Study on Nielsen Fixed Point Theorem (A Review)

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ABSTRACT
Fixed-point theory plays an important role in solving the existence and uniqueness of solutions of differential equation, in solving Eigen value Problems and Boundary Value problems. Fixed-point theory also contributes in characterization of the completeness of matric spaces. Due to its applications in various disciplines of mathematical sciences, the Banach contraction and fixed-point theorems have been established. The ideas have a much wider scope than might be suspected and can be applied to establish many other existence theorem in the theory of differential and integral equations. There arenumerous extension of Banach's fixed point theorem by generalization its hypothesis while retaining the convergence property of successive iterations the unique fixed point of mapping.Inthepresent paper we discuss about fixed point and Nielsen fixed point theorems with some review
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INTRODUCTION:

Let f be a continuous mapping of the closed interval [-1, 1] into itself. Figure suggests that the graph of f must touch or cross the indicated diagonal, or more precisely, that there must be a point x_0 in [-1, 1], with the property that f (x_0) $=$ x_0.The proof is easy. We consider the continuous function F defined on [-1, 1], by F(x) =f(x) = $-$x, and we observed that F $(-1) \geq 0$ and that F (1) $\leq$ 0.It now follows from the Weierstrass intermediate value theorem that there exists a point x_0 in [-1, 1], such that F(x_0) =0 or f (x_0) $=$ x_0.

It is convenient to describe this phenomenon by means of the following terminology. A topological space X is called a fixed point space if every continuous mapping f of X into itself has a fixed point, in the sense that f (x_0) $=$ x_0 for the some x_0 in X .The remarks in the above paragraph show that [-1, 1] is a fixed point space.

The idea of "fixed points" was first introduced by Poincare [43] in1866, while studying the classical problems of vector distribution on surface. In 1912 Brouwer [7] has shown that if D^n is a closed n-dimensional disk then each continuous mapping f:$D^n \rightarrow D^n$has a fixed point. Although this theorem did not indicate about uniqueness of fixed point, but it has been proved very useful application in Analysis and Algebraic Topology. In 1922 the first fixed point theorem generally known as the "Banach Contraction Principal" [3], appeared in explicit from in the Banach's thesis. It was used to establish the existence of a solution for integral equation. The contraction mapping defined by Banach was follows:

Let (X,d) be a metric space. A mapping F : X→X is called a Contraction mapping if there exists K, 0≤ K <1 for all x,y ϵ X we have d (Fx, Fy) ≤ K d (x,y). The explicit form of this theorem is as follows:

Let (X,d) be a complete metric space. F:X→X is a contraction mapping, then there exists a unique fixed point in X. Now fixed point theory has become an important branch of nonlinear Analysis as well as of Topology. The study of fixed point theorems and their applications initiated long ago, still continue to be a highly interesting and useful area of investigations.

A point which is invariant under any transformation is called fixed point. Let T:X→X be a transformation. A fixed point x, under the transformation T is a solution of the functional equation T(x)=x. Theorem which are used to prove existence and uniqueness of fixed points are called fixed point theorems.

Fixed-point theory plays an important role in solving the existence and uniqueness of solutions of differential equation, in solving Eigen value Problems and Boundary Value problems. Fixed-point theory also contributes in characterization of the completeness of matric spaces. Due to its applications in various disciplines of mathematical sciences, the Banach contraction and fixed-point theorems have been established. The ideas have a much wider scope than might be suspected and can be applied to establish many other existence theorems in the theory of differential and integral equations. There are numerous extension of Banach's fixed point theorem by generalization its hypothesis while retaining the convergence property of successive iterations the unique fixed point of mapping.

In 1930, Caccioppoli [12] extended Banach's principal as,

$d(T^n(x), T^n(y)) \leq \|T^n\| d(x, y)$, $\sum_{n=1}^{\infty} \|T^n\| < \infty$

In 1962, Rakotch[44] first time replace the Lipschitz constant K by a function K: $R^+ \rightarrow R^+$ as follows:
$d(Fx,Fy) \leq K\, d(x,y)\, d(x,y)$.

In 1968, Kanan [38] removed the continuity of contraction mapping by using the inequality and proved that "If T is self-mapping of a complete metric space X into itself satisfying.

$$d(Tx,Ty) \leq \eta\, [d(Tx,x) + d(Ty,y)] \text{ for all } x,y \in X,$$

Where $\eta\ \varepsilon[0,\frac{1}{2}]$, then T has unique fixed point X.

The mathematical Fisher [23] also extends this principal as:

$$d(Tx,Ty) \leq \mu\, [d(Tx,x) + d(Ty,y)] + \delta\, d(x,y) \text{ for all } x,y \in X,$$

Where $\mu, \delta \in [0,\frac{1}{2}]$, then T has unique fixed point in X.

Ciric [15] extended the Banach's principal in this form,

$$d(Tx,Ty) \leq \eta\, [d(x,T(x)) + d(y,T(y))] + \mu[d(x,T(y)) + d(y,T(x))] + \delta d(x,y)$$

where$\eta,\mu,\delta \varepsilon[0,1[$, $x,y \in X$, Then T has unique fixed fixed point.

In 1971 Reich [45] unified the mapping of Banach and Kannan and proved the result,

$d(Tx,Ty) \leq \mu[d(x,T(y)) + d(y,T(x))] + \delta\, d(x,y)$
where$\mu,\delta \varepsilon[0,1[$, $x,y \in X$, then T has unique fixed point in X.

A similar conclusion was also obtain by Chaterjee [14] in 1972 and expressed

$$d(Tx,Ty) \leq \mu[d(Ty,x) + d(Tx,y)], \text{for all } x,y \in X,$$

where $\mu \in \left(0,\frac{1}{2}\right)$,then T has unique fixed point in X.

In 1973, a more general contractive condition was considered by Hardy and Roger [28] As follows:

$$d(Tx,Ty) \leq \alpha\, d(x,Tx) + \beta\, d(y,Ty) + \gamma d(x,Ty) + \delta\, d(y,Tx) + \eta\, d(x,y),$$

$\forall \alpha,\beta,\gamma,\delta,\eta,\mu\ \epsilon(0,1)$, then T have unique fixed point.

In 1974 again Ciric [15], considered a generalized contraction condition defined as,

$$d(Tx,Ty) \leq \alpha \max\{ d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx), d(x,y)\}, \forall\ \alpha(0,1).$$

then T have unique fixed point.

In 1977, the researcher Jaggi [30] introduced the rational expression first

$d(Tx,Ty) \leq \delta\, d(x,y) + \beta\frac{d(x,Tx), d(y,Ty)}{d(x,y)}$ for all $x,y \varepsilon X, x \neq y, 0 \leq \delta + \beta < 1$.

Then T has unique fixed point in X.
In 1980 the mathematicians Jaggi and Das[31] obtained some fixed point theorem with the mapping satisfying:
$d(Tx,Ty) \leq \alpha\, d(x,y) + \beta\frac{d(x,Tx), d(y,Ty)}{d(x,y) + d(x,Ty) + d(y,Tx)}$ for all $x,y\ \varepsilon\ X, x \neq y, 0 \leq \alpha + \beta < 1$.

Further Jungck[37] obtained the following extension of Banach contraction principal. A continuous self mapping T of complete metric space(X, d) has a fixed point, if and only,

$\exists p\varepsilon$ {0,1[and a map $G:X\rightarrow X$ which commute with T and G satisfies $G(X)\subset T(X)$.
$d(Gx,Gy)\leq p\ d(Tx,Ty)$ $\forall x,y$ in X. Then G and T have unique common fixed point.

In this way there were many results obtained for common fixed point.
Further Wong [56] derived out a common fixed point theorem with two mappings G and T which neither necessary continuous nor comminuting and satisfies:

$$d(Gx,Ty) \leq \alpha\ d(x,Gx) + \beta\ d(y,Ty) + \gamma d(x,Ty) + \delta\ d(y,Gx) + \eta\ d(x,y),$$

$\forall \alpha,\beta,\gamma,\delta,\eta<1$, $\alpha=\beta$ or $\gamma = 0$,then G and T have common fixed point.

All above generalization give Banach’s fixed point theorem by choosing suitable constants.

Nielsen theory is a branch of mathematical research with its origins in topologicalfixed point theory. Its central ideas were developed by Danish mathematician Jakob Nielsen [42].

Nielsen theory, named in honor of its founder Jakob Nielsen [42] (1890 - 1959), is concerned with finding the minimum number of solutions to certain equations involving maps, minimized among all the maps in a given homotopy class. The classical setting for the subject is Nielsen fixed point theory which studies a map $f: X \rightarrow X$ on a compact and seeks to find the minimum number MF[f] of solutions to the fixed point equation $g(x) = x$ among all maps g homotopic to f.

The customary procedure of Nielsen theory consists of defining an equivalence relation on the set of solutions and then identifying the "essential" equivalence classes in such a way that the Nielsen number, defined as the number of essential classes, is a homotopy invariant lower bound for the minimum number. In Nielsen fixed point theory, fixed points x and x' of a given map $f: X \rightarrow X$ are equivalent if there is a path w connecting them such that the paths w and f o w are homotopic relative to the endpoints. The fixed point equivalence classes are classified as essential or not by means of the fixed point index of algebraic topology. The resulting Nielsen number N(f) has the required property that N(f) is less than or equal to MF[f].

One goal of Nielsen theory is to calculate the Nielsen number in specific cases, either by finding explicit formulas where that is possible, or else by means of algorithms. For instance, if $f: X \rightarrow X$ is a map of a compact Lie group, a theorem of Boju Jiang implies that N(f) is the order of the cokernel of the homomorphism.

Some other areas of Nielsen theory are periodic point theory, relative fixed point theory, coincidence theory and root theory.

Fixed point :-Thefixed point of a function, f is any value, x for which f (x) = x. A function may have any number of fixed points from none (e.g. f x = x+1) to infinitely many (e.g. f x = x). The fixed pointcombinator, written as either "fix" or "Y" will return the fixed point of a function.

In mathematics, a fixed point of a function is a point that is mapped to itself by the function. For example, if *f* is defined on the real numbers by $f(x) = x^2 - 3x + 4$, then 2 is a fixed point of *f*, because $f(2) = 2$.

In computing, a fixed-point number representation is a real data type for a number that has a fixed number of digits after the decimal (or binary or hexadecimal) point. For example, a fixed-point number with 4 digits after the decimal point could be used to store numbers such as 1.3467, 281243.3234 and 0.1000, but would round 1.0301789 to 1.0302 and 0.0000654 to 0.0001. These fixed-point presentations are usually used, if either the executing processor does not have any floating point unit (FPU) or performance out rules exactness.

A fixed point is a point that does not change upon application of a map, system of differential equations, etc.

If a variable is slightly displaced from a fixed point, it may (1) move back to the fixed point ("asymptotically stable" or "super stable"), (2) move away ("unstable"), or (3) move in a neighborhood of the fixed point but not approach it ("stable" but not "asymptotically stable"). Fixed points are also called critical points or equilibrium points. If a variable starts at a point that is not a critical point, it cannot reach a critical point in a finite amount of time. Also, a trajectory passing through at least one point that is not a critical point cannot cross itself unless it is a closed curve, in which case it corresponds to a periodic solution.

A fixed point can be classified into one of several classes using linear stability analysis and the resulting stability matrix.

In mathematics, a fixed point (sometimes shortened to fixpoint) of a function is a point that is mapped to itself by the function. That is to say, x is a fixed point of the function f if and only if f(x) = x. For example, if f is defined on the real numbers then 2 is a fixed point of *f*, because *f* (2) = 2.

Not all functions have fixed points: for example, if f is a function defined on the real numbers as f(x) = x + 1, then it has no fixed points, since x is never equal to x + 1 for any real number. In graphical terms, a fixed point means the point (x, f(x)) is on the line y = x, or in other words the graph of f has a point in common with that line. The example f(x) = x + 1 is a case where the graph and the line are a pair of parallel lines. Points which come back to the same value after a finite number of iterations of the function are known as periodic points; a fixed point is a periodic point with period equal to one.

In periodic point theory, a map f: X → X is iterated, that is, f^2: X → X is defined by $f^2(x) = f(f(x))$ and, in general, $f^k(x) = f(f^{k-1}(x))$. A periodic point is a solution to the equation $f^k(x) = x$ and Nielsen periodic point theory studies the minimum number of solutions to the equation $g^k(x) = x$ among all maps g homotopic to f.

Relative fixed point theory is concerned with maps f: (X, A) → (X, A) of compact ANR pairs and seeks the minimum number of fixed points among all maps of pairs homotopic to f through homotopies of pairs.

Coincidence theory requires two maps f, g: X → Y where, in the classical setting, X and Y are closed manifolds of the same dimension. The Nielsen coincidence number is a lower bound for the minimum number of solutions to the equation f'(x) = g'(x) among all maps f' homotopic to f and g' homotopic to g and, if n is not 2, it equals that minimum.

For root theory, there is just one map f: X →Y between closed manifolds of the same dimension and c in Y is any point. The Nielsen root number is a lower bound for the number of solutions to g(x) = c for all maps g homotopic to f.

There is much more to Nielsen theory than the topics mentioned above. For a somewhat broader survey, along with historical background and references, see R. Brown[8], Fixed Point Theory in History of Topology (I. James, ed.), Elsevier, 1999, pages 271 - 299.

Nielsen theory has found applications within topology as well as in other areas of mathematics. Classical Nielsen fixed point theory has contributed to numerical analysis, see W. Forster, Some computational methods for systems of nonlinear equations and systems of polynomial equations, 2 (J. Global Optimization 1992), pages 317 - 356. Although the classical setting of Nielsen fixed point theory is maps of compact ANRs, there are extensions of the theory that do not require that the spaces even be locally compact. This more general theory has been used in nonlinear analysis to establish the existence of multiple solutions to differential and integral equations, thus advancing a program for Nielsen theory first proposed by Jean Leray. Nielsen periodic point theory has been applied with considerable success in dynamics, see for instance B. Jiang, Applications of Nielsen theory to dynamics in Nielsen Theory and Reidemeister Torsion (J. Jezierski ed.), Banach Centre Publications 49 (1999), pages 203 - 221. Within topology, Nielsen root theory is used to extend the concept of topological degree, see R. Brown and H. Schirmer, Nielsen root theory and Hopf degree theory, Pacific J. Math., to appear.

Brouwer's fixed-point theorem: Under a continuous mapping f : S→S of an n-dimensional simplex into itself there exists at least one point x E S such that f (x)=x.

EPSILON NIELSEN FIXED POINT THEORY

Let f: X→X be a map of a compact, connected Riemannian manifold, with or without boundary. For ε> 0 sufficiently small, we introduce an ε-Nielsen number Nε(f) that is a lower bound for the number of fixed points of all self-maps of X that are ε-homotopic to f. We prove that there is always a map g:X→X that is ε-homotopic to f such that g has exactly Nε(f) fixed points. We describe procedures for calculating Nε(f) for maps of 1-manifolds.

> **Historical remarks:**Unlike any standard fixed-point principle, Nielsen theory gives us additional information about the lower estimate of the fixed points number. It was originated by the Danish mathematician Jakob Nielsen in 1927 [42], who studied with this respect self-maps of compact surfaces. Later on, finite polyhedra were systematically treated by Franz Wecken in 1941-42 [55], who also gave an alternative definition of the Nielsen relations which we use below. The crucial step in this development was also accomplished by Andrzej Granas who defined a strictly related fixed-point index for arbitrary (i.e. also infinite-dimensional) ANRs in 1972 [25].

Thus, a sufficiently general fixed-point principle, preserving the number of essential classes of fixed points under homotopy, could be formulated by U. K. Scholz [46], Boju Jiang [6], R. F. Brown [8], and some others (see the references in [1], [6], [8]). By sufficiently, we mean the appropriate form which was suitable for answering the question posed already by Jean Leray [40] at the first International Congress of Mathematicians held after World War II in Cambridge, Mass., in 1950.Namely, he suggested the problem of adapting the Nielsen theory to the needs of nonlinear analysis and, in particular, of its application to differential systems for obtaining multiplicity results (for more details see e.g. [10], [11]).Since that time only several papers have been devoted to this problem (see e.g. [5], [9], [10], [11], [21], [22]), but either additional parameters had to be involved as in the quoted papers by R. F. Brown and M. Feckan, or a simple alternative approach could be used for the same goal as in [5] .

The theory developed in the study of the so-called minimal number of a map f from a compact space to itself, denoted MF[f]. This is defined as: where ~ indicates homotopy of mappings, and Fix(g) indicates the number of fixed points of g. The minimal number was very difficult to compute in Nielsen's time, and remains so today. Nielsen's approach is to group the fixed point set into classes, which are judged "essential" or "nonessential" according to whether or not they can be "removed" by a homotopy. Nielsen's original formulation is equivalent to the following: We define an equivalence relation on the set of fixed points of a self-map f on a space X. We say that x is equivalent to y if and only if there exists a path c from x to y with f(c) homotopic to c as paths. The equivalence classes with respect to this relation are called the Nielsen classes of f, and the Nielsen number N(f) is defined as the number of Nielsen classes having non-zero fixed point index sum. Nielsen proved that making his invariant a good tool for estimating the much more difficult MF[f]. This leads immediately to what is now known as the Nielsen fixed point theorem: Any map f has at least N(f) fixed points. Because of its definition in terms of the fixed point index, the Nielsen number is closely related to the Lefschetz number. Indeed, shortly after Nielsen's initial work, the two invariants were combined into a single "generalized Lefschetz number" (more recently called the Reidemeister trace) by Wecken and Reidemeister.

A nontrivial example of application of the Nielsen fixed-point theory to differential systems: Problem of Jean Leray Nielsen theory is the part of algebraic topology that is included within topological fixed point theory, a broad subject that has substantial interactions with nonlinear analysis, classical functional analysis, differential equations, dynamics and other branches of mathematics, as well as topology. Nielsen theory, named in honor of Jacob Nielsen who introduced the basic concepts, now called Nielsen numbers, which are lower bounds on the number of solutions to fixed point (and other) equations among all the (continuous) maps homotopic to a given map. A central issue in Nielsen theory is the problem of computing Nielsen numbers in specific cases. In 1999, Joyce Wagner published an algorithm, based on techniques of combinatorial group theory, for computing the Nielsen fixed point numbers of a large class of maps on surfaces with boundary. The further development and extension of Wagner’s ideas has become a very active area within Nielsen theory. In 1990, Helga Schirmer initiated the study of conditions that characterize the subsets of a space that can be the fixed point sets of maps homotopic to a given map. In a recent paper, Christina Soderlund studied the corresponding problem in the setting of fiber-preserving maps of fiber bundles.The fixed point theory of multivalued functions is a big enough subject to merit a book by Lech Gorniewicz in 1999. On the other hand, the corresponding Nielsen theory is largely undeveloped: the entire literature of the Nielsen theory of multimaps is summarized in the Handbook of Topological Fixed Point Theory.

APPLICATIONS: The topological theory of fixed points has grown into a substantial area of mathematics, a significant portion of which is now called Nielsen theory in honor of a pioneer of the subject whose ideas have proved particularly fruitful. The story of the early development of topological fixed point theory is told in [9]. A theorem about multiple fixed points, published by Hirsch in 1940 [29], can be viewed as the birth of mod H

Nielsen theory. Let X be a connected finite polyhedron and f:X → X a map. In 1976, McCord [Mc] considered more general connected finite regular covering spaces p: $\tilde{X}$ → X of a finite polyhedron. Letting H = $p_\pi(\pi_1(\tilde{X}))$, he introduced what is now called the mod H Nielsen number $N_H(f)$. McCord defined $N_H(f)$ for a map f:X → X to be the number of conjugacy classes (with respect to covering transformations) of lifts of f to $\tilde{X}$ that have nonzero Lefschetz numbers and he proved that $N_H(f) \leq N(f)$. The excuse for introducing a homotopy invariant that is a less precise lower bound than the Nielsen number for the number of fixed points of all maps homotopic to f is that it might be easier to compute than N(f), but still give some useful information. McCord illustrated this feature of the mod H Nielsen number by presenting examples of maps of compact, connected manifolds such that L(f) = 0 but $N_H(f) = 2$, where $[\pi_1(X) : H] = 2$. The converse Lefschetz theorem, that L(f) = 0 implies that f is homotopic to a fixed point free map, was known to hold for a large class of simply-connected polyhedra, including simply-connected triangulated manifolds. It is easy to construct examples on non-simply-connected polyhedra for which the converse Lefschetz theorem fails, but the first manifold examples of the failure of the converse Lefschetz theorem were those of McCord.A paper of Wang [53] published in 1982 presented the mod H theory without the requirement that the covering space be finite. That is, let f:X → X be a map on a compact ANR and suppose H is a normal subgroup of the fundamental group of X that is mapped into itself by the homomorphism induced by f (taking a bit of care about basepoints). It is not required that H be of finite index in the fundamental group. The mod H fixed point classes are then the projections of the conjugacy classes, with respect to covering transformations, of lifts of f to the regular covering space corresponding to H. Equivalently, fixed points x and x′ of f are in the same mod H fixed point class if there is a path c from x to x′ such that the loop c(fc)−1 is in H. The mod H Nielsen number $N_H(f)$ is the number of mod H fixed point classes with nonzero fixed point index. An exposition of Wang's theory can be found in the monograph of Jiang [34].In 1982 paper of You [57] was the first in which the mod H Nielsen number no longer served as merely a lower bound for the classical Nielsen number that might be easier to calculate, but instead it was essential to the development of fixed point theory. Given a space X and a fixed point x of a map h:X → X, Suppose p:E → B is an orientable fiber space where E, B and all the fibers are compact, connected ANRs and the fundamental group of E is abelian. Let f:E → E be a fiber-preserving map which induces $\bar{f}$:B → B.

THEORETICAL BACKGROUND: All topological spaces are assumed to be metric, and by a topology in function spaces (in particular, the Frechet spaces $C(J; R^n)$) we mean the one of a uniform convergence on compact intervals. By a (metric) ANR-space Y we understand, as usual, the one such that for any (metric) space X, its closed subset S⊂X and a continuous mapping f f: S →Y , there exists an extension of f onto some neighbourhood of S in X.Following [46], the basic notion of the Nielsen number will be de_ned for a socalled admissible class A of self-maps f : X →X, namely those satisfying:

(i) f has a generalized Lefschetz number (for its definition and more details see e.g. [8], [6]),

(ii) the set of fixed points, Fix(f) = { $\hat{x}$ ∈ X: f($\hat{x}$) = $\hat{x}$ } is compact,

(iii) X is a (metric) ANR-space.

Similarly, an admissible homotopy is a map h : X ×[0; 1] → X such that h_t∈ A and $H_{r,s}$∈ A for all r, s, t ∈ [0, 1], where $h_t(x) = h(x, t)$ and $H_{r,s}(x, t) = [h(x, (1 - t)r + ts)]; (1 - t)r + ts]$, i.e. Fix(h) = $U_{t\in[0,1]}$ Fix(h_t) is compact.

Definition.[1]. ([46], [55]). Given f∈ A and x, y ∈Fix(f), we say that x and y are Nielsen-related (written x ~ y) if there exists a path u : [0, 1] → X so that u(0) = x, u(1) = y and u, f(u) are homotopic keeping endpoints fixed. One can readily check that the relation " $\tilde{N}$ " is equivalence. Furthermore, it is known (see [46]) that each fixed-point class is open in Fix(f) and hence, in view of (ii), the number of such classes is finite. Therefore, we can give

Definition [2]([6], [46]). If, for a Nielsen class C ⊂Fix(f), we have ind(C, f) ≠ 0, i.e. if the associated fixed-point index in the sense of [G] (see also [Bo1]) is nontrivial, then C is called essential. The Nielsen number N(f) is then defined to be the number of essential Nielsen classes.

Lemma 1:-([46]). If f ∈ A is admissible, then f admits at least N(f) fixed points, i.e. N(f) ≤ #Fix(f). Moreover, N(f) is invariant under admissible homotopy h : X ×[0; 1] →X, i.e. if h(0) = f and h(1) = g, then N(f) = N(g).

The following lemma is essentially due to JJezierski [32]..

Lemma 2 (reduction):- Let X and its closed subset Y be ANR-spaces. Assume that $f : X \to X$ is a compact map, i.e. $\overline{f(X)}$ is compact, such that $f(X) \subset Y$. Denoting by $f` : Y \to Y$ the restriction of f, we have

(i) Fix(f`) = Fix(f),

(ii) the Nielsen relations coincide,

(iii) ind(C, f`) = ind (C, f) for any Nielsen class $C \subset$ Fix(f).

Thus, N(f`) = N(f).

Definition 3. We say that a mapping $T : Q \to S$ is retractible onto Q if there is a retraction $r : P \to Q$, where P is an open subset of $C(J, R^n)$ (i.e. a set of continuous maps from J to R^n) containing $Q \cup S$, and $p \in P/Q$, $r(p) = q$ implies that $p \neq T(q)$.

The main advantage of the above definition, already employed in [9], [10], [11], states that $r \circ T : Q \to Q$ has a fixed point $\hat{q} \in Q$ if and only if $\hat{q} = T(\hat{q})$.

The following statement, having for us a character of the method, represents only a special (single-valued) case of a more general (set-valued) version developed in [2]. Let us note that it can also be derived as a combination of the results in [46], Theorems 1, 2], [[13], Theorem 1.1], Definition 3 and the fact that a neighborhood retract in Frechet spaces (in particular $C(J, R^n)$) is at the same time an ANR.

MORE ABOUT NIELSEN THEORIES AND THEIR APPLICATIONS

Let $q: X \to Y$ be a map. Fixed points x and x′ of a map $f:X \to X$ are said to be in the same q-fixed point class if there is a path c in X from x to x′ such that qc is homotopic to qfc by a homotopy that fixes the endpoints. The q fixed point classes are unions of the usual fixed point classes so a q-Nielsen number Nq(f) can be defined as the number of q-fixed point classes with nonzero fixed point index. Then Nq(f) is a homotopy invariant lower bound for N(f).

Also in 1992, Woo and Cho [54] introduced the mod H concept into relative Nielsen theory. Consider (X, A) where X is a finite connected polyhedron and A is a subpolyhedron. In this theory, H is the kernel of the homomorphism of fundamental groups induced by the inclusion of A in X. Let f: $(X, A) \to (X, A)$ be a map of pairs, then a fixed point class of $f:X \to X$ is called a common mod H fixed point class if it contains an essential mod H fixed point class of the restriction f_A: $A \to A$. By analogy with the relative Nielsen number N(f; X, A) introduced by Schirmer in [46], and more appropriately called the Schirmer number, they defined $N_H(f; X, A)$ which we shall call the mod H Schirmer number by

$$N_H(f; X, A) = N(f) + N_H(f_A) - N_H(f; f_A)$$

where $N_H(f_A)$is the mod H Nielsen number and $N_H(f; f_A)$ is the number of common mod H fixed point classes that are essential. They prove that the mod H Schirmer number is a lower bound for the Schirmer number and that it shares the properties of (relative) homotopy invariance, commutativity and homotopy type invariance. They then assume that $p:E \to B$ is a fiber space and that $Fb = p^{-1}(b)$ is a fiber that is mapped to itself by a fiber-preserving map $f:E \to E$. They study conditions under which a product theorem of the form $N(f) = N(\bar{f})N_H(f; fb)$ holds, where $\bar{f}:B \to B$ is induced by f and f_b: $F_b \to F_b$ denotes the restriction of f.

In their paper [16] published in 1995, the same authors investigated further the relationship between mod H Nielsen theory and the Schirmer number.

The local Nielsen number The fixed point index can be thought of as a local version of the Lefschetz number. A non-zero Lefschetz number for a map $f:X \to X$ implies the existence of a fixed point for any map homotopic to f. A nonzero index for f on an open subset U of X implies the existence of a fixed point that lies in U, for any map homotopic to f through homotopies that have no fixed points on the boundary of U and satisfy an appropriate compactness property. In 1981, Fadell and Husseini [18] introduced the local Nielsen number, which produces a homotopy invariant lower bound for the number of fixed points on U. The space X is required to be a finite dimensional ANR (but not necessarily compact) and the set of fixed points of a map $f: U \to X$ must be compact, in which case f is called a compactly fixed map. Fixed points x and x′ of f are in the same (local) fixed point class if there is a path p in U such that p and fp are homotopic in X by a homotopy keeping endpoints fixed. The number of fixed point classes is finite by the compactness of the fixed point set. The classical fixed point index can be applied to define the local Nielsen number n(f, U) as the number of fixed point classes with nonzero index.

A homotopy H: U × I → X is said to be compactly fixed if there is a compact subset of U that contains all solutions to H(x, t) = x for all t. Such a homotopy preserves the local Nielsen number, so n(f, U) is a lower bound for the number of fixed points of all maps of U into X that are homotopic to f by a compactly fixed homotopy. Fadell and Husseini introduced a local obstruction index and used it to prove that if X is a connected PL manifold of dimension at least three then, in this setting, the condition n(f, U) = 0 is sufficient as well as necessary for the existence of a fixed point free map g: U → X that is homotopic to f by a compactly fixed homotopy.

In a subsequent paper of 1983 [19], they showed that this last result does not extend to surfaces by exhibiting a map f: U → X on a surface such that n(f, U) = 0 yet every map homotopic to f by a compactly fixed homotopy has at least two fixed points. By using braid theory arguments, additional examples of surfaces X for which there are maps f: U → X such that n(f, U) = 0 yet every map homotopic to f by a compactly fixed homotopy has fixed points were obtained by Goncalves in 1986 [24].

In 1992, Wong [55] showed that the local Nielsen number could be related in a useful way to another, apparently quite different, Nielsen theory. Suppose (X, A) is a pair consisting of a connected finite polyhedron and a subpolyhedron, and a map of pairs f: (X,A) → (X,A) is given. The extension Nielsen number $N(f \mid f_A)$, introduced in [4], is a lower bound for the number of fixed points among all maps of pair g: (X,A) → (X,A) such that g(x) = f(x) for all x ∈ A, homotopic to f through maps with the same property. Thus, for f_A:A → A the restriction of f, the map g is, like f, an extension to X of fA, as is each stage of the homotopy.

The number $N(f \mid f_A)$ is defined to be the number of fixed point classes F of f that do not intersect the boundary of A and that have the property that their index does not equal the index of the fixed point class F ∩ A of fA. Even though this definition of the extension Nielsen number does not resemble that of the local Nielsen number, Wong showed that the concepts can be related. For instance, he proved that if X − A is not a 2-manifold and has no local cut points, A can be bypassed and f: (X, A) → (X, A) is compactly fixed on X − A, then $N(f|f_A)$ equals the local Nielsen number n(f|X−A,X−A). Results of this kind allow the existing computational theory for the local Nielsen number to be used for the extension Nielsen number as well.

Another connection between the local Nielsen number and a Nielsen theory concerned with extensions of maps was presented by Zhao in 1994 [58]. Consider the universal covering space p: $\tilde{X}$ → X of a connected finite polyhedron X. For C an open subset of X and $\tilde{C}$ component of p−1(C), a lift of the restriction f|C of a map f:X → X is a map $\tilde{f}_C$: $\tilde{C} \to \tilde{X}$ such that p $\tilde{f}_C$ = (f|C)p. The local fixed point classes in C, in the sense of [18], are the projections of the fixed point sets of lifts.

Now suppose f: (X, A) → (X, A) is a map of pairs where A is a subpolyhedron and takes C to be a component of X − A. There is a unique lift ˜ f of f to the universal covering space whose restriction to $\tilde{C}$ is $\tilde{f}_C$ The local fixed point class F that is the image under p of the fixed point set of $\tilde{f}_C$ is said to be special if the corresponding lift $\tilde{f}$ has no fixed points on the boundary of $\tilde{C}$ and non-special otherwise. The number of essential special local fixed point classes in X −A is denoted $SN(f|f_A)$. It is an extension Nielsen number because a map g: (X, A) → (X, A) whose restriction to A is f_A and that is homotopic as such to f has at least $SN(f|f_A)$ fixed points in X − A; see [59]. Moreover, if A can be by-passed, then $SN(f|f_A) = N(f|f_A)$, the extension Nielsen number of [4]. Zhao showed in [58] that the local Nielsen number n(f|X − A,X − A) is bounded below by $SN(f|f_A)$ and bounded above by the sum of $SN(f|f_A)$ and the number of non-special fixed point classes of f in X − A.

A 1994 paper of Fares and Hart [20] placed the local Nielsen number in the context of the Reidemeiester trace with the purpose, as in the classical theory, of relating that Nielsen number to a trace-like concept as an aid to calculation. They still consider a compactly fixed map f: U → X with U open in X, where now X is a connected, finite dimensional locally compact polyhedron. A compact subset K of U containing the fixed points of f in its interior is chosen that has the property that the local fixed point classes using paths in K are the same as those, as in [FH1], that use paths in U. Choose lifts $\tilde{f}$ and ˜i, to the universal covering spaces, of the restriction f|Kof f to K and of the inclusion of K in X. A local setting for (f, U) consists of those choices: K, $\tilde{f}$ and ˜i. Let $\tilde{\pi}_K$ and $\tilde{\pi}_X$ denote the groups of covering transformations of K and X respectively, then $\tilde{f}$ and ˜i induce homomorphisms φ, ξ: $\tilde{\pi}_K \to \tilde{\pi}_X$ by the formulas $\tilde{f}(\tau \tilde{y}) = \varphi(\tau)\tilde{f}(\tilde{y})$ and $\tilde{i}(\tau \tilde{y}) = \xi(\tau)\tilde{i}(\tilde{y})$ for $\tau \in \tilde{\pi}K$ and $\tilde{y} \in \tilde{K}$. The Reidemeister action of ˜πK on ˜πX induced by f is defined for $\tau \in \tilde{\pi}K$ and $\alpha \in \tilde{\pi}X$ by $\tau . \alpha = \xi(\tau)\alpha\varphi(\tau^{-1})$. Representing a lift of f by α ˜ f for some $\alpha \in \tilde{\pi}X$, where ˜ f is the lift of the setting.

In 1995 paper of Hart [29] presented techniques for using the local Reidemeister trace to calculate the local Nielsen number. These can be applied, for instance, when ˜πK is finite. The theory presented in [18] has been simplified for the purposes of this description. Covering spaces other than the universal covering space are used there in order to include the mod H version of the local Nielsen number, for H a normal subgroup of ˜πX (that need not be invariant under the fundamental group homomorphism induced by the map).

The examples in [18] and [24] of maps f: U → X on surfaces where n (f, U) = 0 yet f cannot be homotoped to maps that are fixed point free on U can be viewed as local versions of the examples of Jiang in [33] of maps of surfaces with Nielsen number zero that cannot be homotoped to fixed point free maps. None of the examples in [18] and [24] embeds U in X and it follows from [36] that an embedding of a surface (with boundary) in it with zero Nielsen number is homotopic to a fixed point free map.

In 2000 Ferrario and Goncalves provided examples to show that, on all but finitely many surfaces X, there exist homeomorphisms f:X → X and an open subset U of X for which n(f, U) = 0, but no compactly fixed homotopy can eliminate all the fixed points on U.

A recent (2003) paper of Cardona and Wong [16] makes use both of a mod H Nielsen number and its relative version, in the setting of fiber-preserving maps. However, a discussion of their results would require a more extensive digression into the Nielsen theory of fiber-preserving maps.

NIELSEN THEORIES FOR MULTIVALUED FUNCTIONS

The topological fixed point theory of multivalued functions is a very well-developed area that was the subject of the recent (1999) book [25]. What I want to focus on here are the Nielsen theories that exist in the multivalued function setting. Denoting by 2^X the set of subsets of a set X, a fixed point of a function $\varphi: X \to 2^X$ is a point $x \in X$ such that $x \in \varphi(X)$. A Nielsen theory in this setting should obtain a lower bound on the number of points satisfying $x \in \psi(X)$ among $\psi: X \to 2^X$ homotopic to φ is some appropriate sense. In order to obtain results about such fixed points it is of course necessary to impose topological conditions on the function φ and also on the subsets φ(x). One such hypothesis is that the sets φ(x) be closed and we will always make that assumption. A commonly used continuity hypothesis on a multivalued function is that it be upper semi-continuous (usc), that is, if $\varphi(x) \subset V$ where V is open in X, then there is an open neighborhood U of x such that $\varphi(U) \subset V$.

The first Nielsen theory for multivalued functions was developed in 1975 by Schirmerfor what she called small multivalued functions φ on a connected finite polyhedron X. By that she meant that φ(x) must lie in the star of a vertex of X for each $x \in X$. The previous year [47] she had published a simplicial approximation theorem for small use multivalued functions. She proved that there is a (singlevalued) simplicial map f such that, for v any vertex of an appropriately chosen subdivision, φ(v) is in the star of f(v) with respect to the original triangulation of X. She defined the Nielsen number N(φ) to be the usual Nielsen number of f and she proved that it is independent of the choice of simplicial approximation. If the sets φ(x) are acyclic, that is, have trivial reduced ˇCech rational cohomology, then N(φ) is a sharp fixed point lower bound in the sense that there is single-valued map g that is related to φ by a homotopy of small acyclic valued maps such that g has exactly N(φ) fixed points, provided that X is of type S. A finite polyhedron X is type S if X is of dimension at least three and the boundary of the star of every vertex is connected.

Schirmer developed another Nielsen theory, for n-valued functions on connected finite polyhedra, in 1984 [48]. In that theory, $\varphi: X \to 2^X$ is such that φ(x) consists of exactly n points for all $x \in X$. The function φ is now required to be continuous, that is, it is usc and lower semi-continuous as well, which means that, for each $x \in X$ and open subset V of X that intersects φ(s), there exists an open neighborhood U of x such that φ(x′) intersects V for all x′ ∈ U. The key to this theory is a "splitting lemma" from a previous paper [49] that follows from a much earlier (1957) result of O'Neill . The lemma states that a continuous n-valued function φ on a simply-connected space can be split into n single-valued maps. Thus the lemma implies that the n-valued map φ on the polyhedron X splits locally.

In [49] it was also proved that φ can be homotoped so that its fixed points are finite in number and lie in maximal simplices of X. With the fixed points thus isolated then, splitting φ in a neighborhood of a fixed point x, exactly one of the n distinct maps has x as a fixed point and the index of φ at x is defined to be the usual fixed point index of that map at x. Fixed points x and x′ of φ are defined to be equivalent if there is a path p between them such that some splitting of φ restricted to p includes a map for which x and x′ are equivalent fixed points in the sense of traditional Nielsen fixed point theory. Thus fixed point classes can be defined for a continuous n-valued function and the Nielsen number N(φ) defined to be the number of classes of nonzero index. Schirmer proved in [48] that this procedure produces a well-defined lower bound for the number of fixed points of all continuous n-valued functions homotopic to φ by an n-valued continuous homotopy. [49] thatWecken's theorem for manifolds extends to the setting of continuous n-valued functions. That is, if $\varphi:X \to 2^X$ is a continuous n-valued function on a compact connected triangulated manifold of dimension at least three, then there is a continuous n-valued function φ′ homotopic to φ such that φ′ has exactly N(φ) fixed points.

A 1985 monograph by Dzedzej [17] contains an extension of Schirmer's Nielsen theory for n-valued maps to (continouous) maps $\varphi:X \to 2^X$ on compact connected ANRs such that φ(x) consists of n acyclic sets rather than n points, provided that each component of φ(x) has a neighborhood with the property that every loop in it can be contracted, keeping endpoints fixed, in the entire space. The focus of the monograph is a fixed point index theory for a class of multivalued functions considerably more general than these. Thus, to define a Nielsen number, what is required is an appropriate definition of fixed point class. Let p: $\tilde{X} \to X$ be the universal covering space of the compact ANR on which φ is defined and consider φp: $\tilde{X} \to X$. The splitting lemma of Schirmer in [49] applies in this case because it requires only that each φ(x) consist of exactly n connected sets. Thus φp splits into n disjoint acyclic valued maps ψ_j for j = 1, . . . , n, each with the property that, for all x ∈ X, there is a neighborhood of $\psi_j(x)$ such that every loop in it can be contracted in X. With that hypothesis, a result of Jezierski (see below) implies that the ψj have lifts to eX , that is, there are acyclic-valued functions e ψj on eX such that p($\tilde{\psi}_j(\tilde{x})) = \tilde{\psi}_j(\tilde{x})$ for all $\tilde{x} \in \tilde{X}$. This allowed Dzedzej to extend the covering space definition of fixed point classes in this case: Fixed points x and x′ of φ are in the same class if there is some lift $\tilde{\psi}_j$ of one of the ψ_j such that both x and x′ ie in the image under p of the fixed point set of $\tilde{\psi}_j$.

The concept of Jezierski that Dzedzej made use of to define his Nielsen theory was published in 1987 [35]. Jezierski defined an m-map to be an upper semicontinuous multivalued function φ from a space Y to the subsets of a space X such that each subset φ(x) has a neighborhood with the property that every loop in it can be contracted in X keeping endpoints fixed. Suppose X admits a universal covering space p: $\tilde{X} \to X$. Jezierski proved that if Y is simply-connected, then for $y_0 \in Y$ and $\tilde{x}_0 \in \tilde{X}$ with $p(\tilde{x}_0) \in \varphi(y_0)$, there is a unique lift $\tilde{\varphi}$ of φ, that is, $\tilde{\varphi}$ is an m-map from Y to the subsets of eX such that p $\tilde{\varphi}(x) = \varphi(x)$ for x ∈X and with $\tilde{x}_0 \in \tilde{\varphi}(y_0)$. Then the definition of fixed point class that Dzedzej used makes sense for any m-map φ on a space X with a universal covering space. Since $\tilde{X}$ is simply-connected, the m-map φp has lifts and two fixed points of φ are in the same class if they lie in the image under p of the fixed points of a lift of φp. If X is a compact connected ANR and φ is an m-map on X such that each φ(x) is acyclic with respect to homology with rational coefficients, then the index theory of Dzedzej can be applied to define a Nielsen number of φ that is a homotopy invariant (with respect to homotopiesthat are m-maps) lower bound for the number of fixed points of φ. A detailed development of this Nielsen theory can be found in the book of Gorniewicz [25].

Gorniewicz, Granas and Kryszewski introduced a class of multivalued functions on compact ANRs that they called J-maps in 1988 [26][27]. Denote the set of points in a metric space X that are within ϱ of a subset A by U(A, ϱ).

Single-valued map f:X → Y in the sense that the graph of f lies in the ϱ-neighborhood of the graph of φ. Moreover, any two such single-valued approximations are homotopic through these approximations. They defined an A-map: X → X to be a multivalued function that can be expressed as a finite composition of J-maps (not necessarily with X itself as domain or range) that has the property introduced by Jezierski, that is, for each x ∈ X there is a neighborhood of (x) with the property that every loop in it can be contracted in X keeping endpoints fixed. The class of A-maps includes not only those usc functions such that (x) is compact and contractible but also those for which (x) = T∞ j=1 Kj where the Kj are compact and contractible with Kj+1 ⊂Kj. The next year,Kryszewski and Miklaszewski [39] proved that a valid Nielsen theory for A-maps can be obtained by replacing the J-maps in a composition by their single-valued approximations.

In 1990, Schirmer [51] introduced a bimap, which she defined to be a continuous (that is both upper and lower semicontinuous) multivalued function φ:X → Y such that, for each point x in a space X, the image φ(x) consists of either one or two points of Y and she introduced a fixed point index for such functions. A bihomotopy is a continuous family of bimapsφ_t:X → Y . A bimap α: I → X is called a bipath in X. A bihomotopy {α_t} of bipaths is required to fix the endpoints, that is, $\alpha_t(0) = \alpha_0(0)$ and $\alpha_t(1) = \alpha_0(1)$ for all t∈I. The next year, she produced a Nielsen number for such bimaps in [52]. Fixed points x and x′ of a bimap φ:X → X are in the same fixed point class if there is a single-valued path p: I → X such that the bipathφp is bihomotopic to a bipath α such that α(t) = {p(t), q(t)} for some single-valued path q. A bimap may be viewed as an n symmetric product map, that is, a map from a space X to the space of unordered sets of n not necessarily distinct points of X in the case n = 2. The definition of fixed point classes for a bimap is equivalent to a fixed point class concept for symmetric product maps introduced by Masih in 1979 [41]. Schirmer defined the Nielsen number N(φ) in the usual way, as the number of fixed point classes of nonzero index. However a result of Miklaszewski in 1990 [41(a)] states that all the fixed points of a symmetric product map must lie in a single fixed point class, so N(f) ≤ 1. This result would imply that the Nielsen theory of bimaps uninteresting were it not for the fact that Schirmer proved in [52] that a Wecken theorem for manifolds holds in her theory.Thus any bimap φ on a compact connected triangulated manifold of dimension atleast three is bihomotopic to a bimap ψ with a single fixed point and, if the fixed point index of φ on X is zero, then ψ can be made fixed point free.

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