1 + a_1 \sin \left[ \frac{2\pi}{\lambda} (\bar{X} - c \bar{T}) \right], \quad \text{upperwall},
\]

\[
Y = \bar{h}_2(\bar{X}, \bar{T}) = -d_2 - a_2 \sin \left[ \frac{2\pi}{\lambda} (\bar{X} - c \bar{T}) + \phi \right], \quad \text{lowerwall}.
\]

Here \(a_1\) and \(a_2\) are the amplitude of the waves, \(\lambda\) is the wavelength and \(\phi \in [0,\pi]\) is the phase difference. Note that \(\phi = 0\) corresponds to an asymmetric channel with waves out of phase and \(\phi = \pi\) describes the case where waves are in phase. Moreover \(a_1, a_2, d_1, d_2\) and \(\phi\) satisfy the following inquailty

\[
a_1^2 + a_2^2 + 2a_1a_2\cos \phi \leq (d_1 + d_2)^2
\]
The equations governing the flow of an incompressible fluid are
\[
\text{div } \vec{V} = 0, \quad \text{div } \vec{\sigma} + \rho \vec{f} = \rho \frac{d\vec{V}}{dt},
\]
where \( \vec{V} \) is the velocity, \( \vec{f} \) is the body force per unit mass, \( \rho \) is the fluid density, \( \frac{d}{dt} \) is the material derivative and \( \vec{\sigma} \) is the Cauchy stress tensor given by
\[
\vec{\sigma} = -p \vec{I} + \vec{T},
\]
\[
\vec{T} = 2\mu \vec{D} + \vec{S},
\]
\[
\vec{S} + m \left[ \frac{d\vec{S}}{dt} + \vec{S} \left( \vec{W} - e\vec{D} \right) + \left( \vec{W} - e\vec{D} \right)^T \vec{S} \right] = 2\eta \vec{D},
\]
\[
\vec{D} = \frac{1}{2} \left[ \vec{L} + \vec{L}^T \right], \quad \vec{W} = \frac{1}{2} \left[ \vec{L} - \vec{L}^T \right], \quad \vec{L} = \text{grad } \vec{V}
\]
and bars indicate the quantities in the dimensional coordinates. The equations above include the scalar pressure \( p \), the identity tensor \( \vec{I} \), the extra stress \( \vec{S} \), the dynamic viscosities \( \mu \) and \( \eta \), the relaxation time \( m \), the slip parameter \( e \) and the respective symmetric and skew symmetric part of the velocity gradient, \( \vec{D} \) and \( \vec{W} \). Note, that model (5) reduces to the Maxwell fluid model for \( e = 1 \) and \( \mu = 0 \), and for \( m = 0 = \mu \) we receive the classical Navier-Stokes fluid model.
The velocity for unsteady tow-dimensional flows is defined as

\[ \vec{V} = [\vec{U}(X,Y,t), \vec{V}(X,Y,t), 0]. \] (9)

From Eqs. (4)-(9) we obtain,

\[ \frac{\partial \vec{U}}{\partial X} + \frac{\partial \vec{V}}{\partial Y} = 0 \] (10)

\[ \rho \left( \frac{\partial \vec{U}}{\partial t} + \frac{\partial \vec{U}}{\partial X} + \frac{\partial \vec{V}}{\partial Y} \right) = -\frac{\partial \vec{p}}{\partial X} + \mu \left( \frac{\partial^2 \vec{U}}{\partial X^2} + \frac{\partial^2 \vec{U}}{\partial Y^2} \right) + \frac{\partial s_{xx}}{\partial X} + \frac{\partial s_{tx}}{\partial Y} - \frac{\mu}{\kappa} \vec{U} + \rho g \sin \beta, \] (11)

\[ \rho \left( \frac{\partial \vec{V}}{\partial t} + \frac{\partial \vec{U}}{\partial X} + \frac{\partial \vec{V}}{\partial Y} \right) = -\frac{\partial \vec{p}}{\partial Y} + \mu \left( \frac{\partial^2 \vec{V}}{\partial X^2} + \frac{\partial^2 \vec{V}}{\partial Y^2} \right) + \frac{\partial s_{yy}}{\partial X} + \frac{\partial s_{ty}}{\partial Y} - \frac{\mu}{\kappa} \vec{V} - \rho g \cos \beta, \] (12)

\[ 2\eta \frac{\partial \vec{U}}{\partial X} = \vec{s}_{xx} + m \left[ \frac{\partial \vec{U}}{\partial t} + \frac{\partial \vec{U}}{\partial X} + \frac{\partial \vec{V}}{\partial Y} \right] \vec{s}_{xx} - 2em \frac{\partial \vec{U}}{\partial X} + m \left[ (1-e) \frac{\partial \vec{V}}{\partial X} - (1+e) \frac{\partial \vec{U}}{\partial Y} \right] \vec{s}_{xt}, \] (13)

\[ \eta \left( \frac{\partial \vec{U}}{\partial Y} + \frac{\partial \vec{V}}{\partial X} \right) = \vec{s}_{xt} + m \left[ \frac{\partial \vec{U}}{\partial t} + \frac{\partial \vec{U}}{\partial X} + \frac{\partial \vec{V}}{\partial Y} \right] \vec{s}_{xt} + m \left[ (1-e) \frac{\partial \vec{U}}{\partial X} - (1+e) \frac{\partial \vec{V}}{\partial Y} \right] \vec{s}_{xt}, \] (14)

\[ 2\eta \frac{\partial \vec{V}}{\partial Y} = \vec{s}_{yy} + m \left[ \frac{\partial \vec{U}}{\partial t} + \frac{\partial \vec{U}}{\partial X} + \frac{\partial \vec{V}}{\partial Y} \right] \vec{s}_{yy} - 2em \frac{\partial \vec{V}}{\partial Y} + m \left[ (1-e) \frac{\partial \vec{U}}{\partial X} - (1+e) \frac{\partial \vec{V}}{\partial Y} \right] \vec{s}_{yt}, \] (15)

\[ \rho c_p \left( \frac{\partial T}{\partial t} + \frac{\partial T}{\partial X} + \frac{\partial T}{\partial Y} \right) = \vec{s}_{xx} \frac{\partial \vec{U}}{\partial X} + \vec{s}_{xt} \frac{\partial \vec{V}}{\partial Y} + \vec{s}_{ty} \frac{\partial \vec{V}}{\partial X} + \vec{s}_{yy} \left( \frac{\partial \vec{U}}{\partial Y} + \frac{\partial \vec{V}}{\partial X} \right) + k \left( \frac{\partial^2 \vec{U}}{\partial X^2} + \frac{\partial^2 \vec{T}}{\partial Y^2} \right). \] (16)

In the fixed frame \((X,Y)\) the motion is unsteady, while it becomes steady in the wave frame \((x,y)\) given by

\[ \bar{x} = X - ct, \quad \bar{y} = Y, \quad \bar{u} = U - c, \quad \bar{v} = V, \] (17)

when moving away in direction of the wave from the fixed frame \((\bar{X}, \bar{Y})\) with speed \(c\). Here \(\bar{u}, \bar{v}\) and \(\bar{U}, \bar{V}\) are the velocity components in the wave frame and in the fixed frame, respectively.
We put Eq. (17) into Eqs. (11)-(16) and to write continuity and motion equations in dimensionless form we introduce the following new quantities

\[ x = \frac{\bar{x}}{\lambda}, \quad y = \frac{\bar{y}}{d_1}, \quad u = \frac{\bar{u}}{c}, \quad v = \frac{\bar{v}}{\lambda}, \quad h_1 = \frac{\bar{h}_1}{d_1}, \quad h_2 = \frac{\bar{h}_2}{d_1}, \]

\[ S = \frac{d_1}{\mu c} \bar{S}(\bar{x}), \quad p = \frac{2\pi d_1^2}{\lambda(\mu + \eta)c} \bar{p}(\bar{x}), \quad \delta = \frac{d_3}{\lambda}, \quad d = \frac{d_2}{d_1}, \quad (18) \]

\[ a = \frac{a_1}{d_1}, \quad b = \frac{a_2}{d_1}, \quad R_e = \frac{\rho c d_1}{\mu}, \quad W_i = \frac{mc}{d_1}, \quad \bar{T} = \frac{ct}{\lambda}, \]

\[ E_r = \frac{e^2}{c_p(T_0 - T_1)}, \quad F_r = \frac{c^2}{gd_1}, \quad \rho_{cp} = \frac{mc}{d_1}, \quad \theta = \frac{T - T_1}{T_0 - T_1}, \quad D_r = \frac{k}{d_1^2}, \]

Where \( \rho \) is the density, \( \mu \) is the coefficient of viscosity of the fluid, \( p \) is the pressure, \( \beta \) is the inclination of the channel with the horizontal, \( g \) is the acceleration due to gravity, \( T \) is the temperature, \( c_p \) is the specific heat at constant pressure and \( k \) is the thermal conductivity.

we have

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (19) \]

\[ R_e \delta \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\left( \frac{\mu + \eta}{\mu} \right) \bar{p} + \delta^2 \frac{\partial^2 u}{\partial x^2} + \delta^2 \frac{\partial^2 u}{\partial y^2} + \delta \frac{\partial S_{xx}}{\partial x} + \frac{\partial S_{xy}}{\partial y} \]

\[ -\frac{1}{D_r} (u + 1) + \frac{R_e}{F_r} \sin \beta, \quad (20) \]

\[ R_e \delta^3 \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\left( \frac{\mu + \eta}{\mu} \right) \bar{p} + \delta^2 \left[ \delta^2 \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 \bar{v}}{\partial y^2} \right] + \delta^2 \frac{\partial S_{sy}}{\partial y} \]

\[ + \delta \frac{\partial S_{xy}}{\partial y} - \frac{\partial S_{xy}}{D_r} - \frac{R_e}{F_r} \cos \beta, \quad (21) \]

\[ \left( \frac{2\eta \delta}{\mu} \right) \frac{\partial u}{\partial x} = S_{xx} + W_i \delta \left[ u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right] S_{xx} - 2eW_i \delta S_{xx} \frac{\partial u}{\partial x} \]
\[ +W_i \left[ \delta^2 (1-e) \frac{\partial v}{\partial x} - (1+e) \frac{\partial u}{\partial y} \right] S_{xy}, \]  
(22)

\[ \frac{\eta}{\mu} \left( \frac{\partial u}{\partial y} + \delta^2 \frac{\partial v}{\partial x} \right) = S_{xy} + W_i \delta \left[ u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right] S_{xy} + \frac{W_i}{2} \left[ (1-e) \frac{\partial u}{\partial y} - (1+e) \delta^2 \frac{\partial v}{\partial x} \right] S_{xx} \]

\[ + \frac{W_i}{2} \left[ (1-e) \frac{\partial v}{\partial x} - (1+e) \delta^2 \frac{\partial v}{\partial x} \right] S_{yy}, \]  
(23)

\[ \left( \frac{2\eta\delta}{\mu} \right) \frac{\partial v}{\partial y} = S_{yy} + W_i \delta \left[ u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right] S_{yy} - 2W_i \delta S_{xy} \frac{\partial v}{\partial y} + \left[ (1-e) \frac{\partial u}{\partial y} - (1+e) \delta^2 \frac{\partial v}{\partial x} \right] S_{xy} \]  
(24)

\[ R \delta \left( u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} \right) = E \left[ S_{xx} \delta \frac{\partial u}{\partial x} + S_{xy} \delta \frac{\partial v}{\partial x} + S_{yx} \frac{\partial u}{\partial y} + S_{yy} \delta \frac{\partial v}{\partial y} \right] \]

\[ + \frac{1}{Pr} \left[ \delta^2 \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right]. \]  
(25)

Writing Eqs. (19)-(25) in terms of the stream function \( \psi(x, y) \) defined by

\[ u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \]  
(26)

we get

\[ \delta R \left[ \left( \frac{\partial \psi}{\partial y} \frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} \right) \frac{\partial \psi}{\partial y} \right] = -\left( \frac{\mu + \eta}{\mu} \right) \frac{\partial p}{\partial x} \delta^2 \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^3 \psi}{\partial x^3} + \delta \frac{\partial S_{xx}}{\partial x} \]

\[ + \frac{\partial S_{xy}}{\partial y} - \frac{1}{Dr} \left( \frac{\partial \psi}{\partial y} + 1 \right) \frac{R}{F_r} \sin \beta, \]  
(27)

\[ -\delta^3 R \left[ \left( \frac{\partial \psi}{\partial y} \frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} \right) \frac{\partial \psi}{\partial x} \right] = -\left( \frac{\mu + \eta}{\mu} \right) \frac{\partial p}{\partial y} - \delta^2 \left[ \delta^2 \frac{\partial^3 \psi}{\partial x^3} + \frac{\partial^3 \psi}{\partial x \partial y^2} \right] + \delta^2 \frac{\partial S_{yy}}{\partial x} \]

\[ + \frac{\partial S_{xy}}{\partial y} \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial \psi}{\partial x} - \delta \frac{R}{F_r} \cos \beta, \]  
(28)
\[
\left( \frac{2\eta \delta}{\mu} \right) \frac{\partial^2 \psi}{\partial x \partial y} = S_{xx} + W_i \delta \left[ \left( \frac{\partial \psi}{\partial y} \frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} \right) \right] + S_{xy} - 2eW_i \delta \frac{\partial^2 \psi}{\partial x \partial y} \\
-W_i \left[ \delta^2 (1-e) \frac{\partial^2 \psi}{\partial x^2} + (1+e) \frac{\partial^2 \psi}{\partial y^2} \right] S_{xy},
\]
\[
\eta \left( \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2} \right) - S_{xy} + W_i \delta \left[ \left( \frac{\partial \psi}{\partial y} \frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} \right) \right] S_{xy} \\
+ \frac{W_i}{2} \left[ (1-e) \frac{\partial^2 \psi}{\partial y^2} + \delta^2 (1+e) \frac{\partial^2 \psi}{\partial x^2} \right] S_{xx} + \frac{W_i}{2} \left[ \delta^2 (1-e) \frac{\partial^2 \psi}{\partial x^2} + (1+e) \frac{\partial^2 \psi}{\partial y^2} \right] S_{yy},
\]
\[
\left( \frac{2\eta \delta}{\mu} \right) \frac{\partial^2 \psi}{\partial x \partial y} = S_{xy} + W_i \delta \left[ \left( \frac{\partial \psi}{\partial y} \frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} \right) \right] S_{xy} + 2eW_i \delta S_{xy} \frac{\partial^2 \psi}{\partial x \partial y} \\
+ W_i \left[ (1-e) \frac{\partial^2 \psi}{\partial y^2} + \delta^2 (1+e) \frac{\partial^2 \psi}{\partial x^2} \right] S_{yy},
\]
\[
\delta R_e \left[ \left( \frac{\partial \psi}{\partial y} \frac{\partial \theta}{\partial x} - \delta \frac{\partial \psi}{\partial x} \frac{\partial \theta}{\partial y} \right) \right] = E_r \left[ S_{xx} \delta \frac{\partial^2 \psi}{\partial x \partial y} - S_{xy} \delta \frac{\partial^2 \psi}{\partial x^2} + S_{yy} \frac{\partial^2 \psi}{\partial y^2} - S_{yy} \delta \frac{\partial^2 \psi}{\partial x \partial y} \right] \\
+ \frac{1}{P_r} \left( \delta^2 \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right).
\]

where \( \delta \) is wave number, \( R_e \) is the Renold’s number, \( W_i \) is the Weissenberg number, \( F_r \) is the Froud number, \( P_r \) is the Prandtl number, and \( D_r \) is the Darcy number. The continuity equation is identically satisfied.

### 2.2 Rate of volume flow and boundary conditions

The dimensional rate of fluid flow in the fixed frame \((\tilde{X}, \tilde{Y})\) is
\[
Q = \int_{\tilde{Y}(\tilde{X})} \tilde{U}(\tilde{X}, \tilde{Y}, \tilde{V}) d\tilde{Y}
\]
\[
(33)
\]
In the wave frame \((\tilde{x}, \tilde{y})\) Eq. (33) reduces to
\[
q = \int \tilde{u}(\tilde{x}, \tilde{y}) d\tilde{y}
\]
\[
(34)
\]
By Eq.(17), the above rates are related in the following expression

$$Q = q + c\bar{h}_1 - c\bar{h}_2$$

(35)

Applying the averaged flow

$$\bar{Q} = \frac{1}{\tau} \int_0^\tau Q d\tau$$

(36)

Over a period $\tau = \lambda \phi$ at a fixed position $\bar{X}$, we receive

$$Q = q + c\bar{d}_1 - c\bar{d}_2.$$ 

(37)

With the definition of the dimensionless time averaged flows

$$\Theta \equiv \frac{\bar{Q}}{c\bar{d}_1}, \quad F \equiv \frac{q}{c\bar{d}_1},$$

(38)

in the fixed and moving frames, respectively, we can write Eq(37) as

$$\Theta = F + 1 + d$$

(39)

where

$$F = \int_{h_1(x)}^{h_2(x)} \frac{\partial \psi}{\partial y} dy = \psi(h_1) - \psi(h_2)$$

(40)

and the dimensionless surface of the peristaltic walls are

$$h_1(x) = 1 + a \sin(2\pi x), \quad h_2(x) = -d - b \sin(2\pi x + \phi),$$

(41)

where the inequality

$$a^2 + b^2 + 2ab \cos \phi \leq (1 + d)^2$$

(42)

holds. The dimensionless boundary conditions in the wave frame are

$$\psi = \frac{F}{2}, \quad \frac{\partial \psi}{\partial y} = -1 \quad \text{at} \quad y = h_1(x),$$

$$\psi = -\frac{F}{2}, \quad \frac{\partial \psi}{\partial y} = -1 \quad \text{at} \quad y = h_2(x)$$

(43)

$$\theta = 1 \quad \text{at} \quad y = h_1(x), \quad \theta = 0 \quad \text{at} \quad y = h_2(x),$$
Here it is pointed out that the conditions on \( \psi \) satisfy Eq. (40) and the conditions on \( \frac{\partial \psi}{\partial y} \) are no-slip.

2.3 Model equations

under lubrication approach (i.e., neglecting the terms of order \( \delta \) and \( Re \)), from Eqs. (27)-(32) we get

\[
\left( \frac{\mu+\eta}{\eta} \right) \frac{\partial p}{\partial x} = \frac{\partial S_{xx}}{\partial y} + \frac{\partial^2 \psi}{\partial y^2} - \frac{1}{D_r} \left( \frac{\partial \psi}{\partial y} + 1 \right) + \frac{R_e}{F_r} \sin \beta, \tag{44}
\]

\[
\frac{\partial p}{\partial y} = 0, \tag{45}
\]

\[
S_{xx} = W_i \left( 1 + e \right) \frac{\partial^2 \psi}{\partial y^2} S_{xy}, \tag{46}
\]

\[
\left( \frac{\eta}{\mu} \right) \frac{\partial^2 \psi}{\partial y^2} = S_{xy} + \frac{W_i}{2} \left( 1 - e \right) \frac{\partial^2 \psi}{\partial y^2} S_{xx} - \frac{W_i}{2} \left( 1 + e \right) \frac{\partial^2 \psi}{\partial y^2} S_{yy}, \tag{47}
\]

\[
S_{yy} = -W_i \left( 1 - e \right) \frac{\partial^2 \psi}{\partial y^2} S_{xy}, \tag{48}
\]

\[
E_p S_{xy} \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \theta}{\partial y^2} = 0, \tag{49}
\]

From Eqs. (46)-(48), we write

\[
S_{xy} = \frac{\left( \frac{\eta}{\mu} \right) \left( \frac{\partial^2 \psi}{\partial y^2} \right)}{1 + W_i^2 \left( 1 - e^2 \right) \left( \frac{\partial^2 \psi}{\partial y^2} \right)^2}, \tag{50}
\]

using Eqs. (46), (48) and (50), the Eq. (44) and (49) can be rewritten as

\[
\frac{\partial^2}{\partial y^2} \left[ \left( \frac{\eta}{\mu} + 1 \right) \frac{\partial^2 \psi}{\partial y^2} + W_i^2 \left( 1 - e^2 \right) \left( \frac{\partial^2 \psi}{\partial y^2} \right)^3 \right] = 0, \tag{51}
\]
\[
\frac{\partial p}{\partial x} = \frac{\partial^3 \psi}{\partial y^3} + W_i^2 \alpha_1 \frac{\partial}{\partial y} \left[ \left( \frac{\partial^3 \psi}{\partial y^2} \right)^3 \right] - \mu \frac{\partial \psi}{(\mu + \eta) D_r} \frac{\partial \psi}{\partial y} + \left( \frac{\mu}{\mu + \eta} \right) \frac{R_x}{F_r} \sin \beta, \tag{52}
\]

\[
\frac{\partial^2 \theta}{\partial y^2} = -E_r P_r \left[ \eta \frac{\partial^2 \psi}{\partial y^2} \right] - W_i^2 \eta \left( 1 - e^2 \right) \left( \frac{\partial^2 \psi}{\partial y^2} \right)^4, \tag{53}
\]

where \( \alpha_1 = \frac{(e^2 - 1) \eta}{(\eta + \mu)} \)

3. perturbed systems and perturbation solutions

The Eqs. (51)-(53) are non-linear and its closed form solution is not possible. Thus, we linearize these equations in terms of \( W_i^2 \) since \( W_i \) is small for the type of flow under consideration. So we expand \( \psi, p, F \) and \( \theta \) as

\[
\psi = \psi_0 + W_i^2 \psi_1 + O(W_i^4),
\]

\[
p = p_0 + W_i^2 p_1 + O(W_i^4),
\]

\[
F = F_0 + W_i^2 F_1 + O(W_i^4),
\]

\[
\theta = \theta_0 + W_i^2 \theta_1 + O(W_i^4), \tag{54}
\]

substituting Eq.(54) into Eqs. (51)-(53) and boundary conditions (43), then equating the like powers of \( W_i^2 \) we get:

3.1. perturbed systems

3.1.1. Zeroth-order system

\[
\begin{aligned}
\frac{\partial^4 \psi_0}{\partial y^4} &= \frac{\mu}{(\mu + \eta) D_r} \frac{\partial^2 \psi_0}{\partial y^2}, \\
\frac{\partial p_0}{\partial y} &= 0, \\
\frac{\partial p_0}{\partial x} &= \frac{\partial^3 \psi_0}{\partial y^3} - \frac{\mu}{(\mu + \eta) D_r} \frac{\partial \psi_0}{\partial y} + \left( \frac{\mu}{\mu + \eta} \right) \frac{R_x}{F_r} \sin \beta, \\
\frac{\partial^2 \theta_0}{\partial y^2} &= -E_r P_r \left[ \eta \left( \frac{\partial^2 \psi_0}{\partial y^2} \right)^2 \right].
\end{aligned} \tag{55}
\]
\[ \psi_0 = \frac{F_0}{2}, \quad \frac{\partial \psi_0}{\partial y} = -1, \quad \text{at} \quad y = h_1(x), \]

\[ \psi_0 = -\frac{F_0}{2}, \quad \frac{\partial \psi_0}{\partial y} = -1, \quad \text{at} \quad y = h_2(x), \]

\[ \theta_0 = 1, \quad \text{at} \quad y = h_1(x), \quad \theta_0 = 0, \quad \text{at} \quad y = h_2(x), \]  

(56)

3.1.2. First-order system

\[ \frac{\partial^3 \psi_1}{\partial y^3} = \frac{\mu}{(\mu + \eta)} D_r \frac{\partial^2 \psi_1}{\partial y^2} - \alpha_i \frac{\partial^2}{\partial y^2} \left[ \left( \frac{\partial^2 \psi_0}{\partial y^2} \right)^3 \right], \]

\[ \frac{\partial p_1}{\partial y} = 0, \]

\[ \frac{\partial p_1}{\partial x} = \frac{\partial^3 \psi_1}{\partial y^3} + \alpha_i \frac{\partial^2}{\partial y^2} \left[ \left( \frac{\partial^2 \psi_0}{\partial y^2} \right)^3 \right] - \frac{\mu}{(\mu + \eta)} D_r \frac{\partial \psi_1}{\partial y}, \]  

(57)

\[ \frac{\partial^2 \theta}{\partial y^2} = -E_r P_r \left[ \frac{2\eta}{\mu} \frac{\partial^2 \psi_0}{\partial y^2} \frac{\partial^2 \psi_1}{\partial y^2} - \eta \left( 1 - e^2 \right) \left( \frac{\partial^2 \psi_0}{\partial y^2} \right)^4 \right], \]

\[ \psi_1 = \frac{F_1}{2}, \quad \frac{\partial \psi_1}{\partial y} = 0, \quad \text{at} \quad y = h_1(x), \]

\[ \psi_1 = -\frac{F_1}{2}, \quad \frac{\partial \psi_1}{\partial y} = 0, \quad \text{at} \quad y = h_2(x), \]  

(58)

\[ \theta_1 = 0, \quad \text{at} \quad y = h_1(x), \quad \theta_1 = 0 \quad \text{at} \quad y = h_2(x), \]

3.2 Perturbation solutions

3.2.1. Zeroth-order solution

The solution of Eqs.(55) of zeroth order system, satisfying (56) can be written

\[ \psi_0 = A_7 + A_8 y + \frac{(A_s + A_b) \cosh(\sqrt{A_1} y) + (A_s - A_b) \sinh(\sqrt{A_1} y)}{A_1}, \]

(59)

\[ \frac{dp_0}{dx} = A_2 - A_1 A_8, \]  

(60)
\[ \theta_0 = \frac{A_1(\cosh(2\sqrt{A_1} y) - \sinh(2\sqrt{A_1} y))(A_5^2 + A_6^2(\cosh(4\sqrt{A_1} y) + \sinh(4\sqrt{A_1} y)))}{4A_1} \]

\[ + \frac{4A_1(A_5 + y(A_{10} + A_4 A_5 A_6 y)))}{4A_1}, \quad (61) \]

where

\[ A_1 = \frac{\mu}{(\mu + \eta)D}, \quad A_2 = \frac{\mu R \sin(\beta)}{(\mu + \eta)F}, \quad A_3 = \frac{-E_F \eta}{\mu}, \]

\( A_5, A_6, A_4, A_3 \) and \( A_{10} \) are constants can be determinates by using the boundary conditions Eq.(56).

### 3.2.2. First-order solution

Substituting the zeroth-order solution (59)-(61) into Eq.(57) and then solving the resulting system with the corresponding boundary conditions Eq.(58), we get

\[ \psi_1 = B_3 + B_4 y + \frac{1}{8A_1}(8((B_1 + B_2) \cosh(\sqrt{A_1} y) + (B_1 - B_2) \sinh(\sqrt{A_1} y)) \]

\[ -6A_4 A_6 (A_4 (-5 + 2\sqrt{A_1} y) - A_6 (5 + 2\sqrt{A_1} y)) \cosh(\sqrt{A_1} y) \]

\[ +(A_5^3 + A_6^3) \cosh(3\sqrt{A_1} y) + (A_5^3 - A_6^3) + 6A_4^2 A_6 (-5 + 2\sqrt{A_1} y) \]

\[ +6A_4 A_6^2 (5 + 2\sqrt{A_1} y) + 2(A_5^3 - A_6^3) \cosh(2\sqrt{A_1} y)) \sinh(2\sqrt{A_1} y) \alpha_i, \]

\[ (62) \]

\[ \frac{dp_1}{dx} = -A_4 B_1, \quad (63) \]

\[ \theta_1 = B_5 + \frac{1}{4}\left( (4B_5 y + 4A_1 B_2 B_1 y^2 - 4A_1 \cosh(\sqrt{A_1} y) - \sinh(\sqrt{A_1} y) \)

\[ \left( A_5 + A_3 \cosh(2\sqrt{A_1} y) + A_5 \sinh(2\sqrt{A_1} y) \right) \]

\[ + \frac{1}{4A_1} A_5 \left( \cosh(2\sqrt{A_1} y) - \sinh(2\sqrt{A_1} y) \right) \]

\[ \left( B_2^2 + B_1^2 \cosh(4\sqrt{A_1} y) + B_2^2 \sinh(4\sqrt{A_1} y) \right) \right) \]

\[ \left( \left( B_2^2 + B_1^2 \cosh(4\sqrt{A_1} y) + B_2^2 \sinh(4\sqrt{A_1} y) \right) \right) \]
\[ + \frac{1}{256A_i} A_i(\cosh(6\sqrt{A_i} y) - \sinh(6\sqrt{A_i} y))\alpha_i \]

\[ (9A_i^\delta \alpha_i + 9A_i^\delta \cosh(12\sqrt{A_i} y)\alpha_i + 9A_i^\delta \sinh(12\sqrt{A_i} y)\alpha_i) \]

\[ + 18A_i^3(\cosh(10\sqrt{A_i} y) + \sinh(10\sqrt{A_i} y))(-2B_i - 3A_i^2A_i\alpha_i + 3\sqrt{A_i}A_i^2A_i^2\alpha_i) \]

\[ - 18A_i^3(\cosh(2\sqrt{A_i} y) + \sinh(2\sqrt{A_i} y))(2B_i + 3A_i^2A_i^2\alpha_i + 3\sqrt{A_i}A_i^2A_i^2\alpha_i) \]

\[ + 24A_i^2(\cosh(8\sqrt{A_i} y) + \sinh(8\sqrt{A_i} y))(-6B_2A_i + 12B_1A_i - 8\sqrt{A_i}B_1A_i^2y + 21A_i^2A_i^2\alpha_i \]

\[ - 27\sqrt{A_i}A_i^2A_i^2\alpha_i + 6A_iA_i^2\alpha_i + 6A_iA_i^2\alpha_i^2 + 24A_i^2(\cosh(4\sqrt{A_i} y) \]

\[ + \sinh(4\sqrt{A_i} y))(12B_2A_i - 6B_1A_i + 8\sqrt{A_i}B_1A_i^2y \]

\[ + 21A_i^2A_i^2\alpha_i + 27\sqrt{A_i}A_i^2A_i^2\alpha_i + 6A_iA_i^2A_i^2\alpha_i^2 \]

\[ - 4A_iA_i^2A_i^2\alpha_i + 24A_iA_i^2A_i^2\alpha_i^2 \]

\[ (48B_2A_i - 48B_1A_i + 32\sqrt{A_i}B_2A_i^2y - 32\sqrt{A_i}B_1A_i^2y \]

\[ - 117A_i^2A_i^2\alpha_i + 24A_iA_i^2A_i^2\alpha_i^2 ) \]

(64)

Where

\[ \alpha_i = \frac{(n^2 - 1)\eta}{(\mu + \eta)}, \quad A_i = A_3(1 - n^2), \]

\[ B_1, B_2, B_3, B_4, B_5 \] and \[ B_6 \] are constants can be determinates by using the boundary conditions Eq. (58).

Summing up the perturbation results, we find that

\[ \psi = A_i + A_i B_i y + \left( A_i + A_i \right) \cosh(\sqrt{A_i} y) + \left( A_i - A_i \right) \sinh(\sqrt{A_i} y) \]

\[ W_i(B_i + B_i y + \frac{1}{8A_i} (8(B_i + B_i) \cosh(\sqrt{A_i} y) + (B_i - B_i) \sinh(\sqrt{A_i} y)) \]

192
\[-6A_5A_6(A_5(-5+2\sqrt{A_1}y)-A_6(5+2\sqrt{A_1}y))cosh(\sqrt{A_1}y)\]

\[+(A_5^3+A_6^3)cosh(3\sqrt{A_1}y)+(A_5^3-A_6^3+6A_5^2A_6(-5+2\sqrt{A_1}y))\]

\[+6A_5A_6^2(5+2\sqrt{A_1}y)+2(A_5^3-A_6^3)cosh(2\sqrt{A_1}y))sinh(\sqrt{A_1}y))\alpha_1,\]

(65)

\[\frac{dp}{dx} = A_2 - A_1A_6 - Wi(A_1B_1),\]

(66)

\[\theta = \frac{A_1(cosh(2\sqrt{A_1}y)-sinh(2\sqrt{A_1}y))(A_6^2+A_5^2(cosh(4\sqrt{A_1}y)+sinh(4\sqrt{A_1}y))}{4A_1}\]

\[+W_1(B_5 + \frac{1}{4}(4B_6y+4A_3B_1B_2y^2-4A_4cosh(\sqrt{A_1}y)-sinh(\sqrt{A_1}y)\]

\[\left(A_6+A_5cosh(2\sqrt{A_1}y)+A_5sinh(2\sqrt{A_1}y)\right)\]

\[+\frac{1}{4A_1}A_3\left(cosh(2\sqrt{A_1}y)-sinh(2\sqrt{A_1}y)\right)\]

\[\left(B_2^2+B_1^2cosh(4\sqrt{A_1}y)+B_1^2sinh(4\sqrt{A_1}y)\right)\]

\[+\frac{1}{256A_1}A_3(cosh(6\sqrt{A_1}y)-sinh(6\sqrt{A_1}y))\alpha_1\]

(9A_6^6\alpha_1 + 9A_5^6cosh(12\sqrt{A_1}y)\alpha_1 + 9A_6^6sinh(12\sqrt{A_1}y)\alpha_1)

+18A_5^3(cosh(10\sqrt{A_1}y)+sinh(10\sqrt{A_1}y))(-2B_1 - 3A_5^2A_6\alpha_1 + 3A_1A_5^2A_6y\alpha_1)

-18A_6^3(cosh(2\sqrt{A_1}y)+sinh(2\sqrt{A_1}y))(2B_2 + 3A_6^2\alpha_1 + 3\sqrt{A_1A_5^2A_6}y\alpha_1)

+24A_5^2(cosh(8\sqrt{A_1}y)+sinh(8\sqrt{A_1}y))(-6B_2A_5 + 12B_1A_6 - 8\sqrt{A_1B_1A_6}y + 21A_5^2A_6^2\alpha_1

-27\sqrt{A_1A_5^2A_6^2}\alpha_1 + 6A_1A_5^2A_6^2y^2\alpha_1) + 24A_6^2(cosh(4\sqrt{A_1}y)\]

+sinh(4\sqrt{A_1}y)(12B_2A_5 - 6B_1A_6 + 8\sqrt{A_1B_2A_5}y

+21A_6^2\alpha_1 + 27\sqrt{A_1A_5^2A_6^2}\alpha_1 + 6A_1A_5^2A_6^2y^2\alpha_1)\]
\[-4A_i^2A_5^2A_6y^2(\cosh(6\sqrt{A_i}y) + \sinh(6\sqrt{A_i}y))\]
\[\left(-48B_2A_5 - 48B_1A_6 + 32\sqrt{A_i}A_5A_6y - 32\sqrt{A_i}B_1A_6y\right.\]
\[\left.-117A_5^2A_6^2\alpha_1 + 24A_iA_5^2A_6^2y^2\alpha_1\right)\].

(67)

The non-dimensional pressure rise per wavelength \(\Delta P\) is defined as
\[
\Delta P = \int_0^1 \frac{dp}{dx} \, dx,
\]
(68)

Where \(\frac{dp}{dx}\) is defined through Eq.(66).

Not that, if we assume that the Darcy number was closer to infinity, then the results of the problem reduce exactly to the same as that found by Nadeem and Akbar [28].

**4. Results and discussions**

This section represents the graphical results in order to be able to discuss the quantitative effects of the sundry parameters involved in the analysis.

**4.1. Pumping characteristics**

We plot the expression for \(\Delta P\) in Eq. (68) against \(\Theta\) for various values of parameters of interest in Figs.(2-9). In Fig.2 the effects of region channel width \(d\) on \(\Delta P\) are seen. Observe that, in the pumping (\(\Delta P > 0\)) and free pumping (\(\Delta P = 0\)) for the J-S fluid, an increase in \(d\) causes a decrease in pumping. While in the co-pumping (\(\Delta P < 0\)), the pumping increases with an increase in \(d\). Fig.3 the effects of Weissenberg number on \(\Delta P\). Observe that an increase in \(W_i\) causes a decrease in the pumping region (\(\Delta P > 0\)) and behaves oppositely in the free pumping (\(\Delta P = 0\)) and co-pumping. The observations regarding the effects of upper and lower wave amplitudes \(b\) and \(a\) (Fig.4 and 5) on \(\Delta P\) are quite opposite to those in the case of phase difference \(\phi\) Fig.7. Observe that an increase in \(\phi\) causes a decrease in the pumping region (\(\Delta P > 0\)). It is also noted from Fig.7 that in free pumping and co-pumping \(\Delta P\) increases with an increase in \(\phi\). The effects of \(F_r\) on pressure rise \(\Delta P\) are illustrated in Fig.6. It is seen that with the increase in \(F_r\) the pressure rise decreases in the all pumping regions (\(\Delta P > 0\), (\(\Delta P = 0\)) and (\(\Delta P < 0\)). The pressure rise increases with increase in \(\beta\) and \(D_r\). But this increasing in \(\Delta P\) occurs in the co-pumping region which are displayed in (Fig.8 and 9) respectively.
Fig. 2. Effect of the width of channel $d$ on variation of $\Delta p$ versus $\Theta$ for: $a=0.5$, $b=0.4$, $W_i=0.25$, $\phi=0.2$, $\mu=0.03$, $\eta=0.04$, $d=0.5$, $D_r=1$, $F_r=1$, $Re=10$, $\beta=0.5$, $e=0.2$.

Fig. 3. Effect of the Weisenberg number $W_i$ on variation of $\Delta p$ versus $\Theta$ for: $a=0.5$, $\phi=0.2$, $\mu=0.03$, $\eta=0.04$, $d=0.9$, $D_r=1$, $F_r=1$, $Re=10$, $\beta=0.5$, $e=0.2$.

Fig. 4. Effect of the lower wave amplitude $b$ on variation of $\Delta p$ versus $\Theta$ for: $a=0.5$, $\phi=0.2$, $\mu=0.03$, $\eta=0.04$, $W_i=0.25$, $d=0.5$, $D_r=1$, $F_r=1$, $Re=10$, $\beta=0.5$, $e=0.2$.

Fig. 5. Effect of the upper wave amplitude $a$ on variation of $\Delta p$ versus $\Theta$ for: $b=0.5$, $\phi=0.5$, $\mu=0.03$, $\eta=0.04$, $W_i=0.01$, $d=0.1$, $D_r=1$, $F_r=1$, $Re=10$, $\beta=0.5$, $e=0.2$.

Fig. 6. Effect of the Froud number $F_r$ on variation of $\Delta p$ versus $\Theta$ for: $b=0.5$, $a=0.5$, $\mu=0.03$, $\eta=0.04$, $W_i=0.01$, $d=0.1$, $D_r=1$, $\phi=0.2$, $Re=10$, $\beta=0.5$, $e=0.2$.

Fig. 7. Effect of the phase difference $\phi$ on variation of $\Delta p$ versus $\Theta$ for: $b=0.5$, $a=0.5$, $\mu=0.03$, $\eta=0.04$, $W_i=0.01$, $d=0.1$, $D_r=1$, $F_r=1$, $Re=10$, $\beta=0.5$, $e=0.2$. 


4.2. Pressure gradient characteristics

The variation of the axial pressure gradient $dp/dx$ with $x$ for various values $\phi, a, b, \mu, \eta, d, D_r, F_r, R_e, \beta, \Theta$ and $W_i$ are shown in Figs.(10-21). Fig.10 studies the effects of phase shift $\phi$ on the variation of pressure gradient $dp/dx$, and it is noticed that the axial pressure gradient decreases by increasing $\phi$. Fig.11 and 12 show the variation of the axial pressure gradient $dp/dx$ with $b$ and $a$ respectively. It is clear that the axial pressure gradient increases with an increase in amplitudes of the waves. Fig.13 shows the effect of $\mu$ on $dp/dx$. It is notted that, the magnitude of $dp/dx$ increase with increasing $\mu$ and versa to versa in Fig.14 the magnitude of $dp/dx$ decreases with increasing $\eta$. The effect the Weissenberg number $W_i$ on the variation of pressure gradient $dp/dx$ with $x$ is shown in Fig.15. It is observed that, the magnitude of the $dp/dx$ increases with increasing $W_i$. Fig.(16-21) illustrates the variation of the axial pressure gradient $dp/dx$ with $x$ for different values of $D_r, \beta, d, R_e, F_r$ and $\Theta$. It is clear that the axial pressure gradient $dp/dx$ decreases with an increase in $D_r$. Fig.16. Also $dp/dx$ increases by increasing $\beta$ and $R_e$ (Fig.17 and 19). The situation is reversed in Figs.18, 20 and 21, the axial pressure gradient is decreased with an increase in $d, F_r$ and $\Theta$. 

![Figure 8: Effect of inclined angle $\beta$ on variation of $\Delta p$ versus $\Theta$ for: $b=0.5,a=0.5,\mu=0.03,\eta=0.04,W_i=0.01$, $d=3.5,D_r=1,F_r=1,R_e=10,\phi=0.2,e=0.2$.](image1)

![Figure 9: Effect of the Darcy number $F_r$ on variation of $\Delta p$ versus $\Theta$ for: $b=0.5,a=0.5,\mu=0.03,\eta=0.04,W_i=0.01$, $d=0.1,F_r=1,\phi=0.2,R_e=10,\beta=0.5,e=0.2$.](image2)
Fig.10. Effect of $\phi$ on the variation of pressure gradient $dp/dx$ with $x$: $b=0.4$, $a=0.2$, $\mu=1$, $\eta=1$, $W_i=0.01$, $D_r=0.5$, $F_r=1$, $R_e=10$, $\beta=0.5$, $d=0.3$, $e=0.8$, $\Theta=0.6$.

Fig.11. Effect of $b$ on the variation of pressure gradient $dp/dx$ with $x$: $\phi=0.2$, $a=0.2$, $\mu=1$, $\eta=1$, $W_i=0.01$, $D_r=0.5$, $F_r=1$, $R_e=10$, $\beta=0.5$, $d=0.3$, $e=0.8$, $\Theta=0.6$.

Fig.12. Effect of $a$ on the variation of pressure gradient $dp/dx$ with $x$: $b=0.4$, $\phi=0.2$, $\mu=1$, $\eta=1$, $W_i=0.01$, $D_r=0.5$, $F_r=1$, $R_e=10$, $\beta=0.5$, $d=0.3$, $e=0.8$, $\Theta=0.6$.

Fig.13. Effect of $\mu$ on the variation of pressure gradient $dp/dx$ with $x$: $b=0.4$, $a=0.2$, $\phi=0.2$, $\eta=1$, $W_i=0.01$, $D_r=0.5$, $F_r=1$, $R_e=10$, $\beta=0.5$, $d=0.3$, $e=0.8$, $\Theta=0.6$.

Fig.14. Effect of $\eta$ on the variation of pressure gradient $dp/dx$ with $x$: $b=0.4$, $a=0.2$, $\mu=1$, $\phi=0.2$, $W_i=0.01$, $D_r=0.5$, $F_r=1$, $R_e=10$, $\beta=0.5$, $d=0.3$, $e=0.8$, $\Theta=0.6$.

Fig.15. Effect of $W_i$ on the variation of pressure gradient $dp/dx$ with $x$: $b=0.4$, $a=0.2$, $\mu=1$, $\phi=0.2$, $\eta=1$, $D_r=0.5$, $F_r=1$, $R_e=10$, $\beta=0.5$, $d=0.3$, $e=0.8$, $\Theta=0.6$. 
Fig. 16. Effect of $D_r$ on the variation of pressure gradient $dp/dx$ with $x$: $b=0.4, a=0.2, \phi=0.2, \mu=1, \eta=1, W_i=0.01$, $F_r=1, R_e=10, \beta=0.5, d=0.3, e=0.8, \Theta=0.6$.

Fig. 17. Effect of $\beta$ on the variation of pressure gradient $dp/dx$ with $x$: $b=0.4, a=0.2, \phi=0.2, \mu=1, \eta=1, W_i=0.01$, $D_r=0.5, F_r=1, R_e=10, d=0.3, e=0.8, \Theta=0.6$.

Fig. 18. Effect of $d$ on the variation of pressure gradient $dp/dx$ with $x$: $b=0.4, a=0.2, \phi=0.2, \mu=1, \eta=1, W_i=0.01$, $D_r=0.5, F_r=1, R_e=10, \beta=0.5, d=0.3, e=0.8, \Theta=0.6$.

Fig. 19. Effect of $R_e$ on the variation of pressure gradient $dp/dx$ with $x$: $b=0.4, a=0.2, \phi=0.2, \mu=1, \eta=1, W_i=0.01$, $D_r=0.5, F_r=1, \beta=0.5, d=0.3, e=0.8, \Theta=0.6$.

Fig. 20. Effect of $F_r$ on the variation of pressure gradient $dp/dx$ with $x$: $b=0.4, a=0.2, \phi=0.2, \mu=1, \eta=1, W_i=0.01$, $D_r=0.5, R_e=10, \beta=0.5, d=0.3, e=0.8, \Theta=0.6$.

Fig. 21. Effect of $\Theta$ on the variation of pressure gradient $dp/dx$ with $x$: $b=0.4, a=0.2, \phi=0.2, \mu=1, \eta=1, W_i=0.01$, $D_r=0.5, F_r=1, R_e=10, \beta=0.5, d=0.3, e=0.8, \Theta=0.6$. 
4.3. Trapping phenomenon

Another interesting phenomenon in peristaltic motion is trapping. In the wave frame, streamlines under certain conditions split to trap a bolus which moves as a whole with the speed of the wave. The effect of Weissenberg number $W_i$ on trapping can be seen through Fig.22 for an inclined asymmetric channel. Furthermore, Fig.22 shows that the bolus is anti-symmetric about the center line and its size decreases with an increase in $W_i$. It is observed from Figs.23 and 24 that the trapped bolus which are moving as whole increases in size with the increase in b and a. The effects of phase shift $\phi$ on trapping can be seen from Fig.8. It is depicted that increase in $\phi$ the trapping bolus which is moving as a whole decreases. The effect of Darcy number $D_r$ on the trapping is illustrated in Fig.26 and it is observed that the size of trapped bolus rapidly increases with increasing $D_r$. Fig.27 depicts the effects of channel width $d$ on trapping. The trapped bolus exists for small values of $d$, its size decreases with increasing $d$.

![Fig.22. streamlines for $b=0.7, \phi=0.5, a=0.7, \mu=0.01, \eta=0.01, D_r=0.7, \Theta=1.2, d=0.5, e=0.5$, and for different $W_i$; (a) $W_i=1.1$, (b) $W_i=1.2$; (c) $W_i=1.3$.](image)

![Fig.23. streamlines for $W_i=1.3, \phi=0.5, a=0.7, \mu=0.01, \eta=0.01, D_r=0.7, \Theta=1.2, d=0.5, e=0.5$, and for different $b$; (a) $b=0.6$, (b) $b=0.7$, (c) $b=0.8$.](image)
Fig. 24. Streamlines for $W=1.3, \phi=0.5, b=0.7, \mu=0.01, \eta=0.01, Dr=0.7, \Theta=1.2, d=0.5, e=0.5$, and for different $a$; (a) $a=0.6$, (b) $a=0.7$, (c) $a=0.8$.

Fig. 25. Streamlines for $W=1.3, a=0.7, b=0.7, \mu=0.01, \eta=0.01, Dr=0.7, \Theta=1.2, d=0.5, e=0.5$, and for different $\phi$; (a) $\phi=0.3$, (b) $\phi=0.6$, (c) $\phi=0.9$.

Fig. 26. Streamlines for $W=1.3, a=0.7, b=0.7, \mu=0.01, \eta=0.01, \phi=0.5, \Theta=1.2, d=0.5, e=0.5$, and for different $Dr$; (a) $Dr=0.6$, (b) $Dr=0.7$, (c) $Dr=0.9$. 

(a) (b) (c)
Fig. 27. Streamlines for $W_i=1.3$, $a=0.7$, $b=0.7$, $\mu=0.01$, $\eta=0.01$, $\phi=0.5$, $\Theta=1.2$, $D_r=0.7$, $e=0.5$, and for different $d$; (a) $d=0.3$, (b) $d=0.4$, (c) $d=0.5$.

4.4 Temperature characteristics

The expressions for temperature are given by Eq.(67). To explicitly see the effects of various parameters on temperature, Eq.(67) has been numerically evaluated and the results are presented in Fig.28-34. From Fig.28, it can be found that the temperature profiles are almost parabolic and temperature increases with increase of channel width $d$. Further, it can be noticed that the increase in Weissenberg number $W_i$ causes a decrease in temperature are plotted in Fig.29. Also Fig.30 display the influence of amplitude ratio of the upper wall on the temperature distribution. We note that temperature increases with increasing $a$. Fig.31 displays the influence of amplitude ratio of the lower wall on the temperature distribution. It can be noticed that temperature increases when $(1<\Theta<0.4)$ and decreases when $(1>\Theta>0.4)$ that occurs by increasing $b$. The effects increasing Darcy number $D_r$ and Eckert number $E_r$ on the temperature are plotted in Figs.32 and 33, we note that the increasing in $D_r$ and $E_r$ causes increases in temperature. While with the increase in Prandtl number $P_r$, the temperature field decreases Fig.34.

Fig.28. Effect of $d$ on temperature for $b=0.2$, $\phi=\pi/2$, $a=0.2$, $\mu=0.3$, $\eta=0.4$, $W_i=0.7$, $D_r=0.1$, $E_r=-4$, $P_r=1$, $e=0.01$, $x=1$, $\Theta=0.2$.

Fig.29. Effect of $W_i$ on temperature for $b=0.2$, $\phi=\pi/2$, $a=0.2$, $\mu=0.3$, $\eta=0.4$, $d=0.3$, $D_r=0.1$, $E_r=-4$, $P_r=1$, $e=0.01$, $x=1$, $\Theta=0.2$. 
Fig. 30. Effect of $a$ on temperature for $b=0.2$, $\phi=\pi/2$, $\mu=0.3$, $\eta=0.4$, $d=0.3$, $D_r=0.1$, $E_r=-4$, $P_r=1$, $W_i=0.8$, $e=0.01$, $x=1$, $\Theta=0.2$.

Fig. 31. Effect of $b$ on temperature for $a=0.2$, $\phi=\pi/2$, $\mu=0.3$, $\eta=0.4$, $d=0.3$, $D_r=0.1$, $E_r=-4$, $P_r=1$, $W_i=0.8$, $e=0.01$, $x=0.9$, $\Theta=0.2$.

Fig. 32. Effect of $D_r$ on temperature for $a=0.2, b=0.2$, $\phi=\pi/2$, $\mu=0.3$, $\eta=0.4$, $d=1.3$, $E_r=-4$, $P_r=1$, $W_i=0.8$, $e=0.01$, $x=0.9$, $\Theta=0.2$.

Fig. 33. Effect of $E_r$ on temperature for $a=0.2, b=0.2$, $\phi=\pi/2$, $\mu=0.3$, $\eta=0.4$, $d=0.3$, $D_r=0.1$, $P_r=1$, $W_i=0.8$, $e=0.01$, $x=1$, $\Theta=0.2$.

Fig. 34. Effect of $P_r$ on temperature for $a=0.2, b=0.2$, $\phi=\pi/2$, $\mu=0.3$, $\eta=0.4$, $d=0.3$, $D_r=0.1$, $x=1$, $W_i=0.8$, $e=0.01$, $E_r=-4$, $\Theta=0.2$. 
5. Concluding remarks

In this paper, we investigated the peristaltic transport of a Johnson-Segalman fluid through a porous medium in an inclined asymmetric 2-D channel under the assumptions of long-wavelength and low-Reynolds number. A perturbation solution for small Weissenberg number is obtained for the stream function, axial pressure gradient, pressure rise and temperature field over a wavelength. It is found that:

- The pressure rise over a wavelength \( \Delta P \) decreases with an increase in \( d \) in the pumping (\( \Delta P > 0 \)) and free pumping (\( \Delta P = 0 \)), while the situation is reversed in the co-pumping region (\( \Delta P < 0 \)).
- The pressure rise over a wavelength \( \Delta P \) decreases with an increase in Wessenberg number \( W_i \) and phase difference \( \phi \) in the pumping region (\( \Delta P > 0 \)), while the situation is reversed in the co-pumping region (\( \Delta P < 0 \)) and free pumping (\( \Delta P = 0 \)).
- The pressure rise over a wavelength \( \Delta P \) decreases by increasing \( F_r \), while increases by increasing \( D_r \) and \( \beta \).
- The axial pressure gradient decreases by increasing \( d, F_r, \phi, \eta \) and \( \Theta \), while increases by increasing \( a, b, W_i, R_e, \beta \) and \( \mu \).
- The size of the trapped bolus increases with an increase in \( a, b, D_r \), while decreases with an increase in \( W_i, \phi \) and \( d \).
- The temperature field increases with the increase in \( d, a, D_r \) and \( E_r \), while with the increase in \( W_i, b \) and \( P_r \) the temperature field decreases.

References

35. Reddy. MS., Raju GSS (2010), Non-linear peristaltic pumping of Johnson- Segalman fluid in an asymmetric channel under the effect of a magnetic field. European J. os scientific research 1, 147-164.