

Some Fixed Point Results in b-Metric Spaces

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Abstract: In this paper, we have obtained some fixed point and common fixed point results on b-metric space.

Keywords: b-metric space, fixed point, common fixed point, contractive mapping.

2. Introduction & Preliminaries

In 1993, The concept of b-metric space was introduced by Czerwik [5]. Using this idea, he presented a generalization of the renowned Banach fixed point theorem in the b-metric spaces (see [6,7,8]) Many researches Aydi[1], chugh[9], Mehmet [11] and Rao[12] studied the extension of fixed point theorems in b-metric space.

Before starting the main results first we are giving some basic concepts.

Definition 2.1[5]: If X is a non-empty set and let $s \geq 1$ be a given real number. A function $d: X \times X \to \mathbb{R}_+$, is called a b-metric if the following conditions are satisfied:

(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, z) \leq s.[d(x, y) + d(y, z)]$. for all $x, y, z \in X$

The pair (X, d) is called b-metric spaces. It is clear that definition of b-metric space is a extension of usual metric space.

Definition 2.2[5]: Let (X, d) be a b-metric space. Then a sequence $\{x_n\}$ in X is called a Cauchy sequence if and only if for all $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that for each $m, n \geq n(\varepsilon)$ we have $d(x_n, x_m) < \varepsilon$.

Definition 2.3[5]: Let (X, d) be a b-metric space. Then a sequence $\{x_n\}$ in X is called a convergent sequence if and only if there exist $x \in X$ such that for all there exists $n(\varepsilon) \in \mathbb{N}$ such that for each $n \geq n(\varepsilon)$ we have $d(x_n, x) < \varepsilon$. In this case write $\lim_{n\to\infty} x_n = x$.

Definition 2.4[5]: The b-metric is complete if every Cauchy Sequence converges.

3. Main Result

Theorem 3.1: Let (X, d) be a complete b-metric space with metric d and let $T: X \to X$ be a function with the following property

$$d(Tx, Ty) \leq a_1 \frac{\max\{d^2(x,y), d^2(x,Tx), d^2(y,Ty)\}}{d(x,Tx)+d(x,Ty)}$$

$$+a_2 \frac{\max\{d(x,Tx)d(y,Tx), d(x,Ty)d(y,Ty)\}}{d(x,Tx)+d(x,Ty)}$$

$$+a_3 \frac{d(x,Ty)d(y,Tx)}{d(x,Tx)+d(x,Ty)} \qquad \ldots (3.1.1)$$

For all $x, y \in X$ where a_1, a_2, a_3 are non-negative real number and satisfy

$a_1 + 2sa_2 < 1$ & $s(a_1 + a_2) < 1$ for $s \geq 1$ Then T has a unique fixed point.

Proof: Let $x_0 \in X$ and $\{x_n\}$ be a sequence in X, such that

$$x_n = Tx_{n-1} = T^n x_0$$

Now

$$d(x_{n+1},x_n)=d(Tx_n,Tx_{n-1})$$

$$\leq a_1\frac{\max\left\{\begin{matrix}d^2(x_n,x_{n-1}),d^2(x_n,Tx_n),\\ d^2(x_{n-1},Tx_{n-1})\end{matrix}\right\}}{d(x_n,Tx_n)+d(x_n,Tx_{n-1})}$$

$$+a_2\frac{\max\left\{\begin{matrix}d(x_n,Tx_n)d(x_{n-1},Tx_n),\\ d(x_n,Tx_{n-1})d(x_{n-1},Tx_{n-1})\end{matrix}\right\}}{d(x_n,Tx_n)+d(x_n,Tx_{n-1})}$$

$$+a_3\frac{d(x_n,Tx_{n-1})d(x_{n-1},Tx_n)}{d(x_n,Tx_n)+d(x_n,Tx_{n-1})}$$

$$\leq a_1\frac{\max\left\{\begin{matrix}d^2(x_n,x_{n-1}),d^2(x_n,x_{n+1}),\\ d^2(x_{n-1},x_n)\end{matrix}\right\}}{d(x_n,x_{n+1})+d(x_n,x_n)}$$

$$+a_2\frac{\max\left\{\begin{matrix}d(x_n,x_{n+1})d(x_{n-1},x_{n+1}),\\ d(x_n,x_n)d(x_{n-1},x_n)\end{matrix}\right\}}{d(x_n,x_{n+1})+d(x_n,x_n)}$$

$$+a_3\frac{d(x_n,x_n)d(x_{n-1},x_{n+1})}{d(x_n,x_{n+1})+d(x_n,x_n)}$$

$$\leq a_1\frac{\max\{d^2(x_n,x_{n-1}),d^2(x_n,x_{n+1})\}}{d(x_n,x_{n+1})}$$

$$+a_2\frac{\max\{d(x_n,x_{n+1})d(x_{n-1},x_{n+1}),0\}}{d(x_n,x_{n+1})}$$

$$\Rightarrow d(x_{n+1},x_n).\,d(x_{n+1},x_n)$$

$$\leq a_1\max\{d^2(x_n,x_{n-1}),d^2(x_n,x_{n+1})\}$$

$$+a_2 d(x_n,x_{n+1})d(x_{n-1},x_{n+1})$$

$$\leq a_1\max\{d^2(x_n,x_{n-1}),d^2(x_n,x_{n+1})\}$$

$$+a_2 d(x_n,x_{n+1})s\begin{bmatrix}d(x_{n-1},x_n)\\ +d(x_n,x_{n+1})\end{bmatrix}$$

If $d(x_n,x_{n+1})\geq d(x_n,x_{n-1})$ then we have

$$d^2(x_n,x_{n+1})\leq(a_1+2sa_2)d^2(x_n,x_{n+1})$$

This is a contradiction. Thus

$$d(x_n,x_{n+1})\leq kd(x_{n-1},x_n)$$

Where $k=\sqrt{(a_1+2sa_2)}<1$

Continuing this process n times we can easily

$$d(x_n,x_{n+1})\leq k^n d(x_0,x_1)$$

This implies that T is a contraction mapping.

Now, it is to show that $\{x_n\}$ is a Cauchy sequence in X.

Let $m,n>0$, with $m>n$

$$d(x_n,x_m)\leq s[d(x_n,x_{n+1})+d(x_{n+1},x_m)]$$

$$\leq sd(x_n,x_{n+1})+s^2d(x_{n+1},x_{n+2})$$

$+s^3 d(x_{n+2}, x_{n+3}) + \cdots ..$

$\leq sk^n d(x_0, x_1) + s^2 k^{n+1} d(x_0, x_1)$

$+s^3 k^{n+2} d(x_0, x_1) + \cdots ..$

$\leq sk^n d(x_0, x_1)[1 + (sk) + (sk)^2$

$+(sk)^3 + \cdots ...]$

$\leq \frac{sk^n}{1-sk} d(x_0, x_1)$

Taking limit $n \to \infty$ we get

$\lim_{n\to\infty} d(x_n, x_m) = 0$

Therefore $\{x_n\}$ is a Cauchy sequence in X is complete, we consider that $\{x_n\}$ converses to x^*.

Now, we show that x^* is fixed point of T.

We have

$d(x^*, Tx^*) \leq s[d(x^*, x_n) + d(x_n, Tx^*)]$

$\leq s[d(x^*, x_n) + d(Tx_{n-1}, Tx^*)]$

$\leq s[d(x^*, x_n) + d(Tx^*, Tx_{n-1})]$

$$\leq s\left[\begin{array}{c} d(x^*, x_n) \\ +a_1 \frac{\max\left\{\begin{array}{c} d^2(x^*, x_{n-1}), d^2(x^*, Tx^*), \\ d^2(x_{n-1}, Tx_{n-1}) \end{array}\right\}}{d(x^*, Tx^*) + d(x^*, Tx_{n-1})} \\ +a_2 \frac{\max\left\{\begin{array}{c} d(x^*, Tx^*) d(x_{n-1}, Tx^*), \\ d(x^*, Tx_{n-1}) d(x_{n-1}, Tx_{n-1}) \end{array}\right\}}{d(x^*, Tx^*) + d(x^*, Tx_{n-1})} \\ +a_3 \frac{d(x^*, Tx_{n-1}) d(x_{n-1}, Tx^*)}{d(x^*, Tx^*) + d(x^*, Tx_{n-1})} \end{array}\right]$$

Taking $n \to \infty$, we get

$\Rightarrow [1 - s(a_1 + a_2)] d(x^*, Tx^*) \leq 0$

Which is contraction.

$\Rightarrow d(x^*, Tx^*) = 0 \Rightarrow Tx^* = x^*$

$\Rightarrow x^*$ is the fixed point of T.

Uniqueness: Let y_0 and z_0 be two fixed point of T such that $y_0 \neq z_0$.

Putting $x = y_0$ and $y = z_0$ in (3.1.1) we have

$d(y_0, z_0) = d(Ty_0, Tz_0)$

$\leq a_1 \frac{\max\{d^2(y_0, z_0), d^2(y_0, Ty_0), d^2(z_0, Tz_0)\}}{d(y_0, Ty_0) + d(y_0, Tz_0)}$

$+a_2 \frac{\max\{d(y_0, Ty_0) d(z_0, Ty_0), d(y_0, Tz_0) d(z_0, Tz_0)\}}{d(y_0, Ty_0) + d(y_0, Tz_0)}$

$+a_3 \frac{d(y_0, Tz_0) d(z_0, Ty_0)}{d(y_0, Ty_0) + d(y_0, Tz_0)}$

$d(y_0, z_0) \leq (a_1 + a_3) d(y_0, z_0)$

$[1-(a_1+a_3)]d(y_0,z_0)\leq 0$

This is a contraction.

$d(y_0,z_0)=0\Longrightarrow y_0=z_0$.

Hence the fixed point of T is unique.

Theorem 3.2: let X be a complete b-metric space and A be a self map of X. The mapping A satisfying the following condition:

$$d(Ax,Ay)\leq\left[\begin{matrix}\alpha d(x,Ax).d(y,Ay)\\ \beta\max\begin{Bmatrix}d(x,Ax).d(y,Ay),\\ d(x,Ax).d(y,Ax),\\ d(x,Ay).d(y,Ay),\\ d(x,Ay).d(y,Ax)\end{Bmatrix}\end{matrix}\right]^{1/2}\quad(3.2.1)$$

For all $x,y\in X$

Where α,β are non negative real numbers $\alpha+2s\beta<1$ and $s^2\beta<1$ for $s\geq 1$.
Then A has a unique fixed point.

Proof: Let $x_0\in X$ and $\{x_n\}$ be a sequence in X, such that

$$x_n=Ax_{n-1}=A^nx_0$$

Now $d(x_n,x_{n+1})=d(Ax_{n-1},Ax_n)$

$$\leq\left[\begin{matrix}\alpha d(x_{n-1},Ax_{n-1}).d(x_n,Ax_n)\\ +\beta\max\begin{Bmatrix}d(x_{n-1},Ax_{n-1}).d(x_n,Ax_n),\\ d(x_{n-1},Ax_{n-1}).d(x_n,Ax_{n-1}),\\ d(x_{n-1},Ax_n).d(x_n,Ax_n),\\ d(x_{n-1},Ax_n).d(x_n,Ax_{n-1})\end{Bmatrix}\end{matrix}\right]^{1/2}$$

$$\leq\left[\begin{matrix}\alpha d(x_{n-1},x_n).d(x_n,x_{n+1})\\ +\beta\max\begin{Bmatrix}d(x_{n-1},x_n).d(x_n,x_{n+1}),\\ d(x_{n-1},x_n).d(x_n,x_n),\\ d(x_{n-1},x_{n+1}).d(x_n,x_{n+1}),\\ d(x_{n-1},x_{n+1}).d(x_n,x_n)\end{Bmatrix}\end{matrix}\right]^{1/2}$$

$$\leq\left[\begin{matrix}\alpha d(x_{n-1},x_n).d(x_n,x_{n+1})\\ +\beta\max\begin{Bmatrix}d(x_{n-1},x_n).d(x_n,x_{n+1}),\\ 0,\\ d(x_{n-1},x_{n+1})d(x_n,x_{n+1}),\\ 0\end{Bmatrix}\end{matrix}\right]^{1/2}$$

$$\Rightarrow[d(x_n,x_{n+1})]^2\leq d(x_n,x_{n+1})\left[\begin{matrix}\alpha d(x_{n-1},x_n)\\ +\beta d(x_{n-1},x_{n+1})\end{matrix}\right]$$

$$\Rightarrow d(x_n,x_{n+1})\leq\left[\begin{matrix}\alpha d(x_{n-1},x_n)\\ +\beta d(x_{n-1},x_{n+1})\end{matrix}\right]$$

$$\leq\left[\begin{matrix}\alpha d(x_{n-1},x_n)\\ +\beta s\{d(x_{n-1},x_n)+d(x_n,x_{n+1})\}\end{matrix}\right]$$

$$\Rightarrow d(x_n,x_{n+1})\leq\frac{\alpha+\beta s}{1-\beta s}d(x_{n-1},x_n)$$

$$\Rightarrow d(x_n,x_{n+1})\leq kd(x_{n-1},x_n)$$

Where $k=\frac{\alpha+\beta s}{1-\beta s}<1$.

Similarly,
$\Rightarrow d(x_{n-1}, x_n) \le kd(x_{n-2}, x_{n-1})$
Continue this process n times, we get
$\Rightarrow d(x_n, x_{n+1}) \le k^n d(x_0, x_1)$
This implies that T is a contraction mapping.

Now, it is to show that $\{x_n\}$ is a Cauchy sequence in X.

Let $m, n > 0$, with $m > n$

$$\begin{aligned} d(x_n, x_m) &\le s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] \\ &\le sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) \\ &\quad + s^3 d(x_{n+2}, x_{n+3}) + \cdots .. \\ &\le sk^n d(x_0, x_1) + s^2 k^{n+1} d(x_0, x_1) \\ &\quad + s^3 k^{n+2} d(x_0, x_1) + \cdots .. \\ &\le sk^n d(x_0, x_1)[1 + (sk) + (sk)^2 \\ &\quad + (sk)^3 + \cdots \ldots] \\ &\le \frac{sk^n}{1-sk} d(x_0, x_1) \end{aligned}$$

Taking limit $n \to \infty$ we get

$$\lim_{n\to\infty} d(x_n, x_m) = 0$$

Therefore $\{x_n\}$ is a Cauchy sequence in X is complete, we consider that $\{x_n\}$ converses to u.

Now, we show that u is fixed point of A.

We have

$$d(u, Au) \le s[d(u, x_n) + d(x_n, Au)]$$

$$\le s[d(u, x_n) + d(Ax_{n-1}, Au)]$$

$$\le s\left[\begin{array}{c} d(u, x_n) \\ + \left[\begin{array}{c} \alpha d(x_{n-1}, Ax_{n-1}).\, d(u, Au) \\ +\beta \max \left\{\begin{array}{c} d(x_{n-1}, Ax_{n-1}).\, d(u, Au), \\ d(x_{n-1}, Ax_{n-1}).\, d(u, Ax_{n-1}), \\ d(x_{n-1}, Au).\, d(u, Au), \\ d(x_{n-1}, Au).\, d(u, Ax_{n-1}) \end{array}\right\} \end{array}\right]^{1/2} \end{array}\right]$$

$$\le s\left[\begin{array}{c} d(u, x_n) \\ + \left[\begin{array}{c} \alpha d(x_{n-1}, x_n).\, d(u, Au) \\ +\beta \max \left\{\begin{array}{c} d(x_{n-1}, x_n).\, d(u, Au), \\ d(x_{n-1}, x_n).\, d(u, x_n), \\ d(x_{n-1}, Au).\, d(u, Au), \\ d(x_{n-1}, Au).\, d(u, x_n) \end{array}\right\} \end{array}\right]^{1/2} \end{array}\right]$$

Taking $n \to \infty$ we get

$$d(u, Au) \le s[\beta d(u, Au).\, d(u, Au)]^{1/2}$$

$d(u, Au) \le s^2 \beta d(u, Au)$
Which is contraction.
$\Rightarrow d(u, Au) = 0 \Rightarrow Au = u$

$\Rightarrow u$ is the fixed point of A.

Uniqueness: Let u and v be two fixed point of A such that $u \neq v$.

Putting $x = u$ and $y = v$ in (3.2.1) we have

$d(u, v) = d(Au, Av)$

$$\leq \left[\begin{array}{c} \alpha d(u, Au). d(v, Av) \\ +\beta \max \begin{Bmatrix} d(u, Au). d(v, Av), \\ d(u, Au). d(v, Au), \\ d(u, Av). d(v, Av), \\ d(u, Av). d(v, Au) \end{Bmatrix} \end{array} \right]^{1/2}$$

$\Rightarrow d(u, v) \leq \beta d(u, v)$
Which is a contraction.
$\Rightarrow d(u, v) = 0 \Longrightarrow u = v$
Hence the fixed point of A is unique.

Theorem 3.3: Let X be a complete b-metric space with metric d and let $S, T: X \to X$ are two functions with the following property

$$d(Sx, Ty) \leq a_1 \frac{\max\{d^2(x,y), d^2(x,Sx), d^2(y,Ty)\}}{d(y,Sx) + d(y,Ty)}$$

$$+a_2 \frac{\max\{d(x,Sx)d(y,Sx), d(x,Ty)d(y,Ty)\}}{d(y,Sx) + d(y,Ty)}$$

$$+a_3 \frac{d(x,Ty)d(y,Sx)}{d(y,Sx) + d(y,Ty)} \qquad \ldots (3.3.1)$$

For all $x, y \in X$ where a_1, a_2, a_3 are non-negative real number and satisfy $(a_1 + 2sa_2) < 1$ & $s(a_1 + a_2) < 1$ for $s \geq 1$. Then S and T has a unique fixed point.

Proof: Let $x_0 \in X$ and $\{x_n\}$ be a sequence in X, such that

$$x_1 = S(x_0), x_2 = T(x_1), \ldots..$$

$$x_{2n} = T(x_{2n-1}), x_{2n+1} = S(x_{2n})$$

Now

$d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n}, Tx_{2n+1})$

$$\leq a_1 \frac{\max\begin{Bmatrix} d^2(x_{2n},x_{2n+1}), d^2(x_{2n},Sx_{2n}), \\ d^2(x_{2n+1},Tx_{2n+1}) \end{Bmatrix}}{d(x_{2n+1},Sx_{2n}) + d(x_{2n+1},Tx_{2n+1})}$$

$$+a_2 \frac{\max\begin{Bmatrix} d(x_{2n},Sx_{2n})d(x_{2n+1},Sx_{2n}), \\ d(x_{2n},Tx_{2n+1})d(x_{2n+1},Tx_{2n+1}) \end{Bmatrix}}{d(x_{2n+1},Sx_{2n}) + d(x_{2n+1},Tx_{2n+1})}$$

$$+a_3 \frac{d(x_{2n},Tx_{2n+1})d(x_{2n+1},Sx_{2n})}{d(x_{2n+1},Sx_{2n}) + d(x_{2n+1},Tx_{2n+1})}$$

$$\leq a_1 \frac{\max\begin{Bmatrix} d^2(x_{2n},x_{2n+1}), d^2(x_{2n},x_{2n+1}), \\ d^2(x_{2n+1},x_{2n+2}) \end{Bmatrix}}{d(x_{2n+1},x_{2n+1}) + d(x_{2n+1},x_{2n+2})}$$

$$+a_2 \frac{\max\begin{Bmatrix} d(x_{2n},x_{2n+1})d(x_{2n+1},x_{2n+1}), \\ d(x_{2n},x_{2n+2})d(x_{2n+1},x_{2n+2}) \end{Bmatrix}}{d(x_{2n+1},x_{2n+1}) + d(x_{2n+1},x_{2n+2})}$$

$$+a_3 \frac{d(x_{2n},x_{2n+2})d(x_{2n+1},x_{2n+1})}{d(x_{2n+1},x_{2n+1}) + d(x_{2n+1},x_{2n+2})}$$

$$\leq a_1 \frac{\max\{d^2(x_{2n},x_{2n+1}),d^2(x_{2n+1},x_{2n+2})\}}{d(x_{2n+1},x_{2n+2})}$$

$$+a_2 \frac{\max\{0,\mathrm{d}(x_{2n},x_{2n+2})d(x_{2n+1},x_{2n+2})\}}{d(x_{2n+1},x_{2n+2})}$$

$$\Rightarrow d(x_{2n+1},x_{2n+2}).\,d(x_{2n+1},x_{2n+2})$$

$$\leq a_1 \max\begin{bmatrix} d^2(x_{2n},x_{2n+1}), \\ d^2(x_{2n+1},x_{2n+2}) \end{bmatrix}$$

$$+a_2\mathrm{d}(x_{2n},x_{2n+2})d(x_{2n+1},x_{2n+2})$$

$$\leq a_1 \max\begin{bmatrix} d^2(x_{2n},x_{2n+1}), \\ d^2(x_{2n+1},x_{2n+2}) \end{bmatrix}$$

$$+a_2 d(x_{2n+1},x_{2n+2})s\begin{bmatrix} \mathrm{d}(x_{2n},x_{2n+1}) \\ +\mathrm{d}(x_{2n+1},x_{2n+2}) \end{bmatrix}$$

If $\mathrm{d}(x_{2n+1},x_{2n+2}) \geq \mathrm{d}(x_{2n},x_{2n+1})$ then we have

$$d^2(x_{2n+1},x_{2n+2})$$

$$\leq (a_1 + 2sa_2)d^2(x_{2n+1},x_{2n+2})$$

This is a contradiction. Thus

$$d(x_{2n+1},x_{2n+2})$$

$$\leq \left[\sqrt{(a_1 + 2sa_2)}\right]d(x_{2n},x_{2n+1})$$

From the above inequality we deduce that

$$d(x_{n+1},x_{n+2})$$

$$\leq \left[\sqrt{(a_1 + 2sa_2)}\right]d(x_n,x_{n+1})$$

Continuing this process we can in general

$$d(x_n,x_{n+1}) \leq k^n d(x_0,x_1)$$

This implies that T is a contraction mapping.

Now, it is to show that $\{x_n\}$ is a Cauchy sequence in X.

Let $m,n > 0$, with $m > n$

$$d(x_n,x_m) \leq s[d(x_n,x_{n+1}) + d(x_{n+1},x_m)]$$

$$\leq sd(x_n,x_{n+1}) + s^2d(x_{n+1},x_{n+2})$$

$$+s^3d(x_{n+2},x_{n+3}) + \cdots ..$$

$$\leq sk^n d(x_0,x_1) + s^2k^{n+1}d(x_0,x_1)$$

$$+s^3k^{n+2}d(x_0,x_1) + \cdots ..$$

$$\leq sk^n d(x_0,x_1)[1 + (sk) + (sk)^2$$

$$+(sk)^3 + \cdots ...]$$

$$\leq \frac{sk^n}{1-sk}d(x_0,x_1)$$

Taking limit $n \to \infty$ we get

$$\lim_{n\to\infty} d(x_n, x_m) = 0$$

Therefore $\{x_n\}$ is a Cauchy sequence in X is complete, we consider that $\{x_n\}$ converses to u.

Now, we show that u is fixed point of S & T.

We have

$$d(u, Su) \leq s[d(u, x_n) + d(x_n, Su)]$$

$$\leq s[d(u, x_n) + d(Tx_{n-1}, Su)]$$

$$\leq s[d(u, x_n) + d(Su, Tx_{n-1})]$$

$$\leq s\begin{bmatrix} d(u, x_n) \\ +a_1 \frac{\max\left\{ d^2(u,x_{n-1}), d^2(u,Su), \atop d^2(x_{n-1},Tx_{n-1}) \right\}}{d(x_{n-1},Su)+d(x_{n-1},Tx_{n-1})} \\ +a_2 \frac{\max\left\{ d(u,Su)d(x_{n-1},Su), \atop d(u,Tx_{n-1})d(x_{n-1},Tx_{n-1}) \right\}}{d(x_{n-1},Su)+d(x_{n-1},Tx_{n-1})} \\ +a_3 \frac{d(u,Tx_{n-1})d(x_{n-1},Su)}{d(x_{n-1},Su)+d(x_{n-1},Tx_{n-1})} \end{bmatrix}$$

$$\leq s\begin{bmatrix} d(u, x_n) \\ +a_1 \frac{\max\left\{ d^2(u,x_{n-1}), d^2(u,Su), \atop d^2(x_{n-1},x_n) \right\}}{d(x_{n-1},Su)+d(x_{n-1},x_n)} \\ +a_2 \frac{\max\left\{ d(u,Su)d(x_{n-1},Su), \atop d(u,x_n)d(x_{n-1},x_n) \right\}}{d(x_{n-1},Su)+d(x_{n-1},x_n)} \\ +a_3 \frac{d(u,x_n)d(x_{n-1},Su)}{d(x_{n-1},Su)+d(x_{n-1},x_n)} \end{bmatrix}$$

Taking $n \to \infty$, we get

$$\Rightarrow [1 - s(a_1 + a_2)]d(u, Tu) \leq 0$$

Which gives $d(u, Su) = 0 \Rightarrow Su = u$.

Hence u is a fixed point of S.

Similarly we can show that u is a fixed point of T.

Hence u is a common fixed point of S & T.

Uniqueness: Let u and v be two fixed point of S & T. such that $u \neq v$.

Putting $x = u$ and $y = v$ in (3.3.1) we have

$$d(u, v) = d(Su, Tv)$$

$$\leq a_1 \frac{\max\{d^2(u,v), d^2(u,Su), d^2(v,Tv)\}}{d(v,Su)+d(v,Tv)}$$

$$+a_2 \frac{\max\{d(u,Su)d(v,Su), d(u,Tv)d(v,Tv)\}}{d(v,Su)+d(v,Tv)}$$

$$+a_3 \frac{d(u,Tv)d(v,Su)}{d(v,Su)+d(v,Tv)}$$

$$d(u, v) \leq (a_1 + a_3)d(u, v)$$

This is a contraction.

$d(u, v) = 0 \Rightarrow u = v.$

Hence common fixed point of S and T is unique.

This complete the proof.

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