

On Minimal Semi neat Subgroups

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Abstract

In [1] Abdulla Hattem gave some new results of minimal neat subgroups of Abelian G .

"L. Fuchs " poses the problem of characterizing the subgroups of an Aealian group G which are intersections of finitely many pure subgroups of G (problem 13, p. 134) . This problem has been solved by "Khalid Benabdallah" and John Irwin (see[2]) .

In this paper we shall give the generalization of the problem solved by Khalid Benabdallah . Firstly we shall give the definition of such subgroups which are called almost almostdense in G .

Introduction:

We start with the following definitions:

Definition 1:

A subgroup H of G is said to be neat in G, if $\forall$ prime number P. PG see[4]

Definition 2: A subgroup H of G is said to be pure in G, if $\forall$ P and $\forall$ k ($k \in Z^{+}$)

$P^{k}GH=P^{k}H$ (see[4])

Definition 3: A subgroup H of G is said to be almost-dense in G (abbrevinated a.b)

If , for every pure subgroup K of G containing H , G/K is divisible (see[2])

We shall give an example of a.b. subgroup :

Example : Take G = c_{2}^{∞} A= {0 , 1/2} . Clearly A is a.d in G , because there is no near (pure) subgroup of G (except G) , moreover G is divisible , so G/A is divisible . Hence A is a.b in G .

Now we shell give the following definition of almost-almost- dense subgroup :

Definition 4:A : subgroup H of G, we said to be almost-almost- dense (abbreviated ; a.a.d.) if , for every semi neat subgroup H of G containing

H , G/K is divisible .

Definition 5: A _subgroup H , is said to be semi neat subgroup of G ; if PG for some P.

Clearly , every neat is semi neat but the converse is not true . We can show that by the following example .

Example : Take G = Z_8 and H = c{5,4} it's clear that $3G4^{-}$}=3H

So H is semi near in G but H is not neat, because 2G H = 4 } and 4 } 2H = {

Remark: In this paper we denoted the following notations by:

P. Prime number

K. Positive integer

G. Abelian Group

Remark:

- Every group G is a.a.d. itself
- Every a.a.d. subgroup is a.d.
- In torsion-free groups or in divisible groups ,the subgroups are a.d. if there are a.a.d. subgroups

Notation : let G be any group . We denote by G_k the following

G_k = {x P^k G / o(x) = P for some K Z^+ } = P^k G [p].

The following , shows some properties of a.a.d. subgroups

THEOREM 1 : In a primary group G , if every neat subgroups N H such a subgroups H if and only if H is a.a.d. N G_k .

Proof . Suppose H is a.a.d. in G' and N G_k , then every semi neat subgroups B of G contains H , contains also G_k . Claim P^kG B

Let x P^kG . Then

(1)x = P^k g for some g G .

Since G is a p-group , then o(x) = p^m (m Z^+) .So by (1) we have p^m x = 0 if m = 1 then we get the result . if m 1 then

p^m x =p (p^{m-1} x) = 0 , but p^{m-1} x p^k G , so p^{m-1} x G_k . Hence p^{m-1} x by assumption we have , PB is pure , thus p^{m-1} x G PB = p^{m-1} (PB) . So p^{m-1} x = p^{m-1} b(Pb) for sonic b B

hence P(P^{m-2} x – P^m b) = 0 but P^m b P^k B P^k G then

P^{m-2} x – P^m b G_k H B

So P^{m-2} x – B . but this way we get $P^{m-(m-1)}$ x- Px B So Px BPG = PB , hence

Px pB PG = p(pG)

Thus px = p^2b_0 some b_0 B Then we get p(x-pbo) = 0 since pbo pB P^k G . therefore x-pbo G_k B . consequently x B .

Thus it proves that . p^k G B.So G/B is at the same time divisible and bounded.

G/B = B, i.e. G = B.

Conversely , if no proper semi neat subgroup of G containing H and GIG ={0} is divisible . Therefore H is a.a.d. in G . Now ,

Since no proper semi neat subgroup of G contains H , so no peoper pure subgroup of G contains H , thus by Lemma 4.1 and theorem 3.7 in [2] , HG_k for some K Z^+ .

In view of the preceding theorem , we need only characterize an subgroup of G . For this purpose we need the following lemmas:

LEMMA 1: In a primary group G if S is a subgroup of G[p] such that S p^n G = 0 for some it Z^+ , then there exists a neat subgroup K of G such that K[p] = S. Furthermore (Kp^n G)/p^nG is neat in G/p^nG .

Proof . By Lemma 1.4 of [1] , there exists a pure subgroup K of G such that K[P] = S , also we have (Kp^n G) p^nG is neat in G/P^n G .

LEMMA 2 : Let N be a subgroup of a primary group G such that for some n P^n G_{n-1} .Then Mere exists a proper subgroup of G such theft R p^aG and N + p^n R_{n-1} (see semi neat [1])

LEMMA 3 : In a primary group G (for every semi neat subgroup Acontaining G[p] , A = G

Proof . Let A be a semi subgroup of G and let x since G is a p group , so p^kx = 0 for some K Z^+ . (If k =1 we get the result.

Assume K 1)

So p^kx = p(p^{k-1}x) = 0 thus p^{k-1}x [p] A and p(p^{k-2}x) A p then p^{k-2}x must belong to G[p] A .

Again , we have p(p^{k-3}x) pG , and so(p^{k-3}x) pa_0 for some , a_0 thus

P(p^{k-3} x-a_0) = 0 and p^{k-3} x-a_0

By this way we obtain px A so P(x-a_1) = 0x-a_1 , which implies that x

We are ready to show that the following :

THEOREM 2 : *n a group G a sub*group N of G is a.a.d. if mid only if

(*) N+ p^n G G_{n-1}

Holds for all n

Proof . Suppose N satisfies (*) and K is any semi neat *sub*group containing N . to show G/K divtalbh , it not (on proof will be by showing the contradiction) .So G/K must ave cyclic summand R/K (see [4] , Theorem 9).

Now G/K = H/K and G/H is finite (say p^n(R/K)= K for some n Z^+ . claim p^nG .

Let x =p^n g p^n G for some n Z^+ .

(2) X+K = (h+K) (r+K)

For some h+k and r+k R/k. since p^n/x in G so p^n/x+k in G/k and hence p^n/r+k in R/k . So p^n(ro+k)=r+k therefore r . By (2) we get x+k=h+k which implies x-h , hence x.

So p^nG for some n Z^+ . Thus H N +p^n G G_{n-1} , after a finite number of steps we see that .

H .

Since K is semi neat in G , and H/K , so H is neat in G . (Because, in g = pg so pg + k = pho + k and hence pg – pho k . But K is neat , thus p(g – ho) = pL for some L and h = pL +pho = p(L + ho).) Thus by Lemma 3 , H = G . Then R/K =0 and this is in contradiction for the fact that R/K . Hence g/k is divisible and thus N is a.a. dense .

Conversely , let N is an a.a.d , if (*) is not satisfied , then we are in the situation of Lemma 2 , there exists a proper neat R in G with

R N + p^n G

Since N is a.a.d , then G/R is divisible , but p^n(G/R) =R This is a concretion , consequently (+) is satisfied . combining theorem 1 and theorem 2 we obtain :

THEOREM 3 : In a p-group G if every semi neat subgroup K containing H , with pk is pure in G , then K is minimal semi neat in G containing N if and only if N k_n for some n Z^+ and N + r^n K k_{r-1} , Vr .

Proof . Let N K_n and N + r^n K K_{r-1} so by Theorem 2 , N is a.a.d. in k . Then by the theorem 1 , there is no proper neat subgroup in K which contains N , so K is minimal semi neat subgroup containing N .

Cinversely , if k is a minimal semi neat subgroup in G K then ther is no proper neat subgroup in K which contains N . By theorem 1 , we get N K_n for some n Z^+ and N is a.a.d. in K . By using theorem 2 , we obtain .

$$N + r^n K \quad K_{r-1} (r) .$$

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