

A numerical Approach for solving classes of Linear and Nonlinear Volterra Integral Equations by Chebyshev polynomial

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Abstract

In this paper we propose a numerical method for solving classes of of linear and nonlinear Volterra integral equations having regular as well as weakly singular kernels. The method is based upon replacing the unknown function by a truncated shifted Chebyshev series. This yeilds either a linear system of algebraic equations that can be solved using matrix algebra or a nonlinear system that can be solved by Newton's iterative method. This method is effective and so easy to apply with low cost of computing operations. The accuracy and efficiency of the method can be shown through the illustrated numerical examples.

keywords: Volterra integral equations, Newton's method, Simpson's method, Chebyshev polynomials, Shifted Chebyshev polynomials.

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1. Introduction

Many problems in Physics, Chemical Kinetics, Fluid mechanics and other applied sciences can be modelled and formulated to integral equations. A comprehensive reference on integral equation can be found in [3,4,5]. Many researchers studied and discussed numerical solution of Volterra integral equation as Adomian decomposition method, Homotopy purturbation method, sucessive approximation method and many others [4]. In (1959) David Elliott [8] and in (1969) S.E. Elgendi [13] presented anumerical method based on Chybyshev polynomials. Recently orthogonal polynomials have been widely used for function approximation [2,9] that help to solve linear and nonlinear Volterra integral equations. Ycheng Lie in [15] presented a numerical method to solve linear Volterra equation of the second kind by orthogonal Legendre polynomial. M.M.R in [11] presented a method to solve classes of linear Volterra integral equations by Hermit and Legendre polynomials. In [6] Chniti presented a method to solve Abel's integral equation by Legendre polynomials. In [13] an approach to numerical solution of Abel's integral equation is presented by M.N. Sahlan using Block pulse function . In the work of the authors a Volterra integral equation is converted to a linear system of algebraic equations that can be solved to obtain the solution of the integral equation.

Numerical solution of nonlinear integral equations has been presented by A. Akyüz-Daşcioğlu, H. Çerdik Yaslan [1] , E. Babolian in [7] and K.Maleknejad, S.shorabi, Y. Rostami in [10] using Chebyshev polynomials. Also in [16] Ordokhani presented a simple method using Bessel functions. In this case Volterra integral equations is converted to a nonlinear system of eauations which has been solved numerically.

In our paper a simple method using orthogonal shifted Chebyshev polynomials is presented to solve classes of linear and nonlinear Volterra integral equations having regular as well as singular kernels . Simpson's method is used to approximate the integra and Newton's iteration methods [14] are used for solving nonlinear system we get when applying the method.

The paper is organized as follows: in section 2 the properties of shifted Chebysheve polynomials are introduced. in section 3, we give the describtion of the method . in section 4, a brief view about convergence is introduced. Section 5 is devoted to the conclusion of this paper through several numerical examples that are presented to illustrate the method.

2. About shifted Chebyshev polynomials

Definition: [5] The Chebyshev polynomial of the first kind $T_n(x)$ is a polynomial in x of degree n defined by

$$T_n(x) = \cos n\ \theta \qquad \text{where } x = \cos\ \theta. \tag{2.1}$$

If the range of the variable x is the interval [-1,1], the range corresponding to the variable θ is the interval $[0, \pi]$.
We may deduce from (2.1) the first few Chebyshev Polynomials

$$T_0(x) = 1,\ T_1(x) = x,\ T_2(x) = 2x^2 - 1,\ T_3(x) = 4x^3 - 3x,\ T_4(x) = 8x^2 - 8x + 1,\ \ldots. \tag{2.2}$$

We can also deduce that

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad n = 2, 3, \ldots, \tag{2.3a}$$

with initial conditions

$$T_0(x) = 1, \quad T_1(x) = x. \tag{2.3b}$$

Since the range [0,1] is quite often more convenient to use than [-1,1], we use the transformation

$$s = 2x - 1. \tag{2.4}$$

This leads to shifted Chebyshev polynomials of the first kind given by

$$T_n^*(x) = T_n(s) = T_n(2x - 1). \tag{2.5}$$

Thus the shifted Chebysheve polynomials are

$$T_0^*(x) = 1, \quad T_1^*(x) = 2x - 1, \quad T_2^*(x) = 8x^2 - 8x + 1,$$
$$T_3^*(x) = 32x^3 - 48x^2 + 18x - 1,\ T_4^*(x) = 32x^4 - 64x^3 + 16x^2 + 16x - 5,\ \ldots \tag{2.6}$$

from (2.4) and (2.5), we can deduce the recurrence relation for

$$T_{n+1}^*(x) = 2(2x - 1)T_n^*(x) - T_{n-1}^*(x), \tag{2.7a}$$

with initial conditions

$$T_0^*(x) = 1,\ T_1^*(x) = 2x - 1. \tag{2.7b}$$

Also from (2.1) and (2.5) we can deduce that

$$T_{2n}(x) = T_n^*(x^2). \tag{2.8}$$

3. Describtion of the Method

Consider the following Volterra integral equation of the second kind

$$y(x) - \int_0^x K(x,t)\ y(t)\ dt = f(x), \qquad 0 \le t \le x \le 1. \tag{3.1}$$

To find the approximate solution of the equation (3.1),
let

$$y(x) = \sum_{i=0}^{N}{}' a_i\ T_i^*(x) = \frac{1}{2}a_0 T_0^*(x) + a_1 T_1^*(x) + \ldots + a_N T_N^*(x), \tag{3.2}$$

where $T_i^*(x)$ are shifted Chebyshev polynomials of degree i defined by (2.6) and a_i are the coefficients to be determined.

Substituting from (3.2) into (3.1), we have

$$\sum_{i=0}^{N}{}' a_i T_i^*(x) = f(x) + \sum_{i=0}^{N} a_i \int_0^x K(x,t)\ T_i^*(t)\ dt, \tag{3.3}$$

multiplying both sides of the equation (3.3) by $T_j^*(x)$, $0 \le j \le n$ and integrate with respect to x from 0 to 1, we obtain

$$\sum_{i=0}^{N}{}' a_i \int_0^1 T_i^*(x) T_j^*(x)\ dx = \int_0^1 T_j^*(x)[f(x) + \sum_{i=0}^{N}{}' a_i \int_0^x K(x,t)\ y(t)\ dt]\ dx. \tag{3.4}$$

Equation (3.4) may be converted into the matrix form

$$TA = F, \tag{3.5}$$

where $T = [t_{ij}]$ is the $(N+1) \times (N+1)$ matrix defined by

$$t_{ij} = \int_0^1\ [T_{j-1}^*(x) - \int_0^x K(x,t)\ T_{j-1}^*(t)\ dt]\ T_{i-1}^*(x)\ dx,$$

$$A = \begin{pmatrix} \frac{a_0}{2} \\ a_1 \\ a_2 \\ \vdots \\ a_{N-1} \\ a_N \end{pmatrix} \quad \textbf{and} \quad F = \begin{pmatrix} \int_0^1\ f(x)T_0(x)dx \\ \int_0^1\ f(x)T_1(x)dx \\ \int_0^1\ f(x)T_2(x)dx \\ \vdots \\ \int_0^1\ f(x)T_{N-1}(x)dx \\ \int_0^1\ f(x)T_N(x)dx \end{pmatrix}. \tag{3.6}$$

Solving (3.5) for A and substituting into (3.2), we obtain the approximate solution of (3.1).

4 The error of a truncated Chebyshev expansion

The useful applications of Chebyshev expansion arise when the expansion converges much faster. If the function to be expanded is continuously differentiable then Chebyshev expansion converges rapidly. Numerically, this means that how fast the coefficients a_n decrease depend on continuity and differentiability properties of the function to be expanded. The following theorem summarizes the point above

Theorem: [5] If the function $f(x)$ has $m+1$ continuous derivatives on $[-1,1]$, then

$|f(x) - S_N^T f(x)| = O(n^{-m})$ for all x in $[-1,1]$.

5 Numerical Examples

Example 1.

Consider the following linear Volterra integral equation

$$y(x) = x + \frac{1}{5}x\int_0^x ty(t)dt, \qquad 0 \le x \le 1, \tag{5.1}$$

with the exact solutions $y = xe^{\frac{x^3}{15}}$.

we find the approximate solution of (5.1) if N =3, N=4 and N=6.

(a) N =3
Let

$$y(x) = {\sum_{i=0}^{3}}' a_i T_i^*(x). \tag{5.2}$$

Multiplying both sides of (5.1) by $T_j^*(x)$, $0 \le j \le 3$ and integrate with respect to x from 0 to 1, we obtain the system (3.5), where

$$T = \begin{pmatrix} \frac{39}{80} & -\frac{1}{600} & -\frac{191}{600} & \frac{171}{175} \\ -\frac{3}{400} & \frac{191}{360} & \frac{107}{12600} & -\frac{37}{175} \\ -\frac{199}{1200} & -\frac{37}{12600} & \frac{167}{360} & -\frac{517}{1575} \\ \frac{86}{175} & -\frac{106}{525} & -\frac{512}{1575} & -\frac{1286}{875} \end{pmatrix} \text{ and } F = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{6} \\ -\frac{1}{6} \\ \frac{2}{5} \end{pmatrix}. \tag{5.3}$$

Solving the linear system $TA = F$, we have

$$A = \begin{pmatrix} \frac{3683590085}{3582492258} \\ \frac{1265783085}{2388328172} \\ -\frac{17654595}{1194164086} \\ \frac{10589075}{2388328172} \end{pmatrix}. \tag{5.4}$$

Substituting from (5.4) into (5.2), we have the approximate solution

$$y(x) = -0.001093037952 + 1.021507860x - 0.09454399217x^2 + 0.1418776548x^3.$$

(b) N=4.

Let

$$y(x) = \sum_{i=0}^{4}{}' a_i T_i^*(x). \tag{5.5}$$

Proceeding as above , we obtain

$$T = \begin{pmatrix} \frac{39}{80} & -\frac{1}{600} & -\frac{191}{600} & \frac{171}{175} & -\frac{289}{4200} \\ -\frac{3}{400} & \frac{191}{360} & \frac{107}{12600} & -\frac{37}{175} & -\frac{23}{37800} \\ -\frac{199}{1200} & -\frac{37}{12600} & \frac{167}{360} & -\frac{517}{1575} & -\frac{1349}{7560} \\ \frac{86}{175} & -\frac{106}{525} & -\frac{512}{1575} & \frac{1286}{875} & -\frac{5692}{86625} \\ -\frac{23}{8400} & -\frac{13}{7560} & -\frac{6977}{37800} & -\frac{5627}{86625} & \frac{10331}{21000} \end{pmatrix} \text{ and } F = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{6} \\ -\frac{1}{6} \\ \frac{2}{5} \\ -\frac{1}{30} \end{pmatrix}. \tag{5.6}$$

Solving the linear system $TA = F$, we have

$$A = \begin{pmatrix} 0.5142879955 \\ 0.5299845196 \\ 0.01518329450 \\ 0.004431228946 \\ 0.0006406557418 \end{pmatrix}. \tag{5.7}$$

Substituting from (5.7) into (4.5), we have the approximate solution

$$y(x) = 0.0001274262791 + .9977638206x + 0.01127228524x^2 - 0.02220854362x^3 + 0.08200393495x^4.$$

(c) N=6

Let

$$y(x) = \sum_{i=0}^{6}{}' a_i T_i^*(x). \tag{5.8}$$

Proceeding as above, we have

$$T = \begin{pmatrix} \frac{39}{160} & -\frac{1}{600} & -\frac{191}{600} & \frac{3}{1400} & -\frac{289}{4200} & -\frac{1}{2520} & -\frac{121}{4200} \\ -\frac{3}{800} & \frac{119}{360} & \frac{107}{12600} & -\frac{11}{56} & -\frac{23}{37800} & -\frac{73}{1512} & -\frac{13}{46200} \\ -\frac{199}{2400} & -\frac{37}{12600} & \frac{167}{360} & \frac{43}{12600} & -\frac{1349}{7560} & -\frac{17}{8316} & -\frac{1157}{27720} \\ \frac{11}{5600} & -\frac{841}{4200} & -\frac{17}{2520} & \frac{3393}{7000} & \frac{307}{99000} & -\frac{1141}{6600} & -\frac{269}{3003000} \\ -\frac{257}{16800} & \frac{13}{7560} & -\frac{6977}{37800} & -\frac{373}{99000} & \frac{10331}{21000} & \frac{641}{200200} & -\frac{139499}{819000} \\ \frac{1}{1440} & -\frac{349}{7560} & -\frac{7}{11880} & -\frac{4901}{27720} & -\frac{887}{360360} & \frac{96029}{194040} & -\frac{23651}{12612600} \\ -\frac{113}{16800} & \frac{1}{1320} & -\frac{1907}{46200} & -\frac{589}{3003000} & -\frac{520123}{3003000} & -\frac{24509}{12612600} & \frac{3478857}{7007000} \end{pmatrix} \text{ and } F = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{6} \\ -\frac{1}{6} \\ -\frac{1}{10} \\ -\frac{1}{30} \\ -\frac{1}{42} \\ -\frac{1}{70} \end{pmatrix} \quad (5.9)$$

Solving the linear system $TA = F$, we have

$$A = \begin{pmatrix} .518703488 \\ 0.5299971857 \\ 0.01513846050 \\ 0.004445469709 \\ 0.0006232955095 \\ 0.00002597807922 \\ 0.000004261010945 \end{pmatrix}. \quad (5.10)$$

Substituting from (5.10) into (5.8) , we obtain the approximate solution

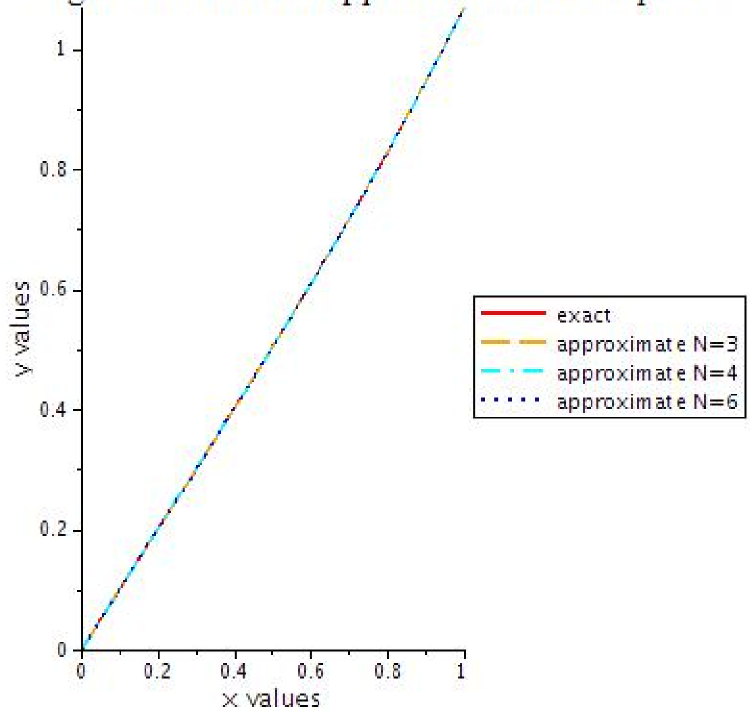
$y(x) = 0.0000008716672502 + 0.999951797\,x + 0.000640436963\,x^2 - 0.003484634234e\,x^3 + 0.07598199146\,x^4 - 0.01287887469\,x^5 +$

Figure 1 and table 1 show fast convergence of the expansion

Table 1: the absolute error for example 1 in case of N=3, N=4 and N=6

x	N=3	N=4	N=6
0	1.093E-2	1.764E-5	8.716672502E-7
0.1	2.475E-4	1.764E-5	2.176127498E-7
0.2	4.55E-4	9.42124E-6	2.847672502E-7
0.3	1.40571E-4	11.5477E-6	1.2327498E-9
0.4	2.47E-4	1.6789E-5	2.657327498E-7
0.5	4.24475E-4	2.98792E-6	1.37327498E-8
0.6	2.81E-4	1.520794E-5	2.695672502E-7
0.7	1.088864E-4	1.710124E-5	2.86672502E-8
0.8	4.684E-4	6.01115E-6	3.056327498E-7
0.9	2.919E-4	2.49858E-5	2.168672502E-7
1	1.19E-3	6.287E-5	9.672327498E-7

figure 1 :exact and approximate of example 1



Example 2.

Consider the weakly singular Volterra integral equation

$$y(x) = 1 - \frac{\pi}{2} + \int_0^x \frac{y(t)}{\sqrt{x^2 - t^2}}\, dt, \tag{5.11}$$

with exact solution $y(x) = 1$.

Let

$$y(x) = {\sum_{i=0}^{3}}{}' a_i T_i^*(x). \tag{5.12}$$

Using the method, we obtain

$$T = \begin{pmatrix} \frac{1}{4} - \frac{\pi}{8} & \frac{\pi}{2} - 1 & 8 - \frac{\pi}{2} - \frac{1}{3} - \frac{2\pi}{3} & \frac{\pi}{2} + 4\pi - \frac{43}{3} \\ 0 & 0 & -\frac{\pi}{3} & 2\pi - \frac{32}{5} \\ \frac{\pi}{24} - \frac{1}{12} & \frac{1}{3} - \frac{\pi}{6} & \frac{7}{15} + \frac{2\pi}{15} - \frac{13}{15} + \frac{\pi}{6} & \frac{151}{45} - \frac{4\pi}{5} - \frac{\pi}{6} \\ 0 & 0 & \frac{\pi}{5} & \frac{151}{45} - \frac{4\pi}{5} - \frac{\pi}{6} \end{pmatrix} \text{ and } F = \begin{pmatrix} -\frac{\pi}{2} \\ 0 \\ \frac{\pi}{6} - \frac{1}{3} \\ 0 \end{pmatrix} \tag{5.13}$$

Solving the linear system $TA = F$, we have

$$A = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{5.14}$$

Substituting from (5.14) into (5.12), we have the approximate solution $y(x) = 1$ which is the same as the exact solution.

Example 3.

consider the weakly singular Volterra integral equation

$$y(x) = 1 - 2x - \frac{32}{21}x^{\frac{7}{4}} + \frac{4}{3}x^{\frac{3}{4}} - \int_0^x \frac{y(t)}{(x-t)^{\frac{1}{4}}}\, dt, \tag{5.15}$$

with exact solution $y(x) = 1 - 2x$.

let

$$y(x) = \sum_{i=0}^{3}{}' a_i T_i^*(x). \tag{5.16}$$

Following the method, we obtain

$$T = \begin{pmatrix} \frac{37}{42} & -\frac{16}{77} & -\frac{2099}{3465} & \frac{2608}{21945} \\ \frac{16}{308} & \frac{121}{315} & -\frac{208}{1463} & -\frac{6353}{26565} \\ \frac{2099}{6930} & \frac{208}{1463} & \frac{9547}{15939} & -\frac{82672}{592515} \\ \frac{1717}{2090} & -\frac{11873}{26565} & -\frac{276257}{592515} & \frac{10344923}{15646785} \end{pmatrix} \text{ and } F = \begin{pmatrix} \frac{16}{77} \\ -\frac{121}{315} \\ -\frac{208}{1463} \\ \frac{11873}{26565} \end{pmatrix}. \tag{5.17}$$

Solving the linear system $TA = F$, we have

$$A = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}. \tag{5.18}$$

Substituting from (5.18) into (5.16), we obtain the approximate solution $y(x) = 1 - 2x$ which is the exact solution.

The following example illustrates bad convergence of the Chebyshev expansion that is caused by the analytic property of the function:$\sqrt{x}$ is not differentiable at the end point $x = 0$

Example 4.
Consider the following weakly singular integral equation

$$y(x) = \sqrt{x} - \frac{\pi}{2}x - \int_0^x \frac{y(t)}{(x-t)^{\frac{1}{2}}}\, dt, \qquad 0 \leq x \leq 1, \tag{5.19}$$

with the exact solution $y(x) = \sqrt{x}$.

we find the approximate solution if N= 3, N=6 and N=10.

(a) N=3

let

$$y(x) = {\sum_{i=0}^{3}}' a_i T_i^*(x). \tag{5.20}$$

Proceeding as before, we obtain

$$T = \begin{pmatrix} \frac{7}{6} & -\frac{4}{15} & -\frac{29}{35} & \frac{44}{315} \\ \frac{4}{30} & \frac{11}{21} & -\frac{28}{135} & -\frac{79}{231} \\ -\frac{52}{105} & -\frac{28}{135} & -\frac{388}{1155} & \frac{20}{91} \\ -\frac{29}{35} & \frac{28}{135} & \frac{309}{385} & -\frac{20}{91} \\ -\frac{44}{315} & -\frac{79}{231} & \frac{20}{91} & \frac{1387}{1925} \end{pmatrix}$$

and

$$F = \begin{pmatrix} \frac{2}{3} + \frac{\pi}{4} \\ \frac{2}{15} + \frac{\pi}{12} \\ -\frac{26}{105} - \frac{\pi}{12} \\ \frac{\pi}{5} + \frac{188}{315} \\ -\frac{\pi}{20} - \frac{22}{315} \end{pmatrix} \tag{5.21}$$

Solving the linear system $TA = F$, we have

$$A = \begin{pmatrix} 1.286588968 \\ 0.4184361908 \\ -0.07054449385 \\ 0.02989779470 \end{pmatrix}. \tag{5.22}$$

Substituting from (5.22) into (5.20), we obtain the approximate solution

$$y(x) = .1244160047 + 1.939388637\ x - 1.999450097\ x^2 + .9567294304\ x^3.$$

(b) N=6

Let

$$y(x) = \sum_{i=0}^{6}{}' a_i T_i^*(x). \tag{5.23}$$

Proceeding as before, we obtain

$$T = \begin{pmatrix} \frac{7}{6} & -\frac{4}{15} & -\frac{29}{35} & \frac{44}{315} & -\frac{95}{693} & \frac{388}{9009} & -\frac{2603}{45045} \\ \frac{4}{30} & \frac{11}{21} & -\frac{28}{135} & -\frac{79}{231} & \frac{92}{2457} & -\frac{691}{10395} & \frac{596}{51051} \\ -\frac{29}{70} & \frac{28}{135} & \frac{309}{385} & -\frac{151}{273} & -\frac{13897}{51975} & \frac{30908}{2297295} & -\frac{193657}{4279275} \\ \frac{691}{630} & -\frac{703}{1155} & -\frac{277}{455} & \frac{2604}{2475} & -\frac{237283}{765765} & \frac{55471}{82593} & -\frac{223753}{5360355} \\ -\frac{95}{1386} & -\frac{92}{2457} & -\frac{13897}{51975} & \frac{7387}{69615} & \frac{79847}{141075} & -\frac{820532}{5360355} & -\frac{24027827}{89423325} \\ -\frac{388}{18018} & -\frac{691}{10395} & -\frac{375}{1463} & \frac{820532}{5360355} & \frac{3669308}{26801775} & \frac{582198473}{885809925} & -\frac{3669308}{26801775} \\ -\frac{2603}{90090} & -\frac{596}{51051} & -\frac{193657}{4279275} & -\frac{26573}{595595} & -\frac{240277827}{98423325} & \frac{3669308}{26801775} & \frac{582198473}{885809925} \end{pmatrix}$$

and

$$F = \begin{pmatrix} \frac{2}{3} + \frac{\pi}{4} \\ \frac{2}{15} + \frac{\pi}{12} \\ -\frac{26}{105} - \frac{\pi}{12} \\ \frac{\pi}{5} + \frac{188}{315} \\ \frac{\pi}{60} + \frac{122}{3465} \\ -\frac{\pi}{84} - \frac{194}{9009} \\ -\frac{94}{6435} - \frac{\pi}{140} \end{pmatrix} \tag{5.24}$$

Solving the linear system $TA = F$, we have

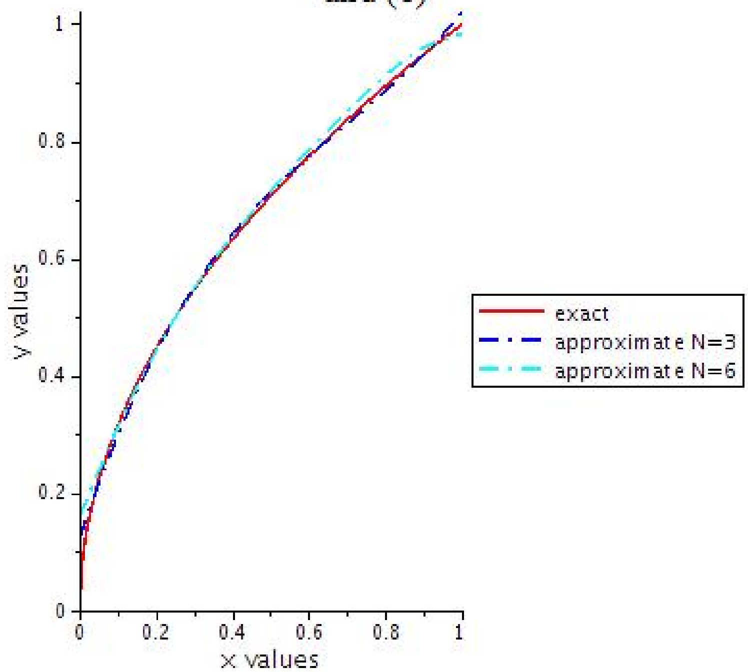
$$A = \begin{pmatrix} 1.2832338582 \\ 0.4045978562 \\ -0.07329374135 \\ 0.0.007199777645 \\ -0.007634498943 \\ -0.001595310723 \\ 0.0005339855088 \end{pmatrix}. \tag{5.25}$$

Substituting from (5.25) into (5.23), we obtain the approximate solution

$y(x) = 1.594478264 + 1.618520812x - 0.652168123x^2 - 2.839775364x^3 + 6.327142711x^4 - 4.726187261x^5 + 1.093602322x^6.$

figure 2 shows the exact and approximate solution if N=3 and N=6

figure 2 :exact and approximate of example 4 (a) and (b)



(c) N=10

Proceeding as before, we have the approximate solution

$$y(x) = 0.03598340522 + 6.533193289x - 82.48136738x^2 + 739.120474x^3 - 3977.0709521x^4 + 13246.38638x^5$$
$$-27968.88659x^6 + 37453.8127x^7 - 30793.02783x^8 + 14168.44744x^9 - 2791.884633x^{10}.$$

figure 3 shows the exact and approximate solution of example 4 if N=10.

Table 2 gives the absolute error for example 4 at the nodes x in case of N=3, N=6 and N=10

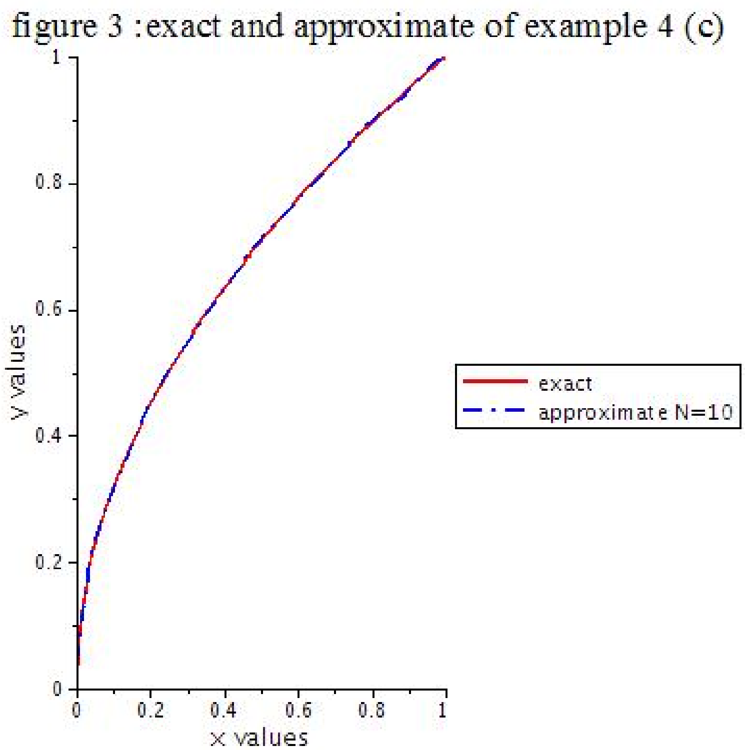


Table 2: absolute error for example 4

x	N=3	N=6	N=10
0	1.244160047E-1	1.594478264E-1	3.59834E-2
0.1	1.691066914E-2	3.70276900E-3	2.37901E-3
0.2	7.244031837E-3	4.185495535E-3	2.18595E-3
0.3	4.39122422E-3	2.474903613E-3	5.1552E-4
0.4	9.03459555E-3	6.366187161E-3	1.8515E-3
0.5	6.7321966E-3	8.42810378E-3	2.9236E-3
0.6	3.040396E-4	1.130412504E-2	1.454E-3
0.7	6.2443287E-3	1.56249296E-2	1.221E-3
0.8	8.3028699E-3	1.80866001E-2	3.07E-3
0.9	9.163446E-4	1.06379686E-2	3.93E-3
1	2.10839754E-2	1.9417077E-2	1.518E-2

It can be seen from the table that the maximum error occurs at the end point $x = 0$ for which the function is not differentiable.

We can also apply our method to solve nonlinear Volterra integral equation as shown in the following example

Example 5.
Consider the following nonlinear Volterra integral equation

$$y(x) = \frac{1}{4}x^5 + \frac{5}{4}x^4 + \frac{5}{2}x^3 + \frac{5}{2}x^2 + 2x + 1 - \int_0^x (x+1)(y(t))^3 \, dt, \qquad 0 \leq x \leq 1 \tag{4.26}$$

with exact solution $y(x) = x + 1$.
Let

$$y(x) = \sum_{i=0}^{2}{}' a_i T_i^*(x), \tag{5.27}$$

Substituting from (5.27) into (5.26) and apply Simpson's rule to the integral in the right hand side of (4.26) , we have

$$\sum_{i=0}^{2}{}' a_i T_i^*(x) = \frac{1}{4}x^5+\frac{5}{4}x^4+\frac{5}{2}x^3+\frac{5}{2}x^2+2x+1-\frac{1}{6}x(x+1)[(a_0-a_1+a_2)^3+4(a_0+a_1(x+1)+a_2(2x^2-4x+1))^3$$

$$+ (a_0 + a_1(2x-1) + a_2(8x^2-8x)+1)^3]. \tag{5.28}$$

Now multiplying both sides of (4.28) by $T_j^*(x)$, $0 \leq j \leq 2$ and then integrate with respect to x from 0 to 1, we have the following nonlinear system of 3 equations in 3 unknowns.

$$-\tfrac{1}{3}a_2+a_0-\tfrac{3}{100}a_0a_1a_2-\tfrac{391}{1890}a_2^3+\tfrac{353}{210}a_2^2a_0-\tfrac{239}{420}a_2^2a_1-\tfrac{3}{20}a_1^3+\tfrac{5}{6}a_0^3-\tfrac{9}{10}a_0^2a_2+\tfrac{4}{5}a_0a_1^2+\tfrac{41}{105}a_1^2a_2-\tfrac{3}{4}a_0^2a_1-\tfrac{15}{4}=0$$

$$\tfrac{1}{3}a_1+a_0-\tfrac{3}{70}a_0a_1a_2-\tfrac{146}{945}a_2^3+\tfrac{31}{35}a_2^2a_0-\tfrac{27}{140}a_2^2a_1-\tfrac{1}{84}a_1^3+\tfrac{1}{3}a_0^3-\tfrac{8}{15}a_0^2a_2+\tfrac{7}{30}a_0a_1^2+\tfrac{37}{210}a_1^2a_2-\tfrac{1}{20}a_0^2a_1-\tfrac{37}{28}=0$$

$$\tfrac{7}{15}a_2-\tfrac{1}{3}a_0+\tfrac{3}{70}a_0a_1a_2-\tfrac{61}{20790}a_2^3-\tfrac{157}{630}a_2^2a_0+\tfrac{283}{1260}a_2^2a_1+\tfrac{11}{140}a_1^3-\tfrac{7}{30}a_0^3+\tfrac{17}{70}a_0^2a_2-\tfrac{8}{35}a_0a_1^2-\tfrac{1}{315}a_1^2a_2+\tfrac{9}{20}a_0^2a_1+\tfrac{6}{7}=0. \tag{4.29}$$

Using Newton's iterative method with initial guess (1,0.5,0) we obtain an approximate solution of the the system (4.29)

$$a_0 = 1.49999860988977,\ a_1 = 0.499999195660806,\ a_2 = -0.00000299304846076335. \tag{4.30}$$

Substituting from (4.30) into (4.28), we have the approximate solution

$$y(x) = .9999964213 + 1.000022336x - 0.00002394438769x^2$$

Numerical result of example 5 are shown in table 3 and figure 4

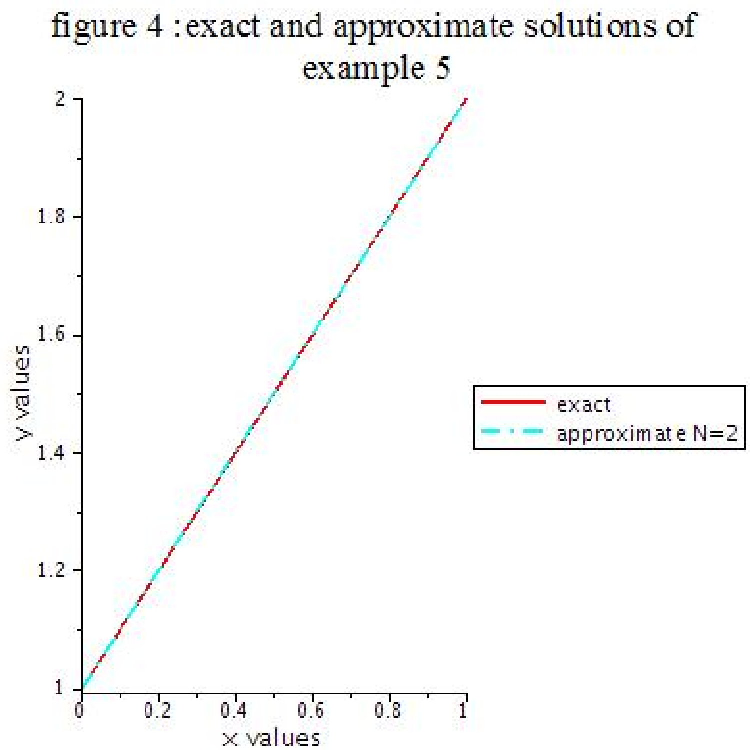


Table 3. The results of Example 5.

x	Exact	Approx	Absolute error
0	1	0.9995403391	.0004596609
0.1	1.1	1.099773664	2.26336E-4
0.2	1.2	1.199958290	4.17103E-5
0.3	1.3	1.300094217	9.421623E-5
0.4	1.4	1.400181443	1.81444E-4
0.5	1.5	1.500219972	2.19971E-4
0.6	1.6	1.600209801	2.09800E-4
0.7	1.7	1.700150931	1.5093E-4
0.8	1.8	1.800043361	4.3361E-5
0.9	1.9	1.899887093	1.1291E-4
1	2	1.999682125	3.1787E-4

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