Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.6, No.4, 2016 www.iiste.org

STUDY OF STATIC SPHERICALLY SYMMETRIC FLUID DISTRIBUTION IN EINSTEIN –CARTAN THEORY

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Abstract

The problem of static fluid sphere in the framework of Einstein-Cartan theory is considered and a new technique to obtain the solution of Einstein –Cartan Field Equations in an analytic form by the method of quadrature is developed. The application of the technique in general cases give some exact solutions in a quite easy manner.

Keywords: Static, Quadrature, Exact Solution

1. Introduction

In the Einstein theory of gravitation, the role of spin is not so important, because the effect of spins of many particles cancel out one another, whereas the effect of mass is additive. Hence only curvature due to mass is considered. Einstein-Cartan theory is the generalization of Einstein's theory. In this both curvature and torsion are incorporated. Spin is a source of torsion. One can consider Einstein-Cartan theory as the theory of two tensor fields, the metric field g and the torsion field Q.

Since the prediction of E-C theory differs from those of general relativity only for matter filled regions, therefore, besides cosmology an important application field for E-C theory is relativistic astrophysics dealing with the interior of stellar objects like neutron starts with some alignment of spins of the constituent particles and under conditions when torsion may produce some observable effects. As such it seems desirable to understand the full implication of the E-C theory for finite distributions like fluid spheres with non-zero pressure. With this view the problem of static-fluid spheres in the E-C theory have been considered by many workers (Prasanna 1975, Kerlick 1975, Kuchowicz 1975 and Skinner and Webb 1977).

In this paper, the problem of static fluid sphere in the framework of Einstein-Cartan theory is considered and a new technique to obtain the solution in an analytic form by the method of quadrature is developed. The application of the technique in general cases give some exact solutions in a quite easy manner.

2. The Einstein –Cartan Field Equations

Let M be a C^∞ four dimensional, oriental, connected Hausdorff differential manifold with a Lorentz metric g defined on it. All geometric objects other than the forms are defined by their components with respect to a field of coframes ν^i (in the contingent space of M) which are linearly independent at each point of M. Since we are interested in spinor fields, we take Θ^i to be in general non-holonomic and the associated tetrad to be orthonormal. Since the manifold is paracompact, there exists a connection on it which we assume to be metric linear connection. The metric g and the connection w are described with respect to the chosen co-frame Θ^i by the metric components g_{ij} and by the set of one form ω_i, defining the covariant derivative respectively.

Hence we have

(2.1) $g = ds^2 = g_{ij}\, \theta^i\, \theta^j$, ij themselves are completely determined by 64 functions Γ^i_{kj} such that

(2.2) $\omega^i_j = \Gamma^i_{kj}\ \ \theta^k$

The Einstein –Cartan field equations are

(2.3) $R_i^j - \frac{1}{2} R\, \delta_i^j = -\chi t_i^j$, where R_j^i is Ricci curvature tensor, R is curvature scalar, δ_j^i metric tensor and T_j^i is stress energy momentum tensor. $\chi = 8\pi$

(2.4) $Q_{jk}^i - \delta_j^i\ Q_{lk}^i - \delta_k^i\ Q_{ji}^i = -\chi\, S_{jk}^i$, where Q_{jk}^i is torsion tensor and S_{jk}^i is spin tensor and t_i^j and S_{jk}^i are defined through the relations

(2.5) $t_i = \eta_j\, t_i^j$, $S_{jk} = \eta_i\, S_{jk}^i$.

A static spherically symmetric matter distribution is considered which is represented by the space –time metric

(2.6) $ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2\,(d\theta^2 + \sin^2\theta\; d\emptyset^2)$, ν and λ being functions of r. If θ^i represents an orthonormal co-frame we have from (2.1) and (2.6)

(2.7) $\theta^1 = e^{\lambda/2}\, dr$, $\theta^2 = r\, d\theta$, $\theta^3 = r \sin\theta\, d\emptyset$, $\theta^4 = e^{\nu/2}\, dt$,

so that g_{ij} = diag (1, -1, -1, -1).

Assuming that the spins of the individual particles composing the fluid are all aligned in the radial direction, we get for the spin tensor S_{ij} the only independent non-zero components to be S_{23} = k (say). Since the fluid is supposed to be static, has the velocity four-vector $u^i = \delta_4^i$.

Thus the non-zero components of S_{jk}^i are

(2.8) $S_{23}^4 = - S_{32}^4 = k$.

Hence from the Cartan equations (2.4), the non-zero components of Q_{jk}^i are obtained.

(2.9) $Q_{23}^4 = - Q_{32}^4 = - x\, k$.

Thus for a perfect fluid distribution with pressure p and density ρ the field equation (2.3) finally reduce to

(2.10) $8\,\pi\, p = 16\,\pi^2\, k^2 - \frac{1}{r^2} + e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r}\right)$,

(2.11) $8\,\pi\, \rho = 16\,\pi^2\, k^2 + \frac{1}{r^2} - e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right)$,

(2.12) $\frac{e^{\lambda}}{r^2} = \frac{1}{r^2} - \frac{\nu'^2}{4} - \frac{\nu''}{2} + \frac{\nu'\,\lambda'}{4} + \frac{\nu' + \lambda'}{2r}$, where dashes denote differentiation with respect to r.

The conservation law gives us the relations

(2.13) $\nabla_i\,[(\rho + p)\ u^i] = 0$ (matter conservation)

(2.14) $\nabla_i\,(k\, u^i) = 0$ (spin conservation) and

(2.15) $p' + \frac{1}{2}(\rho + p)\ \nu' + \lambda\, k\,(k' + \frac{k\,\nu'}{2}) = 0$.

If the equation of hydrostatic equilibrium is used

(2.16) $p' + \frac{1}{2}(\rho + p)\ \nu' = 0$,

The following equation is obtained.

(2.17) $k' + \frac{k\,\nu'}{2} = 0$.

From (2.17) we have

(2.18) $\quad$ k = $A_1\, e^{-\nu/2}$, where A_1 is a constant of integration.

In principle we have a completely determined system if an equation of state is specified. However, as is well known that in practice this set of equations is formidable to solve using a pre-assigned equation of state , except perhaps for the case ρ = p, which may not be physically meaningful. Secondly, we have the equation of boundary conditions to be chosen for fitting the solutions in the interior and the exterior of the star. A very interesting aspect of the Einstein- Cartan theory is that outside the fluid distribution the equations reduce to Einstein's equations for empty space viz. R_{ij} = 0, since there is no spin density.

Now, if we define

(2.19) $\quad \bar{\rho} = \rho - 2\pi\, k^2, \qquad \bar{p} = p - 2\pi\, k^2$, then the equations (2.10) and (2.11) take the usual general relativistic form for a static fluid sphere as given by

(2.20) $\quad 8\pi\,\bar{p} = -\frac{1}{r^2} + e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r}\right)$,

(2.21) $\quad 8\pi\,\bar{\rho} = \frac{1}{r^2} + e^{-\lambda}\left(-\frac{1}{r^2} + \frac{\lambda'}{r}\right)$, along with (2.12).

The equation of continuity (2.15) now becomes

(2.22) $\quad \frac{d\bar{p}}{dr} + \frac{1}{2}\left(\bar{\rho} + \bar{p}\right)\nu' = 0.$

In $\bar{p}$ and $\bar{\rho}$ the square term of spin behaves as the effective repulsive force. The repulsion can be important if the expression $2\pi\, k^2$ is of the same order as the energy density ρ. It is clear from these equations that it is the $\bar{p}$ and not p which is continuous across the boundary r = r_0 of the fluid sphere. The continuity of $\bar{p}$ across the boundary ensures that of $(\nu' e^{\nu})$. Further with $\bar{p}$ and $\bar{\rho}$ replacing p and ρ respectively, we are assured that the metric coefficients are continuous across the boundary. Hence we shall apply the usual boundary conditions to the solutions of equations (2.12), (2.20) and (2.21).

The boundary conditions are

(2.23) $\quad [e^{-\lambda}]_{r=r_0} = [e^{\nu}]_{r=r_0} = \left(1 - \frac{2m}{r_0}\right)$,

$\bar{p}$ = 0 at $r = r_0$, where r_0 is the radius and m is the mass of the fluid sphere. The total mass, as measured by an external observer, inside the fluid sphere of radius r_0 is given by

(2.24) $\quad m = 4\pi \int_0^{r_0} \bar{\rho}\, r^2\, dr = 4\pi \int_0^{r_0} \rho\, r^2\, dr - 8\pi^2 \int_0^{r_0} k^2(r)\, r^2\, dr.$

Thus the total mass of the fluid sphere is modified by the correction,

$$8\pi^2 \int_0^{r_0} k^2(r)\, r^2\, dr.$$

3. Solution of the Field Equations

We have to solve equation (2.12) for ν and λ. This may be fulfilled by quadrature in a number of ways e.g. Tolman specifies various conditions on the functions ν and λ that simplify, the equation and allow immediate integration while Adler in 1974 and Whitman in 1977 find λ by judicious choice of ν (r). We note that λ may be obtained if ν is given and vice-versa. Once ν and λ are obtained, p and ρ follow directly from equations (2.10) and (2.11).

We define

(3.1) $\quad \nu = 2 \log Y$

Then using equation (2.12), we get the differential equation

(3.2) $Y'' - (\frac{1}{r} + \frac{\lambda'}{r}) Y' + (\frac{e^{\lambda}}{r^2} - \frac{\lambda'}{2r} - \frac{1}{r^2})$ Y = 0

It is not always possible to get a traceable solution from the analytic specification of the equation of state. In these cases numerical and graphic technique are easy to apply. Exact solution in terms of known functions are most easily obtained by requiring one of the field variable to satisfy some subsidiary condition which simplify the full set of equations. Once the field equations are solved in this manner, an equation of state then can be extracted. Such solutions may be useful in understanding a system in the extreme, relativistic limit where we cannot specify a priori what the equation of state might be.

As stated above, the set of equations (2.10) to (2.12) cannot be solved without either choosing an equation of state or making a specific assumption on one of the functions p, ρ, ν and λ. For this we assume

(3.3) $e^{\lambda(r)} = A\ r^n$, where A and n are constants.

Substitution of equation (3.3) in (3.2) provides

(3.4) $Y'' - (\frac{1}{r} + \frac{n}{2r}) Y' + (A\, r^{n-2} - \frac{n}{2r^2} - \frac{1}{r^2})$ Y = 0.

This is a second order differential equation in Y for the general value of n and A.

We solve it for n = - 2

Equation (3.4) reduces to

(3.5) $Y'' - \frac{A}{r^4}$ Y = 0

which has the solution

(3.6) Y = $\frac{A}{6r^2} + B_4$ r + C_4.

Thus

(3.7) $e^{\nu} = (\frac{A}{6r^2} + B_4\, r + C_4)^2, \quad e^{\lambda} = \frac{A}{r^2}$, where B_4 and C_4 are constants.

In this case pressure and density are

(3.8) $8\pi r^2 \rho(r) = 1 - \frac{2r^2}{A} + 16\pi^2 A_1^2 (\frac{A}{6r^2} + B_4\, r + C_4)^{-2}$

(3.9) $8\pi r^2 p(r) = -1 + \frac{2}{A}\left[\frac{-\frac{A}{r^3} + B_4\, r^3}{\frac{A}{6r^2} + B_4\, r + C_4}\right]$

$+ 16\pi^2 (\frac{A}{6r^2} + B_4\, r + C_4)^{-2}$.

Spin density K is given by

(3.10) K = $A_1 [\frac{A}{6r^2} + B_4\, r + C_4]^{-1}$.

The constants A, B_4, C_4 and A_1 are given by

(3.11) A = $\frac{1}{R_b}(1 - \frac{2M}{R_b})^{-1}$

(3.12) $B_4 = \frac{1}{3\, R_b^5 (1 - \frac{2M}{R_b})} + \frac{M}{R_b^2 (1 - \frac{2M}{R_b})^{1/2}}$,

(3.13) $C_4 = (1 - \frac{2M}{R_b})^{1/2} - \frac{1}{2\, R_b^4 (1 - \frac{2M}{R_b})} + \frac{M}{R_b (1 - \frac{2M}{R_b})^{1/2}}$

(3.14) $8\, \pi\, R_b^2\, \rho\, (R_b) = 1 - \frac{2R_b^2}{A} + 16\, \pi\, A_1^2 \, (\frac{A}{6R_b^2} + B_4\, R_b + C_4)^{-2}$

4. Conclusion

The Einstein –Cartan Field Equations are written for a perfect fluid distribution with pressure p and density ρ. For solving this, the method of quadrature is used. Since exact solution in terms of known functions can be obtained by requiring one of the field variable to satisfy some subsidiary condition which simplify the full set of equations, we define ν = 2 log Y. Also it is assumed that $e^{\lambda(r)} = A\ r^n$, where A and n are constants. The equation is solved for n = -2. The equations for pressure p, density ρ and spin density K are obtained.

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