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SOME RESULTS ON SEMI-REFLEXIVITY AND REFLEXIVITY IN LOCALLY CONVEX SPACES

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Abstract
In this paper we discuss that if $E[\tau]$ is a barreled space such that every bounded subset of E is relatively compact, then $E[\tau]$ is reflexive, and that a barreled space $E[\tau]$ in which there is a denumerable system of convex compact subsets is reflexive. We also discuss Some hereditary-type properties of reflexive locally convex spaces.
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1. Introduction:

For a locally convex space $E[\tau]$, which we always consider Hausdorff, the dual is denoted as E', strong dual is $E'[\tau_b(E)]$ and the bidual of $E[\tau]$ is $E''=(E'[\tau_b(E)])'$. Locally convex space $E[\tau]$ is called semi-reflexive if $E = E''$. A semi-reflexive locally convex space $E[\tau]$ is called reflexive provided $\tau = \tau_b(E')$. The so called b-reflexivity of Kye, in [1], is nothing but reflexivity. Reflexivity is also discussed in [2] and [3] in terms of V-reflexivity and G-reflexivity, respectively.

We follow the notion of Köthe [3] for notations and terminology, unless specifically mentioned. Recall that a locally convex space $E[\tau]$ is said to be quasi-complete if every bounded closed subset of $E[\tau]$ is complete. If E is quasi-complete for the weak topology $\tau_s(E')$, then we say that it is weakly quasi-complete. $E[\tau]$ is called sequentially complete if every τ-Cauchy sequence has a limit point in E. A closed absorbent absolutely convex set in a locally convex space $E[\tau]$ is called a barrel. A locally convex space $E[\tau]$ is said to be barreled if the barrels form a base of neighborhoods of origin 0. Equivalently, $E[\tau]$ is barreled if and only if topology τ coincides with the strong topology $\tau_b(E')$. $E[\tau]$ is called quasi-barrelled if τ coincides with $\tau_{b*}(E')$, the topology of uniform convergence on strongly bounded sets in E'. Equivalently, $E[\tau]$ is quasi-barreled if and only if every barrel which absorbs all bounded sets of $E[\tau]$ is a neighborhoods of 0. A barreled space $E[\tau]$ is called an (M)-space (Montel space) if every bounded subset of E is relatively compact. A Banach space cannot be an (M)-space unless it is finite-dimensional ([5], §7). A locally convex space is called metrizable if and only if it has a countable base of neighborhoods of 0. A complete metrizable locally convex space is called an (F) space. $E[\tau]$ is called bornological if every absolutely convex set which absorbs all the bounded subsets of E is a neighborhood of 0.

Let us mention the following theorem, from [6], of characterizations of semi-reflexivity for locally convex spaces.

1.1 Theorem: For a locally convex space $E[\tau]$, the following assertions are equivalent:
(i) $E[\tau]$ is semi-reflexive,
(ii) E is weakly quasi-complete,
(iii) Every bounded subset of $E[\tau]$ is relatively weakly compact,
(iv) $E'[\tau_k(E)]$ is barreled.

A characterization of reflexivity is given in the following theorem, which due to [4]:

1.2 Theorem: A locally convex space is reflexive if and only if it is semi-reflexive and quasi-barreled if and only if it is weakly quasi-complete and quasi-barreled.

We note that a semi-reflexive locally convex space $E[\tau]$ is a Banach-Mackey space i.e. $\tau_s(E)$-bounded sets are $\tau_b(E)$-bounded in E' (see [7]). We also note that, like a reflexive Banach space, a semi-reflexive locally convex space contains no copy of $(c_0, \|\ \|_\infty)$ (see [8]).

2. Semi-Reflexivity And Reflexivity In Some Classes Of Locally Convex Spaces

Semi-reflexivity in locally convex spaces is inherited by closed subspaces ([4], §23, 3(5)), but not by quotient spaces, in general. For reflexivity is concerned, in general, it is inherited neither by closed subspaces nor by quotient spaces (see example-2.7). However it is a fact that every closed subspace of a reflexive (F)-space is reflexive ([4], §23, 5(10)). We generalize it in the following:

2.1 Theorem: Let F be a closed subspace of a reflexive locally convex space $E[\tau]$. Then F with the induced topology is reflexive if and only if it is quasi-barreled.

Proof: If F is reflexive, then it is always quasi-barreled. Conversely, if F is quasi-barreled then, since $E[\tau]$ is semi-reflexive and so the closed subspace F is also semi-reflexive, we obtain, by theorem-1.2, that F is reflexive.

Note that a separated quotient of a reflexive locally convex space is not semi-reflexive, in general. However, we have the following:

2.2 Theorem: Let F be a closed subspace of a reflexive locally convex space $E[\tau]$. Then the quotient space E/F is reflexive if and only if it is quasi-complete for the weak topology.

Proof: If E/F is reflexive (and so semi-reflexive), then it is quasi-complete for the weak topology.
Conversely, If E/F is quasi-complete for the weak topology, then it is semi-reflexive (by theorem-1.1). Further, the reflexive locally convex space $E[\tau]$ is always quasi-barreled. So its separated quotient space E/F is also quasi-barreled. Hence E/F is reflexive.

An (F)-space is reflexive if and only if it is semi-reflexive ([4], §23, 5(4)). We have generalized this fact to metrizable locally convex spaces in [9], in corollary-2.7. Here, we discuss it with an independent proof.

2.3 Theorem: A metrizable locally convex space $E[\tau]$ is semi-reflexive if and only if it is reflexive.

Proof: If $E[\tau]$ is metrizable, then, by [4], §21, 5(3), we have we have $\tau = \tau_{b^*}(E')$ and so it is quasi-barreled. Therefore, if $E[\tau]$ is semi-reflexive, then it becomes reflexive. Conversely, reflexive locally convex space is always semi-reflexive by definition.

2.4 Theorem: If $E[\tau]$ is a barreled space such that every bounded subset of E is relatively compact, then $E[\tau]$ is reflexive.

Proof: Let $E[\tau]$ be a barreled space. If every bounded subset of E is relatively compact, then every closed, bounded subset B of E is compact. So B is also weakly compact, and hence it is complete for the weak topology. So E is weakly quasi-complete and so (by theorem-1.1) $E[\tau]$ is semi-reflexive. Consequently, $E[\tau]$ is reflexive.

2.5 Corollary: An (M)-space is always reflexive.
Proof: It is direct from definition of (M)-space and the theorem-2.4.

2.6 Theorem: A barreled space $E[\tau]$ in which there is a denumerable system of convex compact subsets is reflexive.

Proof : From hypothesis, $E[\tau]$ becomes a strong dual of an (M)-space (by [10], proposition-1). So $E[\tau]$ itself is an (M)-space (see [6], 5.9). Hence $E[\tau]$ is reflexive (by corollary-2.5).

2.7 Example: This example is borrowed from[4]. Consider the product spaces $\varphi\omega$ and $\omega\varphi$ (see [4], §13,5.). They contain the vectors of the form $x = \{x_{11}, x_{12},\ldots, x_{21},x_{22},\ldots,\ldots\}$ and the vectors lying in both the products $\varphi\omega$ and $\omega\varphi$ has only finitely many nonzero x_{ij} and therefore we can write $\varphi\omega \cap \omega\varphi = \varphi$. Consider the direct sums $\varphi\omega \oplus \omega\varphi$ and $\omega\varphi \oplus \varphi\omega$. Let H_1 be the linear subspace of $\varphi\omega \oplus \omega\varphi$ consisting of all (x , x) with $x \in \varphi\omega \cap \omega\varphi = \varphi$ and H_2 the linear subspace of $\omega\varphi \oplus \varphi\omega$ consists of all $(x , -x)$, $x \in \varphi$. Let $H_1^{\perp}$ be the orthogonal space of H_1. It is easy to obtain that $H_1^{\perp} = H_2$ and $H_2^{\perp} = H_1$. Note that $\varphi\omega \oplus \omega\varphi$ and $\omega\varphi \oplus \varphi\omega$ are both locally convex spaces having closed subspaces H_1 and H_2 , respectively. Since $\varphi\omega \oplus \omega\varphi$ and $\omega\varphi \oplus \varphi\omega$ are constructed from K by repeatedly forming locally convex direct sums and topological products, both $\varphi\omega \oplus \omega\varphi$ and $\omega\varphi \oplus \varphi\omega$ are reflexive and the dual of $\varphi\omega \oplus \omega\varphi$ is $\omega\varphi \oplus \varphi\omega$. The quotient space $(\varphi\omega \oplus \omega\varphi) /H_1$ is not semi-reflexive. Now, H_2 is a closed subspace of the reflexive space $\omega\varphi \oplus \varphi\omega$ therefore it is semi-reflexive ([4], §23,3(5)). But $H_2^{\perp} = H_1$, so the dual of H_2 coincides with $(\varphi\omega \oplus \omega\varphi) /H_1$. Hence H_2 cannot be reflexive. Thus we obtain that the reflexive locally convex space $\varphi\omega \oplus \omega\varphi$ whose separated quotient $(\varphi\omega \oplus \omega\varphi) /H_1$ is not semi-reflexive. We also obtain that the closed subspace H_2 of the reflexive locally convex space $\omega\varphi \oplus \varphi\omega$ is not reflexive (Note that, here, H_2 is also an example of a semi-reflexive locally convex space which is not reflexive).
In this example, we note that the space ω is an (F)-space as well as an (M)-space. So $\varphi\omega \oplus \omega\varphi$ and $\omega\varphi \oplus \varphi\omega$ are (M)-spaces ([4], §27, 2(4)). Now the closed subspace H_2 of an (M)-space $\omega\varphi \oplus \varphi\omega$ in not reflexive. Further, the separated quotient $(\varphi\omega \oplus \omega\varphi) /H_1$ of the (M)-space $\varphi\omega \oplus \omega\varphi$ is not semi-reflexive. Therefore, we have the following:

2.8 Theorem: Closed subspace of an (M)-space need not be reflexive.

2.9 Theorem: Separated quotient of an (M)-space need not be semi-reflexive.

If a locally convex space E[τ] is quasi-complete for the weak topology $\tau_s(E')$, it is also quasi-complete for the finer topology τ. Therefore, a semi-reflexive locally convex space is always quasi-complete. But its converse is not true, in general. Any nonreflexive Banach space is an example for this. However we have the following result due to [4]:

2.10 Theorem: If E[τ] is quasi-complete and that τ is the Mackey topology and if the strong dual $E'[\tau_b(E)]$ is semi-reflexive, then E[τ] is reflexive.

Criteria for semi-reflexivity in quasi-complete locally convex spaces are also discussed in [11], [12], and [13].

We generalize these results for sequentially complete locally convex spaces. First, we introduce the following definition: A locally convex space E[τ] is said to be boundedly-reflexive if it contains all the linear functional on the dual E′ which are bounded on weakly bounded subsets of E′.

We, further, recall that a linear functional f on a locally convex space E[τ] is called locally bounded if its values remains bounded on any bounded subset of E. We note as well, that locally bounded linear functionals on E[τ] are continuous if and only if $E[\tau_k(E')]$ is bornological ([4],28, 1(3)).Now we have:

2.11 Theorem: A locally convex space E[τ] is boundedly-reflexive if and only if its Mackey dual is bornological.

Proof: Let E[τ] be boundedly-reflexive**.** We prove that $E'[\tau_k(E)]$ is bornological. Let u be a locally bounded linear functional on $E'[\tau_k(E)]$. Then, f is bounded on $\tau_k(E)$-bounded sets and so also bounded on $\tau_s(E)$-bounded sets. Since E[τ] is boundedly-reflexive**,** u ∈ E. Now $E = (E'[\tau_k(E)])'$ implies that u is continuous on $E'[\tau_k(E)]$. Hence, by the remark above, $E'[\tau_k(E)]$ is bornological.

Conversely, let the Mackey dual $E'[\tau_k(E)]$ be bornological. Let u be a linear functional on E′ which is bounded on weakly bounded subsets of E′. Since the Mackey topology $\tau_k(E)$ and the weak topology $\tau_s(E)$ defines same bounded sets, so the linear functional u is bounded on bounded subsets of $E'[\tau_k(E)]$. That is u is a locally bounded linear functional on $E'[\tau_k(E)]$. Since $E'[\tau_k(E)]$ is bornological, so u is continuous and so $u \in (E'[\tau_k(E)])' = E$. It means E is boundedly-reflexive**.**

We establish a result on semi-reflexivity in sequentially complete locally convex spaces:

2.12 Theorem: A boundedly-reflexive locally convex space which is also sequentially complete is semi-reflexive.

Proof: Let E[τ] be a locally convex space which is boundedly-reflexive and sequentially complete. If $u \in E'' = (E'[\tau_b(E)])'$, then u is $\tau_b(E)$-continuous and so u is bounded on $\tau_b(E)$-bounded sets.. Since E[τ] is sequentially complete, weakly bounded subsets of E′ are strongly bounded. So u is also bounded on weakly bounded subsets of E′. Therefore, since E[τ] is boundedly-reflexive, we have u ∈ E. Hence E[τ] is semi-reflexive.

Using theorem-1.2, we have:

2.13 Corollary: A boundedly-reflexive quasi-barreled space is reflexive.

We also have an independent proof of the following result of Hampson and Wilansky [14]:

2.14 Corollary: If E[τ] is a Banach space such that its Mackey dual is bornological, then it is reflexive.

Proof: Since, the Mackey dual $E'[\tau_k(E)]$ is bornological, E[τ] is boundedly-reflexive ,by theorem-2.11. Further, since Banach space E[τ] is always sequentially complete. So , by theorem-2.12,E[τ] is semi-reflexive and therefore, reflexive.

3. Hereditary- Reflexivity in Locally Convex Spaces

Let E[τ] be a locally convex space. Following [15], we define:

Hereditary-reflexivity: E[τ] is called hereditarily-reflexive if its closed subspaces and separated quotients are all reflexive.

Dually Hereditary-reflexivity: E[τ] is called dually hereditarily-reflexive if each of its closed subspaces and each of the closed subspaces of its strong dual are reflexive.

Further, following [16], we define:
Strongly hereditary-reflexivity: E[τ] is called strongly hereditarily-reflexive if its closed subspaces and separated quotients are hereditarily-reflexive.

Dually strongly hereditary-reflexivity: E[τ] is called dually strongly hereditarily-reflexive if it is strongly hereditarily-reflexive and its strong dual is also strongly hereditarily-reflexive.

Now we have:
3.1 Theorem: A reflexive locally convex space E[τ] is hereditarily-reflexive if and only if its closed subspace is quasi-barreled and each of its separated quotient is weakly quasi-complete.

Proof: If E[τ] is hereditarily-reflexive, each of its closed subspaces F and each of its separated quotients E/F are reflexive. Therefore, F is quasi-barreled and E/F is weakly quasi-complete. Conversely, let F be a closed subspace of E[τ] such that F is quasi-barreled. Since E[τ] is reflexive, the closed subspace F is also reflexive by theorem-2.1. Let E/F be a separated quotient of E[τ]. It is (given) weakly quasi-complete, Again, E[τ] is reflexive and so, by theorem-2.2, E/F is reflexive. Hence proved.

3.2 Theorem: Hereditarily-reflexive locally convex space is reflexive.

Proof: If a locally convex space E[τ] is hereditarily-reflexive, each of its closed subspace is reflexive. In particular, the space E, being a closed subspace of it-self, is reflexive.

3.3 Theorem: If E[τ] is a reflexive Banach space, then it is each of (a) hereditarily-reflexive, (b) dually hereditarily-reflexive, (c) strongly hereditarily-reflexive, and (d) dually strongly hereditarily-reflexive.

Proof: We know that:

(i) If E is a Banach space, then closed subspace F and the quotient E/F are also Banach spaces ([4], §14, 3(2), and §14, 4(3), resp.).
(ii) If E[τ] is reflexive Banach space and F is its closed subspace, then F as well as E/F are reflexive [4], §23,5(11).
(iii) The strong dual of a reflexive Banach space is also a reflexive Banach space ([4], §23, 5(11)).

Now assume that E[τ] is a reflexive Banach space. Let F (resp. E/F) be any closed subspace (separated quotient) of E and let H (resp. E′/H) be any closed subspace (resp. separated quotient) of the strong dual $E'[\tau_b(E)]$.

(a) F and E/F are reflexive by (ii). Hence E[τ] is hereditarily-reflexive.

(b) F is reflexive by (ii) and H are reflexive by (iii) and (ii). Hence E[τ] is dually hereditarily-reflexive.

(c) By (i) and (ii), F and E/F are reflexive Banach spaces, so, by (a), they are hereditarily-reflexive and so E[τ] is strongly hereditarily-reflexive.

(d) E[τ] is strongly hereditarily-reflexive (by (c)) and $E'[\tau_b(E)]$ is a reflexive Banach space (by(iii)) and so it is strongly hereditarily-reflexive, by(c). Hence E[τ] is dually strongly hereditarily-reflexive.

There are separated quotients of a reflexive (F)-space which are not reflexive. So reflexive (F)-space is not hereditarily-reflexive, in general. However, we have:

3.4 Corollary: A reflexive (F)-space E[τ] is each of (a) hereditarily-reflexive, (b) dually hereditarily-reflexive, (c) strongly hereditarily-reflexive, and (d) dually strongly hereditarily-reflexive if any one of the following conditions holds:

(i) E[τ] has a fundamental sequence of bounded sets

(ii) E[τ] contains a bounded absorbent set.

Proof: If (i) holds, then E[τ] is an (F)-space and it has a fundamental sequence of bounded sets, so E[τ] is a Banach space, by [4], §29, 1(3). On the other hand, if (ii) holds, then E[τ] is an (F)-space and it contains a bounded absorbent set, so E[τ] is a Banach space, by [4], §29, 1(9). Hence theorem-3.3 completes the proof.

Following [3], a locally convex space E[τ] is called pair-wise-reflexive, if the dual pair (E′, E) is reflexive in the sense of Köthe i.e. $(E'[\tau_b(E)])' = E$ and $(E[\tau_b(E')])' = E'$.

Following result is due to [4]:

3.5 Theorem: A locally convex space E[τ], where τ is the Mackey topology is reflexive if and only if it is pair-wise-reflexive.

We also have:

3.6 Corollary: A reflexive locally convex space is E[τ] is pair-wise-reflexive.

Proof: E[τ] is reflexive, so it is barreled and so $\tau = \tau_k(E')$.

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