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Common Fixed Point Results in Convex 2-Metric Space for Altering Distance Function

Nidhi Gargav, Rajesh Shrivastava*, Rizwana Jamal**
Research Scholar Saifia Science College, Bhopal (M.P.)
*Professor and Head Govt. Science and Commerce College Benazeer Bhopal.
**Professor and Head, Department of Mathematics, Saifia Science College, Bhopal.

Abstract

In the present paper some common fixed point results are obtained for altering distance function which satisfies the (E.A.) property with respect to some $q \in M$, where M is q-starshaped subset of a convex 2-metric space. After that some invariant approximation results as an application are obtained for altering distance function. Our results are the special form in altering distance function of [50] and [51]

MSC: 47H10; 54H25

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2. Introduction & Preliminaries: In 1984, M.S. Khan , M. Swalech and S.Sessa [49] expanded the research of the metric fixed point theory to a new category by introducing a control function which they called an altering distance function.

Definition 2.1 ([49]) A function $\psi : \Re_+ \to \Re_+$ is called an altering distance function if the following properties are satisfied:

(ψ_1) $\psi(t) = 0 \Leftrightarrow t = 0$

(ψ_2) ψ is monotonically non-decreasing.

(ψ_3) ψ is continuous.

By ψ we denote the set of the all altering distance functions.

Theorem2.2 ([49]) Let (M,d) be a complete 2-metric space, let $\psi \in \Psi$ and let $S : M \to M$ be a mapping a > 0 which satisfies the following inequality

$$\Psi[d(Sx,Sy,a)] \leq \alpha\Psi[d(x,y,a)]$$

For all $x, y \in M$ and for some $0 < a < 1$. Then S has a unique fixed point $z_0 \in M$ and moreover for each $x \in M$ $\lim_{n\to\infty} S^n x = z_0$

Lemma 2.3 Let (M,d) be 2- metric space. Let $\{x_n\}$ be a sequence in M such that

$$\lim_{n\to\infty} \Psi[d(x_{n,}x_{n+1,},a)] = 0$$

If $\{x_n\}$ is not a Cauchy sequence in M , then there exist an $\varepsilon_0 > 0$ and sequences of integers positive $\{m(k)\}$ and $\{n(k)\}$ with

$$m(k) > n(k) > k$$

Such that $\Psi\ [d(x_{m(k),}x_{n(k),},a)] \geq \epsilon_0$, $\Psi\ [d(x_{m(k-1),}x_{n(k),},a)] < \epsilon_0$

(i) $\lim_{k\to\infty} \Psi\ [d(x_{m(k-1),}x_{n(k+1),},a)] = \epsilon_0$

(ii) $\lim_{k\to\infty} \Psi\ [d(x_{m(k),}x_{n(k),},a)] = \epsilon_0$

(iii) $\lim_{k\to\infty} \Psi\left[d\left(x_{m(k-1)}, x_{n(k)}, a\right)\right] = \epsilon_0$

Remark 2.4. Form Lemma 2.3 is easy to get

$$\lim_{k\to\infty} \Psi\left[d\left(x_{m(k+1)}, x_{n(k+1)}, a\right)\right] = \epsilon_0$$

In 1976, Jungck [1] established some common fixed point results for a pair of commuting self-maps in the setting of complete metric space. The first ever attempt to relax the commutativity of mappings was initiated by Sessa [2] who introduced a class of non-commuting maps called 'namely' weak commutativity. Further, in order to enlarge the domain of non-commuting mappings, Jungck [3] in 1986 introduced the concept of 'compatible maps' as a generalization of weakly commuting maps.

Definition 2.5 Two self-maps I and T of a 2-metric space (X,d) are called compatible if and only if $\lim_{n\to\infty} d(ITx_n, TIx_n, a) = 0$, Whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$. In 2002, Aamri and Moutawakil [4] obtained the notion of (E.A.) property which enables us to study the existence of a common fixed points of self-maps satisfying nonexpansive or Lipschitz type condition in the setting of non-complete metric space.

Definition 2.6 Two self-maps I and T of 2-metric space (X,d) are said to satisfy the (E.A.) property if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n = t \text{ for some } t \in X.$$

On the other side, in 1970 Takahashi [5] introduced the notion of convexity into the metric space, studied properties of such spaces and proved several fixed point theorems for nonexpansive mappings. Afterward Guay *et al.* [6], Beg and Azam [7], Fu and Huang [8], Ding [9], Ćirić *et al.* [10], and many others have studied fixed point theorem in convex metric spaces. In the recent past, fixed point theorems have been extensively applied to best approximation theory. Meinardus [11] was the first who employed the Schauder's fixed point theorem to prove a result regarding invariant approximation. Later on, Brosowski [12] generalized the result of Meinardus under different settings. Further significant contribution to this area was made by a number of authors (see [13–35]). Many of them considered the pair of commuting or noncommuting mappings in the setting of normed or Banach spaces. In 1992 , Beg *et al.* [36] proved some results on the existence of a common fixed point in the setting of a convex metric space and utilized the same to prove the best approximation results. After that, several authors studied common fixed point and invariant approximation results in the setting of convex metric space (see [36–40]) and references therein).

In this work, we introduce a new class of self-maps which satisfy the (E.A.) property with respect to some $q \in M$, where M is q-starshaped subset of a convex metric space and establish some common fixed point results for this class of self-maps. After that we obtain some invariant approximation results as application. Our results represent a very strong variant of the several recent results existing in the literature.

Firstly, we recall some useful definitions and auxiliary results that will be needed in the sequel. Throughout this paper, $\mathbb{N}$ and $\mathbb{R}$ denote the set of natural numbers and the set of real numbers, respectively.

Definition 2.7[5] Let (X,d) be a 2-metric space. A continuous mapping $W : X \times X \times [0,1] \to X$ is called a convex structure on X if, for all $x, y \in X$ and $\lambda \in [0\ 1,]$, we have

$$d\big(u, W(x,y,\lambda,a)\big) \le \lambda d(u,x,a) + (1-\lambda) d(u,y,a) \qquad (2.1)$$

for all $u \in X$.A metric space (X,d) equipped with a convex structure is called a convex metric space.

Definition 2.8 A subset M of a convex 2-metric space (X,d) is called a convex set if $W(x,y,\lambda) \in M$ for all $x, y \in M$ and $\lambda \in [0,1]$. The set M is said to be q-starshaped if there exists $q \in M$ such that $W(x,q,\lambda) \in M$ for all $x \in M$ and $\lambda \in [0,1]$. A set M is called starshaped if it is q-starshaped with respect to any $q \in M$.

Clearly, each convex set M is starshaped but the converse assertion is not true. Thus, the class of starshaped sets properly contains the class of convex sets.

Definition 2.9 A convex 2- metric space (X,d) is said to satisfy the Property (I), if for all $x, y, z \in X$ *and* $\lambda \in [0,1], a > 0$

$$d(W(x,z,\lambda), W(y,z,\lambda),) \le \lambda d(x,y,a). \qquad (2.2)$$

A normed linear space X and each of its convex subset are simple examples of convex metric spaces with W given by $W(x,y,\lambda) = \lambda x + (1-\lambda)y$ for all $x,y \in X$ and $0 \le \lambda \le 1$. Also, Property (I) is always satisfied in a normed linear space. There are many convex metric spaces which are not normed linear spaces (see [5,6]). For further information on a convex metric space, refer to [5 –10, 36-42].

Definition 2.10 Let (X,d) be a convex 2-metric space and M a subset of X. A mapping $I : M \to M$ is said to be

(1) affine, if M is convex and $I(W(x,y,\lambda)) = W(Ix,Iy,\lambda)$ for all $x,y \in M$ and $\lambda \in [0,1]$;

(2) q-affine, if M is q-starshaped and $I(W(x,q,\lambda)) = W(Ix,q,\lambda)$ for all $x \in M$ and $\lambda \in [0,1]$.

In [43] Pant define the concept of reciprocal continuity as follows.

Definition 2.11 Let (X,d) be a 2-metric space and $I,T : X \to X$. Then the pair (I,T) is said to be reciprocally continuous if and only if
$\lim_{n\to\infty} ITx_n = It$ and $\lim_{n\to\infty} TIx_n = Tt$
whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$. It is easy to see that if I and T are continuous, then the pair (I, T) is reciprocally con-tinuous but the converse is not true in general (see [[44], Example 2.3]). Moreover, in the setting of common fixed point theorems for compatible pairs of self-mappings satisfying some contractive conditions, continuity of one of the mappings implies their reciprocal continuity.

Definition 2.12 [45] A pair (I,T) of self-maps of 2- metric space (X,d) is said to be subcom-patible if there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n = t \text{ for some } t \in X \text{ and } \lim_{n\to\infty} d(ITx_n, TIx_n, a) = 0.$$

Obviously, compatible maps which satisfy the (E.A.) property are sub compatible but the converse statement does not hold in general (see [40], Example 2.5).

Definition 2.13 Let (X,d) be a 2-metric space, M a nonempty subset of X, and I and T be self-maps of M. A point $x \in M$ is a coincidence point (common fixed point) of I and T if
$Ix = Tx$ $(Ix = Tx = x)$. The set of coincidence points of I and T is denoted by $C(I,T)$ and the set of fixed points of I and T is denoted by $F(I)$ and $F(T)$, respectively. The pair $\{I,T\}$ is called:

(1) Commuting if $ITx = TIx$ for all $x \in M$.

(2) Weakly compatible [46] if $ITx = TIx$ for all $x \in C(I,T)$.

(3) Banach operator pair [24] if the set $F(I)$ is T- invariant, *i.e.* $T(F(I)) \subseteq F(I)$.

For more details about these classes, one can refer [27,47].

Definition 2.14 [19] Let M be a q-starshaped subset of convex 2-metric space (X,d) such that $q \in F(I)$ and is both I- and T-invariant. Then the self-maps I and T are called R-subweakly commuting on M if for all $x \in M$, there exists a real number $R > 0$ such that $d(ITx,TIx,a) \le R\ \mathrm{dist}(Ix,[q,Tx])$, where $[q,x] = \{W(x,q,\lambda): 0 \le \lambda \le 1\}$.

Clearly, R-subweakly commuting maps are compatible but the converse assertion is not necessarily true (see [31], Example 15).

For a nonempty subset M of a metric space (X,d) and $p \in X$, an element $y \in M$ is called a best approximation to p if $d(p,y) = dist(p,M)$, where $dist(p,M) = inf\{d(p,z): z \in M\}$. The set of all best approximations to p is denoted by $B_M(p)$.

Definition 2.15 Let M be a q-starshaped subset of a convex 2-metric space (X,d) and let
$I,T: M \to M$ with $q \in F(I)$. The pair (I,T) is said to satisfy the (E.A.) property with respect to q if there exists a sequence $\{x_n\}$ in M such that for all $\lambda \in [0,1]$

$$\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} T_\lambda x_n = t \text{ for some } t \in M, \tag{3.1}$$

where $T_\lambda x = W(Tx,q,\lambda)$.

Obviously, if the pair (I,T) satisfies the (E.A.) property with respect to q, then I and T satisfy the (E.A.) property but the converse assertion is not necessarily true.

Remark 2.16[50] If M is convex subset of a convex 2-metric space X and p is common fixed point of the self-maps I and T of M, then the pair (I,T) satisfies the (E.A.) property with respect to p but converse is not true in general.

The following lemma is a particular case of Theorem 4.1 of Chauhan and Pant [48] for 2- metric spaces.

Lemma 2.17[50] Let *I and T be self-maps of a 2-metric space (X,d). If the pair (I,T) is subcom-patible, reciprocally continuous and satisfy*

$$d(Tx,Ty,a) \le \lambda \max\left\{\begin{matrix} d(Ix,Iy,a), d(Ix,Tx,a), \\ d(Iy,Ty,a), d(Ix,Ty,a), d(Iy,Tx,a)\end{matrix}\right\} \tag{2.17 a}$$

for some $\lambda \in (0,1)$ and all $x,y \in X, a > 0$. Then I and T have a unique common fixed point in X.

Definition (2.18) A 2- metric space is a space X in which for each triple of points x, y, z, there exists a real function d (x,y,z,) such that

[M_1] to each pair of distinct points x,y,z,

d (x,y,z) ≠ 0

[M_2] d (x,y,z) = 0 when at lest two of x,y,z are equal

[M_3] d (x,y,z) = d (y,z,x) = d (x,z,y)

[M_4] d (x,y,z) ≤ d (x,y,v) + d (x,v,z) + d (v,y,z) for all x,y,z, v in X.

Definition (2.19): A sequence $\{x_n\}$ in a 2-metic space (X,d) is said to be convergent at x if

$\lim_{n\to\infty} d(x_n, x, z) = 0$ for all z in X.

Definition (2.20) A sequence $\{x_n\}$ in a 2-metric space, (x, d) is said to be Cauchy sequence if

$\lim_{m,n\to\infty} d(x_n, x, z) = 0$ for all z in X.

Definition (2.21) A 2-metic space (X, d) is said to be complete if every Cauchy sequence in X is convergent.

3. Main results

Theorem 3.1 *Let* A function $\psi : \Re_+ \to \Re_+$ is an altering distance function, *M be a nonempty q-starshaped subset of a convex 2-metric space (X, d) with Property (I) and let I and T be continuous self-maps on M such that the pair (I, T) satisfies the (E.A.) property with respect to q. Assume that I is q-affine, $cl(T(M))$ is compact. If I and T are compatible and satisfy the inequality*

$$\Psi d(Tx, Ty, a) \leq \Psi \max\left\{\begin{matrix} d(Ix, Iy, a), dist(Ix, [Tx, q]), dist(Iy, [Ty, q]), \\ dist\ (Ix, [Ty, q]), dist\ (Iy, [Tx, q]) \end{matrix}\right\}, \qquad (3.1\text{ a})$$

for all $x, y \in M$, a>0 *then $M \cap F(T) \cap F(I) \neq \phi$.*

Proof *:* For each $n \in \mathbb{N}$, we define $T_n : M \to M$ by

$$T_n x = \Psi\, W\,(Tx, q, \lambda_n) \text{ for all } x \in M, \qquad (3.1\text{ b})$$

where λ_n is a sequence in (0,1) such that $\lambda_n \to 1$.

Now, we have to show that for each $n \in \mathbb{N}$, the pair (T_n, I) is subcompatible. Since I and T satisfy the (E.A.)-property with respect to q, there exists a sequence $\{x_m\}$ in M such that for all $\lambda \in [0,1]$

$$\lim_{m\to\infty} Ix_m = \lim_{m\to\infty} T_\lambda x_m = t \in M, \qquad (3.1\text{c})$$

where $T_\lambda x_m = \Psi W(Tx_m, q, \lambda)$.

Since $\lambda_n \in (0,1)$, using above , for each $n \in \mathbb{N}$, we have

$$\lim_{m\to\infty} T_n x_m = \lim_{m\to\infty} \Psi W(Tx_m, q, \lambda_n)$$
$$= \lim_{m\to\infty} T_{\lambda_n} x_m = t \in M.$$

Thus, we have

$$\lim_{m\to\infty} Ix_m = \lim_{m\to\infty} T_n x_m = t \in M. \qquad (3.1\text{d})$$

Now, using the fact that I is q-affine and Property (I) is satisfied, we get

$$\Psi d(T_n Ix_m, IT_n x_m, a) = \Psi d\big(W(TIx_m, q, \lambda_n), I\big(W(Tx_m, q, \lambda_n)\big), a\big)$$
$$= \Psi d(W(TIx_m, q, \lambda_n), W(ITx_m, q, \lambda_n), a)$$
$$\leq \lambda_n \Psi d(TIx_m, ITx_m, a)\,. \qquad (3.1\text{e})$$

Since I and T are compatible, we have

$$\lim_{m\to\infty} \Psi d(ITx_m, TIx_m, a) = 0\,.$$

Now, letting $m \to \infty$ in (3.8), we get
$\lim_{m\to\infty} \Psi d(IT_n x_m, T_n I x_m, a) = 0.$ (3.1f)

Hence, on account of above equations it follows that the pair (T_n, I) is subcompatible for each $n \in \mathbb{N}$. Since I and T are continuous, for each $n \in \mathbb{N}$, the pair (T_n, I) is reciprocally continuous. Also,
$\Psi d(T_n x, T_n y, a) = \Psi d(W(Tx, q, \lambda_n), W(Ty, q, \lambda_n), a)$
$\leq \lambda_n \Psi d(Tx, Ty)$, Property (I)
$\leq \lambda_n \Psi \max \left\{ \begin{matrix} d(Ix, Iy, a), dist(Ix, [Tx, q]), \\ dist(Iy, [Ty, q]), dist(Ix, [Ty, q]), dist(Iy, [Tx, q]) \end{matrix} \right\}$
$\leq \lambda_n \Psi \max\{d(Ix, Iy, a), d(Ix, T_n x, a), d(Iy, T_n y, a), d(Ix, T_n y, a), d(Iy, T_n x, a)\}$ (3.10)

for each $x, y \in M$ and $0 < \lambda_n < 1$ using above results , for each $n \in \mathbb{N}$, there exists $x_n \in M$ such that $x_n = I x_n = T_n x_n$.
Now we will show that $\{x_n\}$ *is cauchy squence*. Suppose it is not so which means that there is a constant ε_0 >0 such that for each positive integer j, there are positive integers m(j) and n(j) with m(j)>n(j)>j such that

that $\Psi\left[d\left(x_{m(j),} x_{n(j),}, a\right)\right] \geq \epsilon_0$, $\Psi\left[d\left(x_{m(j-1),} x_{n(j),}, a\right)\right] < \epsilon_0$

(i) $\lim_{j\to\infty} \Psi\left[d\left(x_{m(k-1),} x_{n(k+1),}, a\right)\right] = \epsilon_0$
(ii) $\lim_{k\to\infty} \Psi\left[d\left(x_{m(k),} x_{n(k),}, a\right)\right] = \epsilon_0$
(iii) $\lim_{j\to\infty} \Psi\left[d\left(x_{m(k+1),} x_{n(k)+1,}, a\right)\right] = \epsilon_0$

For $x=x_{m(j)}$ and $y=x_{n(k)}$ Thus we get a contradiction.

So $\{x_n\}$ is Cauchy sequence in X
Now the compactness of $cl(T(M))$ implies that there exists a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ such that $Tx_m \to z$ as $m \to \infty$. Further, it follows that

$x_m = T_m x_m = \Psi W(Tx_m, q, \lambda_m) \to z$ as $m \to \infty$.

By the continuity of I and T, we obtain $Iz = z = Tz$. Thus, $M \cap F(T) \cap F(I) \neq \phi$.

The following corollaries immediately follow from Theorem 3.1
Corollary 3.2 *Let* A function $\psi : \Re_+ \to \Re_+$ is an altering distance function, *M be a nonempty q-starshaped subset of a convex 2-metric space* (X, d) *with Property* (I)A function *and let I and T be continuous self-maps on M such that the pair* (I, T) *satisfies the* $(E.A.)$ *property with respect to q. Assume that I is q-affine,* $cl(T(M))$ *is compact. If I and T are compatible and satisfy the inequality*

$$\Psi d(Tx, Ty, a) \leq \Psi \max \left\{ \begin{matrix} d(Ix, Iy, a), dist(Ix, [Tx, q]), dist(Iy, [Ty, q]), \\ \frac{1}{2}[\text{dist}(Ix, [Ty, q]) + \text{dist}(Iy, [Tx, q])] \end{matrix} \right\} \quad (3.2a)$$

for all $x, y \in M$, where a>0, *then* $M \cap F(T) \cap F(I) \neq \phi$.

Corollary 3.3 *Let* a function $\psi : \Re_+ \to \Re_+$ is an altering distance function, *M be a nonempty q-starshaped subset of a convex metric space* (X, d) *with Property* (I) *and let I and T be continuous self-maps on M such that the pair* (I, T) *satisfies the* $(E.A.)$ *property with respect to q. Assume that I is q-affine,* $cl(T(M))$ *is compact,* a>0,. *If I and T are R-subweakly commuting and satisfy the inequality*
$$\Psi d(Tx, Ty, a) \leq \Psi \max \left\{ \begin{matrix} d(Ix, Iy, a), dist(Ix, [Tx, q]), dist(Iy, [Ty, q]), \\ \text{dist}(Ix, [Ty, q]), \text{dist}(Iy, [Tx, q]) \end{matrix} \right\} \quad (3.3a)$$
for all $x, y \in M$, *then* $M \cap F(T) \cap F(I) \neq \phi$.
Theorem 3.3 A function $\psi : \Re_+ \to \Re_+$ is an altering distance function .*Let I and T be self-maps of a convex 2-metric space* (X, d) *with Property* (I), $p \in F(I) \cap F(T)$, *and M be a subset of X such that* $T(\delta M \cap M) \subseteq M$, *where* δM *denotes the boundary of M. Suppose that* $B_M(p)$ *is nonempty, q-starshaped with* $I(B_M(p)) \subset B_M(p)$ *and I is q-affine and continuous on* $B_M(p)$. *If the maps I and T are compatible, satisfy the* $(E.A.)$ *property with respect to q on* $B_M(p)$, *and also satisfy for all* $x, y \in B_M(p) \cup \{p\}$, a>0,

$$\Psi d(Tx,Ty,a) \le \begin{cases} \Psi d(Ix,Ip,a) & if\ y=p, \\ \Psi \quad \max\{d(Ix,Iy,a), dist(Ix,[Tx,q]), dist(Iy,[Ty,q]), \\ \quad dist(Ix,[Ty,q]), dist(Iy,[Tx,q])\} & if\ y \in B_M(p), \end{cases} \quad (3.3.a)$$

then I and T have a common fixed point in $B_M(p)$, provided $cl(T(B_M(p)))$ is compact and T is continuous on $B_M(p)$.

Proof: Let $x \in B_M(p)$. Then for all $\lambda \in (0,1)$, we have

$$\Psi d(p, W(x,p,\lambda), a) \le \lambda \Psi d(p,x,a) + (1-\lambda)\Psi d(p,p,a) = \lambda \Psi d(p,x,a) < dist(p,M).$$

Thus, it follows that $\{W(x,p,\lambda): \lambda \in (0,1)\} \cap M = \phi$ and so $x \in \delta M \cap M$. As $T(\delta M \cap M) \subseteq M$, therefore $Tx \in M$. Since $Ix \in B_M(p)$ and $p \in F(I) \cap F(T)$, on account of ((3.3.a)), we have

$$\Psi d(Tx,p,a) = \Psi d(Tx,Tp,a) \le \Psi d(Ix,Ip,a) = \Psi d(Ix,p,a) = dist(p,M),$$

which shows that $Tx \in B_M(p)$, and in all I and T are self-maps on $B_M(p)$.

Corollary 3.4 Let a function $\psi : \Re_+ \to \Re_+$ is an altering distance function, I and T be self-maps of a convex metric space (X,d) with Property (I), $p \in F(I) \cap F(T)$, and M be a subset of X such that $T(\delta M \cap M) \subseteq M$, where δM denotes the boundary of M, a > 0, .Suppose that $B_M(p)$ is nonempty, q-starshaped with $I(B_M(p)) \subset B_M(p)$ and I is q-affine and continuous on $B_M(p)$. If the maps I and T are R-subweakly commuting, satisfy the (E.A.) property with respect to q on $B_M(p)$ and also satisfy for all $x,y \in B_M(p) \cup \{p\}$

$$\Psi d(Tx,Ty,a) \le \begin{cases} \Psi d(Ix,Ip,a) & if\ y=p, \\ \Psi \quad \max\{d(Ix,Iy,a), dist(Ix,[Tx,q]), dist(Iy,[Ty,q]), \\ \quad dist(Ix,[Ty,q]), dist(Iy,[Tx,q])\} & if\ y \in B_M(p), \end{cases} \quad (3.4a)$$

then I and T have a common fixed point in $B_M(p)$, provided $cl\left(T(B_M(p))\right)$ is compact and T is continuous on $B_M(p)$.

We define $D = B_M(p) \cap C_M^I(p)$, where $C_M^I(p) = \{x \in M: Ix \in B_M(p)\}$.

Theorem 3.5 *Let* a function $\psi : \Re_+ \to \Re_+$ is an altering distance function. *I and T be self-maps of a convex 2-metric space (X,d) with Property (I), $p \in F(I) \cap F(T)$, and M be a subset of X such that $T(\delta M \cap M) \subseteq M$, where δM denotes the boundary of M. Suppose that D is nonempty, q-starshaped with $I(D) \subset D$ and I is q-affine and nonexpansive on D,* a>0. *If the maps I and T are compatible, satisfy the (E.A.) property with respect to q on D, and also satisfy for all $x,y \in D \cup \{p\}$*

$$\Psi d(Tx,Ty,a) \le \begin{cases} \Psi d(Ix,Ip,a) & if\ y=p, \\ \Psi \quad \max\{d(Ix,Iy), dist(Ix,[Tx,q]), dist(Iy,[Ty,q]), \\ \quad dist(Ix,[Ty,q]), dist(Iy,[Tx,q])\} & if\ y \in D, \end{cases} \quad (3.5a)$$

then I and T have a common fixed point in $B_M(p)$, provided $cl(T(D))$ is compact and T is continuous on D.

Proof *Let $x \in D$.* Then $x \in B_M(p)$, and therefore, proceeding as in the proof of Theorem, we have $Tx \in B_M(p)$. Since I is nonexpansive and $p \in F(I) \cap F(T)$, it follows from (4.4) that

$$\Psi d(ITx,p,a) = \Psi d(ITx,Ip,a) \le \Psi d(Tx,p,a) = \Psi d(Tx,Tp,a) \le \Psi d(Ix,p,a) = dist(p,M).$$

Thus $ITx \in B_M(p)$ and so $Tx \in C_M^I(p)$, which gives $Tx \in D$. Hence I and T are self-maps on D. Now, using above therems, there exists $z \in B_M(p)$ such that z is a common fixed point of I and T.

Corollary 3.6 *Let* a function $\psi : \Re_+ \to \Re_+$ is an altering distance function. *I and T be self-maps of a convex 2-metric space (X,d) with Property (I), $p \in F(I) \cap F(T)$, and M be a subset of X such that $T(\delta M \cap M) \subseteq M$, where δM denotes the boundary of M. Suppose that D is nonempty, q-starshaped with $I(D) \subset D$, and I is q-affine and nonexpansive on D. If the maps I and T are R-subweakly commuting, satisfy the (E.A.)* Property with respect to q on D, a>0, and also satisfy for all $x,y \in D \cup \{p\}$

$$\Psi d(Tx,Ty,a) \le \begin{cases} \Psi d(Ix,Ip,a) & if\ y=p, \\ \Psi \quad \max\{d(Ix,Iy), dist(Ix,[Tx,q]), dist(Iy,[Ty,q]), \\ \quad dist(Ix,[Ty,q]), dist(Iy,[Tx,q])\} & if\ y \in D, \end{cases} \quad (3.6a)$$

Then I and T have a common fixed point in $B_M(p)$, provided $cl(T(D))$ is compact and T is continuous on D.

Let $D_M^{R,I}(p) = B_M(p) \cap G_{R,I}^M(p)$, Where $G_M^{R,I}(p) = \{x \in M: \Psi d(Ix,p,a) \le (2R+1)dist(p,M)\}$

Theorem 3.7 *Let* a function $\psi : \Re_+ \to \Re_+$ is an altering distance function, *I and T be self-maps of a convex 2-metric space* (X,d) *with Property* (I), $p \in F(I) \cap F(T)$, *and M be a subset of X such that* $T(\delta M \cap M) \subseteq M$, *where* δM *denotes the boundary of* M, a > 0,. *Suppose that* $D_M^{R,I}(p)$ *is nonempty, q-starshaped with* $I\left(D_M^{R,I}(p)\right) \subset D_M^{R,I}(p)$, *and I is q-affine and continuous on* $D_M^{R,I}(p)$, *If the maps I and T are R-subweakly commuting, satisfy the* $(E.A.)$ *property with respect to q on* $D_M^{R,I}(p)$, *and also satisfy for all* $x, y \in D_M^{R,I}(p) \cup \{p\}$,

$$\Psi d(Tx,Ty,a) \le \begin{cases} \Psi d(Ix,Ip,a) & if\ y = p, \\ \Psi \quad \max\{d(Ix,Iy), dist(Ix,[Tx,q]), dist(Iy,[Ty,q]), \\ \quad dist(Ix,[Ty,q]), dist(Iy,[Tx,q])\} & if\ y \in B_M(p), \end{cases} \tag{3.7a}$$

then I and T have a common fixed point in $B_M(p)$, *provided* $cl\left(T\left(D_M^{R,I}(p)\right)\right)$ *is compact and T is continuous on* $D_M^{R,I}(p)$.

Proof Let $x \in D_M^{R,I}(p)$, , we have
$Tx \in D_M^{R,I}(p)$, Since I and T are R-subweakly commuting and $p \in F(I) \cap F(T)$, it follows that

$$\begin{aligned} \Psi d(ITx,p,a) &= \Psi d(ITx,Tp,a) \\ &\le \Psi d(ITx,TIx,a) + \Psi d(TIx,Tp,a) \\ &\le R\ dist(Tx,[q,Ix]) + \Psi d(I^2x,Ip,a) \\ &\le R\Psi d(Tx,Ix,a) + \Psi d(I^2x,Ip,a) \\ &\le R[\Psi d(Tx,Tp,a) + \Psi d(Ix,Tp,a)] + \Psi d(I^2x,Ip,a) \\ &\le R[dist(p,M) + dist(p,M) + dist(p,M) \\ &\qquad = (2R+1)\ dist\ (p,M). \end{aligned}$$

Thus $Tx \in G_{R,I}^M(p)$. Hence I and T are self-maps on $D_M^{R,I}(p)$there exists $z \in B_M(p)$ such that z is a common fixed point of I and$T$.

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