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On The Class of Factored Arrangements

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Abstract

The first main objective of the work was to create a combinatorial answer to an essential question; "How Terao generalization of the class of supersolvable arrangements preserved the tensor factorization of the O-S algebra?", by finding a relation among several bases of the O-S algebra. This was achieved in two parts. First, the class of factored arrangements was classified in two subclasses, the subclass of completely factored arrangements and the subclass of factored arrangement that not completely factored. Our classification criteria was, "the existence of an ordering $\trianglelefteq$ on the hyperplanes of a factored arrangement $\mathcal{A}$ such that the set of all monomials that related to the sections of a factorization π on $\mathcal{A}$ forms an NBC basis of the O-S algebra as a free module". The second part was, a comparison among the structures of the O-S complex, the NBC complex and the partition complex. In spite of, our classification criteria was failed of the second subclass of factored arrangement that not completely factored, the existence of a one to one correspondence between the set of all NBC bases of $\mathcal{A}$ and the set of all sections of a factorization π on $\mathcal{A}$, provides a tensor factorization fashion to the O-S algebra.

The second main aim of the work was to prove that our classification is compatible with the product construction for arrangements, by constructing the O-S complex, the NBC complex and the partition complex of the reducible factored arrangements. Finally, several illustrations and applications were indicated.

Keywords: Hyperplane arrangement, Supersolvable arrangement, factored (Nice) arrangement, Orlik-Solomon algebra, NBC module and Partition tensor module.

Introduction:

A hyperplane arrangement $\mathcal{A}$ is defined to be a finite collection of hyperplanes of a finite dimensional vector space V over a field $F = \mathbb{R}$ or $\mathbb{C}$. The field of hyperplane arrangements becomes increasingly popular during the previous century, from the time when its applications used in numerous areas, including Geometry, robotics, graphics, molecular biology, computer vision. The best general reference here is [8].

The chomological group of the complement $M(\mathcal{A}) = V \setminus \bigcup_{H \in A} H$ of an arrangement $\mathcal{A}$ in a complex space had been studied by E. Fadell, R. Fox and L. Neuwirth (1962, [5,6]), in a connection of the Braid arrangement. They gave a presentation of the cohomological ring of the complement $M(\mathcal{A})$ as generators and relations. In (1973, [4]), E. Brieskorn replaced the Braid arrangement by a Coxeter arrangement and the complexification of its reflection arrangement in order to generalize the previous work and give a presentation to the cohomological ring $H^*(M(\mathcal{A}))$ of the complement $M(\mathcal{A})$. Orlik and Solomon in (1980, [9]) generalized Brieskorn results to construct a graded algebra $A(\mathcal{A})$ (that named by their names and for simplicity denoted by O-S algebra) associated to any complex arrangement $\mathcal{A}$ and their description involves the intersection lattice $L(\mathcal{A}) = \{X \big| X = \bigcap_{H \in B} H \text{ and } B \subseteq \mathcal{A}\}$ of $\mathcal{A}$. They proved that, $A(\mathcal{A})$ is isomorphic to the cohomological ring $H^*(M(\mathcal{A}))$ of the complement $M(\mathcal{A})$.

In (1984, [10]), Orlik, Solomon and Terao showed that a supersolvable (Stanely (1972, [13])) arrangement $\mathcal{A}$ admits a partition π which gives rise to a tensor factorization of O-S algebra $A(\mathcal{A})$. Björner and Ziegler in (1991, [3]), gave a sufficient condition for such factorizations of $A(\mathcal{A})$. In (1992, [14]), Terao was able to capture this tensor factorization property of $A(\mathcal{A})$ purely combinatorially in terms of the underlying partition π to introduce firstly the class of factored arrangements as a generalization of Stanely class of supersolvable arrangements. In section (1), we review some of the standard facts about the Terao class of factored arrangements.

Terao generalization turn raises several questions and conjectures, one of them is: Why does Terao generalization still reserve the tensor factorization of the O-S algebra? In order to create a conjecture to answer this question, we study the notion of "quadratic arrangement". In (2001, [11]), K. J. Pearson showed that: an arrangement $\mathcal{A}$ is supersolvable if, and only if, $\mathcal{A}$ is quadratic via an order $\trianglelefteq$ defined on its hyperplanes. Ali in (2014, [1]) gave a sufficient and necessary condition on the structure of O-S algebra $A(\mathcal{A})$ that induced to a central arrangement, a structure as a supersolvable arrangement. Due their work, section (2) includes a classification to the class of factored arrangements into two subclasses. Our classification criteria is, the existence of an ordering $\trianglelefteq$ on the hyperplanes of a factored arrangement $\mathcal{A}$ such that the NBC monomial basis of the O-S algebra $A(\mathcal{A})$ as free module, is the same basis that provides a tensor factorization fashion to $A(\mathcal{A})$. First, we introduce the subclass of factored arrangements that included arrangements satisfied our criteria, called completely factored arrangements. The second subclass consists factored arrangements fails to satisfied our classification criteria. We showed, how the NBC monomial basis of the O-S algebra $A(\mathcal{A})$, exhibits the tensor factorization fashion to $A(\mathcal{A})$ for a factored arrangement $\mathcal{A}$ that not completely factored.

Hoge and Röhrle in (2014, [7]), showed that the notion of "factored arrangement" is compatible with the product construction for arrangements. Section (3), is motivated to prove that our classification is compatible with the product construction. This was realized by a comparing among the O-S complex, the NBC complex and the partition complex of the product of factored arrangement by using our classification. Section (4), indicates some illustrations to ensure our work.

1. Basic facts

This section goal is to introduce Terao class of arrangements (1992, [14]) with some basics of the structure of O-S algebra, NBC module and Partition tensor module with their complexes. The following assumptions will be needed throughout the Paper; Assume $\mathcal{A}$ be an arrangement of a finite dimensional vector space V over a field $F=\mathbb{R}$ or $\mathbb{C}$ and let $L=L(\mathcal{A})=\{X|X=\bigcap_{H\in B}H \text{ and } B\subseteq\mathcal{A}\}$ be its intersection lattice that ordered by reverse the inclusion, (i.e. $X\leq Y \Leftrightarrow Y\subseteq X$, for $X,Y\in L(\mathcal{A})$), and ranked by $rk(X)=codim(X)=\dim(V)-\dim(X)$, for $X\in L(\mathcal{A})$. When we want to emphasize $rk(\mathcal{A})=rk(\bigcap_{H\in\mathcal{A}}H)=\ell$, we will write $\mathcal{A}$ be an ℓ-arrangement.

1.1. Definition: [8]

Let $\pi=(\pi_1,\dots,\pi_\ell)$ be a partition of an ℓ-arrangement $\mathcal{A}$.

1. A **section** S of π is a subarrangement of $\mathcal{A}$ satisfied for each $1\leq k\leq\ell$, either $S\cap\pi_k$ is empty or a singleton. By $S(\pi)$ we denote the set of all sections of π and the set $S_k(\pi)$ denotes the set of all sections S of π with $|S|=k$, we call such sections of π, k-sections of π. We will agree that the empty section $\emptyset_\ell$ is a 0-sections of π.
2. The integer ℓ is called the **length** of π and denoted by $\ell(\pi)$.
3. $rk(\pi_k)=rk\left(\bigcap_{H\in\pi_1\cup\dots\cup\pi_k}H\right)$.
4. π is called **independent** if for every choice of hyperplanes $H_k\in\pi_k$ for $1\leq k\leq\ell$, the resulting ℓ hyperplanes are independent, i.e. $rk(H_1\cap\dots\cap H_\ell)=\ell$.
5. Let $X\in L$. Let $\pi=(\pi_1,\dots,\pi_\ell)$ be a partition of $\mathcal{A}$. Then the **induced partition** π_X is a partition of $\mathcal{A}_X$, its blocks are the nonempty subsets $\pi_k\cap\mathcal{A}_X$, $1\leq k\leq\ell$.
6. π is called a **factorization** of $\mathcal{A}$ or **nice**, if π is independent and if $X\in L\backslash\{V\}$ then the induced partition π_X contains a block, which is a singleton.

7. $\mathcal{A}$ is called **factored** (or **nice**) arrangement if, it has a factorization $\pi = (\pi_1, \dots, \pi_\ell)$. The vector of integers $d = (d_1, .., d_\ell)$ is said to be the exponent vector of $\mathcal{A}$, if $d_k = |\pi_k|$, $1 \le k \le \ell$.

It is clear that, $S(\pi) = \cup_{k=0}^{\ell} S_k(\pi)$ and in general, $rk(\pi_\ell) = rk(\mathcal{A}) \le \ell(\pi)$. If π is independent, then every section S of π is independent subarrangement of $\mathcal{A}$, i.e. $|S| = rk(S) = k$ and $\ell(\pi) = rk(\mathcal{A})$.

1.2. Definition: [1]

1. A subarrangement C of $\mathcal{A}$ is said to be a **circuit**, if it is a minimal dependent subarrangement of $\mathcal{A}$, i.e. $C\backslash\{H\}$ is linearly independent, for any $H \in C$, i.e. $rk(C) = |C| - 1$.
2. Via a total ordering $\trianglelefteq$ on the hyperplanes of $\mathcal{A}$, the corresponding **broken circuit** of a circuit C is $\bar{C} = C\backslash\{H\}$, where H is the smallest hyperplane in C. If $|\bar{C}| = k$, then $\bar{C}$ is said to be ***k*-broken circuit**. The set of all k-broken circuits of $\mathcal{A}$ will be denoted by $BC_k(\mathcal{A})$ and $BC(\mathcal{A}) = \cup_{k=2}^{\ell} BC_k(\mathcal{A})$.
3. We call $B \subseteq \mathcal{A}$, an ***NBC* base** of $\mathcal{A}$, if it contains no broken circuit. Note that, such a set must be independent and we will write ***k-NBC* base** for B if $|B| = k$ and we will agree that $\emptyset^\ell$ is the 0-NBC of $\mathcal{A}$. By $NBC_k(\mathcal{A})$ we denote the set of all k-NBC bases of $\mathcal{A}$ and $NBC(\mathcal{A}) = \cup_{k=0}^{\ell} NBC_k(\mathcal{A})$.
4. If $X \in L(\mathcal{A})$. Then the *NBC* base $B \subseteq \mathcal{A}_X$, (i.e. $\cap_{H \in B} H = X$) is said to be an NBC base of X.
5. If $\mathcal{A}$ is a factored arrangement with a factorization π. Due a total ordering $\trianglelefteq$ on the hyperplanes of $\mathcal{A}$, define, $p_{\trianglelefteq}(\mathcal{A}) = Max\{k |\, NBC_k(\mathcal{A}) = S_k(\pi)\}$. We remarked that, $1 \le p_{\trianglelefteq}(\mathcal{A}) \le \ell$.

1.3. Remark: [8]

Let $\mathcal{A}$ be a factored ℓ-arrangement. Then:

1. If $d = (d_1, .., d_\ell)$ be the exponent vector of π, it is known that;

$$P(\mathcal{A}, t) = \prod_{k=1}^{\ell}(1 + d_k t) = 1 + (d_1 + .. + d_\ell)t + \left(\sum_{i_1=1}^{\ell-1} \sum_{i_2=i_1+1}^{\ell} d_{i_1} d_{i_2}\right)t^2 + \dots + d_1 \dots d_\ell\, t^\ell.$$

2. Independent of our choice of an ordering $\trianglelefteq$ on the hyperplanes of $\mathcal{A}$, Rota in ([12], 1964) proved that, the k^{th} Betti number of the Poincare polynomial $P(\mathcal{A}, t) = b_k(\mathcal{A}) = |NBC_k(\mathcal{A})|$. Accordingly, we have, $b_k(\mathcal{A}) = |NBC_k(\mathcal{A})| = \sum_{i_1=1}^{\ell-k} \sum_{i_2=i_1+1}^{\ell-k+1} \dots \sum_{i_k=i_{k-1}+1}^{\ell} d_{i_1} d_{i_2} \dots d_{i_k}, 1 \le k \le \ell$
3. For $1 \le k \le \ell$ the number of k- sections of π;

$$|S_k(\pi)| = \sum_{i_1=1}^{\ell-k} \sum_{i_2=i_1+1}^{\ell-k+1} \dots \sum_{i_k=i_{k-1}+1}^{\ell} d_{i_1} d_{i_2} \dots d_{i_k}.$$

That is, $|NBC_k(\mathcal{A})| = |S_k(\pi)|$.

1.4. Definition: [11]

Let $\mathcal{A}$ be an arrangement and $\trianglelefteq$ be an order defined on its hyperplanes. $\mathcal{A}$ is said to be quadratic via $\trianglelefteq$ if, and only if, every broken circuit of $\mathcal{A}$ contains a broken circuit of order 2, i.e. if $B \in BC_k(\mathcal{A})$, $k > 2$, then there exists $B' \in BC_2(\mathcal{A})$ such that $B' \subseteq B$.

1.5. Theorem: [1]

Let $\mathcal{A}$ be a central ℓ-arrangement. $\mathcal{A}$ is supersolvable if, and only if, there exists an ordering $\trianglelefteq$ on the hyperplanes of $\mathcal{A}$ such that every subarrangement of $\mathcal{A}$ which contains no 2-broken circuit from an NBC-base of $\mathcal{A}$.

1.6. Corollary: [11]

A central ℓ-arrangement $\mathcal{A}$ is supersolvable if, and only if, it is quadratic via an ordering $\trianglelefteq$.

1.7. Definition: [8]

Let K be any commutative ring and Let $\trianglelefteq$ be an arbitrary total order that defined on the hyperplanes of an ℓ-arrangement $\mathcal{A}$. The Orlik-Solomon algebra (or for simplicity O-S algebra) $A_*(\mathcal{A})$ is defined to be the quotient of the exterior K-algebra $E_* = \wedge_{k \ge 0}(\oplus_{H \in \mathcal{A}} K e_H)$, by the homogeneous ideal $I_*(\mathcal{A})$ is generated by the relations, $\sum_{j=1}^{k}(-1)^{k-1} e_{H_{i_1}} \dots \widehat{e_{H_{i_j}}} \dots e_{H_{i_k}}$, for all $1 \le i_1 < \cdots < i_k \le n$ such that $\{H_{i_1}, \dots H_{i_k}\}$ is dependent subarrangement of $\mathcal{A}$,i.e. $(rk(H_{i_1}, \dots H_{i_k}) < k)$ and the circumflex ^ means $e_{H_{i_j}}$ is deleted. Define a K-linear mapping $\partial_*^E : E_* \to E_*$ as; $\partial_0^E(e_{\emptyset_\ell}) = 0$, $\partial_1^E(e_H) = 1$, for all $H \in \mathcal{A}$ and for $2 \le k \le \ell$, $\partial_k^E(e_C) = \sum_{j=1}^{k}(-1)^{k-1} e_{H_{i_1}} \dots \widehat{e_{H_{i_j}}} \dots e_{H_{i_k}}$, $C = \{H_{i_1}, \dots H_{i_k}\}$. ∂_*^E is a differentiation on E_* and the chain complex (E_*, ∂_*^E): $\cdots \xrightarrow{\partial_{k+1}^E} E_k \xrightarrow{\partial_k^E} E_{k-1} \xrightarrow{\partial_{k-1}^E} \cdots \xrightarrow{\partial_2^E} E_1 \xrightarrow{\partial_1^E} E_0 \xrightarrow{\partial_0^E} 0$, is called the exterior complex;

1.8. Theorem: [8]

The complex $(\boldsymbol{A}_*(\mathcal{A}), \partial_*^A)$ inherits a structure as acyclic chain complex from the exterior complex (E_*, ∂_*^E), where $\partial_*^A = \psi_* \circ \partial_*^E$ and $\psi_* : E_* \to \boldsymbol{A}_*(\mathcal{A})$ is the canonical chain map. The acyclic chain complex $(\boldsymbol{A}_*(\mathcal{A}), \partial_*^A)$ is called the O-S complex.

$$\begin{array}{ccccccccc} \cdots \xrightarrow{\partial_{\ell+1}^E} & E_\ell & \xrightarrow{\partial_\ell^E} & E_{\ell-1} & \xrightarrow{\partial_{\ell-1}^E} \cdots \xrightarrow{\partial_2^E} & E_1 & \xrightarrow{\partial_1^E} & E_0 & \xrightarrow{\partial_0^E} 0 \\ & \psi_\ell \downarrow & & \psi_{\ell-1} \downarrow & & \psi_1 \downarrow & & \psi_0 \downarrow & \\ 0 \to & \boldsymbol{A}_\ell(\mathcal{A}) & \xrightarrow{\partial_\ell^A} & \boldsymbol{A}_{\ell-1}(\mathcal{A}) & \xrightarrow{\partial_{\ell-1}^A} \cdots \xrightarrow{\partial_2^A} & \boldsymbol{A}_1(\mathcal{A}) & \xrightarrow{\partial_1^A} & \boldsymbol{A}_0(\mathcal{A}) & \xrightarrow{\partial_0^A} 0 \end{array}$$

1.9. Definition: [8]

Let K be any commutative ring. The broken circuit module $\boldsymbol{NBC}_*(\mathcal{A})$ of the exterior K-algebra $E_* = \wedge_{k \ge 0}(\bigoplus_{H \in \mathcal{A}} Ke_H)$, is defined as; $\boldsymbol{NBC}_0(\mathcal{A}) = K$ and for $1 \le k \le \ell$, $\boldsymbol{NBC}_k(\mathcal{A})$ be the free K-module of E_k with NBC (no broken circuit) monomials basis $\{e_C | C \in NBC_k(\mathcal{A})\} \subseteq E_k$, i.e.;

$$\boldsymbol{NBC}_k(\mathcal{A}) = \bigoplus_{C \in NBC_k(\mathcal{A})} Ke_C \text{ and } \boldsymbol{NBC}_*(\mathcal{A}) = \bigoplus_{k=0}^{\ell} \boldsymbol{NBC}_k(\mathcal{A}).$$

1.10. Theorem: [8]

The broken circuit subcomplex $(\boldsymbol{NBC}_*(\mathcal{A}), \partial_*^{NBC})$ inherits a structure as acyclic chain complex from the exterior complex (E_*, ∂_*^E), where $\partial_*^{NBC} = \partial_*^E \circ i_*$ and $i_* : E_* \to \boldsymbol{NBC}_*(\mathcal{A})$ is the inclusion chain map.

$$\begin{array}{ccccccccc} 0 \to & \boldsymbol{NBC}_\ell(\mathcal{A}) & \xrightarrow{\partial_\ell^{NBC}} & \boldsymbol{NBC}_{\ell-1}(\mathcal{A}) & \xrightarrow{\partial_{\ell-1}^{NBC}} \cdots \xrightarrow{\partial_2^{NBC}} & \boldsymbol{NBC}_1(\mathcal{A}) & \xrightarrow{\partial_1^{NBC}} & \boldsymbol{NBC}_0(\mathcal{A}) & \xrightarrow{\partial_0^{NBC}} 0 \\ & i_\ell \downarrow & & i_{\ell-1} \downarrow & & i_1 \downarrow & & i_0 \downarrow & \\ \xrightarrow{\partial_{\ell+1}^E} & E_\ell & \xrightarrow{\partial_\ell^E} & E_{\ell-1} & \xrightarrow{\partial_{\ell-1}^E} \cdots \xrightarrow{\partial_2^E} & E_1 & \xrightarrow{\partial_1^E} & E_0 & \xrightarrow{\partial_0^E} 0 \end{array}$$

Moreover, the restriction of the canonical chain map $\psi_* : E_* \to \boldsymbol{A}_*(\mathcal{A})$ of the broken circuit module $\boldsymbol{NBC}_*(\mathcal{A})$, is a chain isomorphism, defined as; for $1 \le k \le \ell$, $\psi_k(e_C) = e_C + I_k(\mathcal{A}) = a_C$, $C \in NBC_k(\mathcal{A})$.

$$\begin{array}{ccccccccc} 0 \to & \boldsymbol{NBC}_\ell(\mathcal{A}) & \xrightarrow{\partial_\ell^{NBC}} & \boldsymbol{NBC}_{\ell-1}(\mathcal{A}) & \xrightarrow{\partial_{\ell-1}^{NBC}} \cdots \xrightarrow{\partial_2^{NBC}} & \boldsymbol{NBC}_1(\mathcal{A}) & \xrightarrow{\partial_1^{NBC}} & \boldsymbol{NBC}_0(\mathcal{A}) & \xrightarrow{\partial_0^{NBC}} 0 \\ & \psi_\ell \downarrow & & \psi_{\ell-1} \downarrow & & \psi_1 \downarrow & & \psi_0 \downarrow & \\ 0 \to & \boldsymbol{A}_\ell(\mathcal{A}) & \xrightarrow{\partial_\ell^A} & \boldsymbol{A}_{\ell-1}(\mathcal{A}) & \xrightarrow{\partial_{\ell-1}^A} \cdots \xrightarrow{\partial_2^A} & \boldsymbol{A}_1(\mathcal{A}) & \xrightarrow{\partial_1^A} & \boldsymbol{A}_0(\mathcal{A}) & \xrightarrow{\partial_0^A} 0 \end{array}$$

Thus the O-S algebra has the following structure as a free K-module: $\boldsymbol{A}_*(\mathcal{A}) = \bigoplus_{k=0}^{\ell} (\bigoplus_{C \in NBC_k(\mathcal{A})} Ka_C)$.

1.11. Definition: [8]

Let $\pi = (\pi_1, \ldots, \pi_\ell)$ be a partition on an ℓ-arrangement $\mathcal{A}$ and let K be any commutative ring. A partition K-module is defined to be $(\pi)_* = (\pi_1)_* \otimes \ldots \otimes (\pi_\ell)_*$, where for $1 \leq k \leq \ell$, $(\pi_k)_*$ is the free K-module with basis 1 and the elements of π_k. For each $B = \{H_{i_1}, \ldots H_{i_k}\} \in S_k(\pi)$, i.e. $H_{i_m} \in \pi_{i_m}$, $1 \leq i_1 < \cdots < i_k \leq \ell$ and $1 \leq m \leq k$, define;$q_B = x_1 \otimes \ldots \otimes x_\ell \in (\pi)_*$ as;

$$x_j = \begin{cases} H_j & if\ j = i_m \text{ for some } 1 \leq m \leq k \\ 1 & if\ j \neq i_m \text{ for all } 1 \leq m \leq k \end{cases}$$

We agree that each of $q_{\emptyset_\ell} = 1 \otimes \ldots \otimes 1$ and q_B is homogeneous of degree k. We denoting the k^{th} -homogeneous part of $(\pi)_*$ by $(\pi)_k$. Therefore, $(\pi)_* = \bigoplus_{k=0}^{\ell} (\pi)_k = \bigoplus_{k=0}^{\ell} (\bigoplus_{B \in S_k(\pi)} K q_B)$ and $\{q_B | B \in S_k(\pi)\}$ forms a basis to the free K-module $(\pi)_*$. Furthermore, $\{q_{\{H\}} | H \in \pi_k\}$ forms a basis to the free K-module $(\pi_k)_*$, $1 \leq k \leq \ell$. Define a K-linear mapping $\partial_*^\pi : (\pi)_* \to (\pi)_*$ as; $\partial_0^\pi(q_{\{\,\}}) = 0$, $\partial_1^\pi(q_H) = 1$, for all $H \in \mathcal{A}$ and for $2 \leq k \leq \ell$, $\partial_k^\pi(q_B) = \sum_{j=1}^{k} (-1)^{k-1} \widehat{q_B}_j$, where $B = \{H_{i_1}, \ldots H_{i_k}\} \in S_k(\pi)$, $q_B = x_1 \otimes \ldots \otimes x_\ell$ as given in (1.8), and $\widehat{q_B}_j = x_1 \otimes \ldots \otimes \widehat{H_{\iota_j}} \otimes \ldots \otimes x_\ell$ by means of $\widehat{H_{\iota_j}} = 1$. ∂_*^π is a differentiation on $(\pi)_*$ and the chain complex $((\pi)_*, \partial_*^\pi)$ is called the partition complex;

$$0 \to (\pi)_\ell \xrightarrow{\partial_\ell^\pi} (\pi)_{\ell-1} \xrightarrow{\partial_{\ell-1}^\pi} \cdots \xrightarrow{\partial_2^\pi} (\pi)_1 \xrightarrow{\partial_1^\pi} (\pi)_0 \xrightarrow{\partial_0^\pi} 0.$$

1.12. Definition: [8]

For $1 \leq k \leq \ell$, define the a map $\tilde{\varphi}_k : \{q_B | B \in S_k(\pi)\} \to \boldsymbol{A}_*(\mathcal{A})$, as $\varphi_k(q_B) = a_B = e_B + I_k(\mathcal{A})$, $B \in S_k(\pi)$. Let $\varphi_k : (\pi)_k \to \boldsymbol{A}_k(\mathcal{A})$ be the unique K-linear map that extend this assignment as follows:

$$\begin{array}{ccc} \{q_B | B \in S_k(\pi)\} & \xrightarrow{\tilde{\varphi}_k} & \boldsymbol{A}_k(\mathcal{A}) \\ i_k \quad \searrow & & \downarrow \exists!\, \varphi_k \\ & (\pi)_k & \end{array}$$

Accordingly, there is a unique K-chain mapping $\varphi_* : (\pi)_* \to \boldsymbol{A}_*(\mathcal{A})$ between acyclic chain complexes as showed in the following diagram;

$$\begin{array}{ccccccccc} 0 \to (\pi)_\ell & \xrightarrow{\partial_\ell^\pi} & (\pi)_{\ell-1} & \xrightarrow{\partial_{\ell-1}^\pi} \cdots \xrightarrow{\partial_2^\pi} & (\pi)_1 & \xrightarrow{\partial_1^\pi} & (\pi)_0 & \xrightarrow{\partial_0^\pi} 0 \\ \varphi_\ell \downarrow & & \varphi_{\ell-1} \downarrow & & \varphi_1 \downarrow & & \varphi_0 \downarrow & \\ 0 \to \boldsymbol{A}_\ell(\mathcal{A}) & \xrightarrow{\partial_\ell^A} & \boldsymbol{A}_{\ell-1}(\mathcal{A}) & \xrightarrow{\partial_{\ell-1}^A} \cdots \xrightarrow{\partial_2^A} & \boldsymbol{A}_1(\mathcal{A}) & \xrightarrow{\partial_1^A} & \boldsymbol{A}_0(\mathcal{A}) & \xrightarrow{\partial_0^A} 0 \end{array}$$

1.13. Theorem: [8]

The chain map $\varphi_* : (\pi)_* \to \boldsymbol{A}_*(\mathcal{A})$ is a K-isomorphism between chain complexes if and only if the partition π is a factorization of $\mathcal{A}$.

1.14. Remark:

The theorems (1.8.), (1.10.) and (1.13), afford a K-isomorphism, $\chi_* = \psi_*^{-1} \circ \varphi_*: (\pi)_* \to \boldsymbol{NBC}_*(\mathcal{A})$ between the partition complex and broken circuit complex as shown in the following diagram;

$$
\begin{array}{ccccccccccc}
0 \to & (\pi)_\ell & \xrightarrow{\partial_\ell^\pi} & (\pi)_{\ell-1} & \xrightarrow{\partial_{\ell-1}^\pi} \cdots \xrightarrow{\partial_2^\pi} & (\pi)_1 & \xrightarrow{\partial_1^\pi} & (\pi)_0 & \xrightarrow{\partial_0^\pi} 0 \\
& \chi_\ell \downarrow & & \chi_{\ell-1} \downarrow & & \chi_1 \downarrow & & \chi_0 \downarrow & \\
0 \to & \boldsymbol{NBC}_\ell(\mathcal{A}) & \xrightarrow{\partial_\ell^{NBC}} & \boldsymbol{NBC}_{\ell-1}(\mathcal{A}) & \xrightarrow{\partial_{\ell-1}^{NBC}} \cdots \xrightarrow{\partial_2^{NBC}} & \boldsymbol{NBC}_1(\mathcal{A}) & \xrightarrow{\partial_1^{NBC}} & \boldsymbol{NBC}_0(\mathcal{A}) & \xrightarrow{\partial_0^{NBC}} 0
\end{array}
$$

2. A Classification Of The Class Of Factored Arrangement:

This section is devoted to classify Terao class of arrangement ([13], 1992) into two subclasses, in order to study the K-isomorphism, $\chi_*: (\pi)_* \to \boldsymbol{NBC}_*(\mathcal{A})$ technique (as remarked in (1.14.)), that joining the NBC monomial basis and the section monomial basis of the O-S algebra, by a one to one correspondence. We will show that, each one of the subclasses has a different technique to produce several fashions to the O-S algebra. So, we will start with another way to define the class of supersolvable arrangements:

2.1. Construction:

Let $\mathcal{A}$ be a factored ℓ-arrangement with a factorization $\pi = (\pi^1, \dots, \pi^\ell)$ and exponent vector $d = (d^1, \dots, d^\ell)$. Assume that, there exists an ordering $\trianglelefteq$ defined on the hyperplanes of $\mathcal{A}$ such that $NBC_k(\mathcal{A}) = S_k(\pi)$, for $0 \le m < k \le \ell$, i.e. suppose $p_\trianglelefteq(\mathcal{A}) = \ell$. In order to emphasis such property of the class of factored arrangements, we will call such arrangement, a **completely factored arrangement via** $\trianglelefteq$. We will reorder the blocks of π as follows:

i. Put π_1 be the block of π that contains the minimal hyperplane H_1 of $\mathcal{A}$. According to our assumption we have that, every ℓ-section of π contains H_1 is an $\ell - NBC$ base of $\mathcal{A}$. As well as, if $H \in \pi_1$ be another hyperplane of π_1, then every ℓ-section S of π that contains H forms an $\ell - NBC$ base of $\mathcal{A}$. But that contradicts that $\{H_1\} \cup S$ is a circuit of $\mathcal{A}$, (since $H_1 \trianglelefteq H$, $rk(\{H_1\} \cup S) = \ell$ and $|\{H_1\} \cup S| = \ell + 1$). Therefore, the block π_1 must be a singleton which contains the minimal hyperplane H_1 of $\mathcal{A}$ via $\trianglelefteq$, i.e. $\pi_1 = \{H_1\}$.

ii. Actually, for each $H \in \pi^k$, $1 \le k \le \ell$ and $\pi^k \ne \pi_1$, we have $rk\,(\pi_1 \cup \{H\}) = 2$. Accordingly, we can choose π_2 to be the block of π that contains the second hyperplane H_2 via $\trianglelefteq$. The important point to note here that $rk(\pi_1 \cup \pi_2) = 2$. To explain that, assume $|\pi_2| > 1$. Then for each $H \in \pi_2$, we have $\{H_2, H\} \notin S_2(\pi)$. Form our assumption $\{H_2, H\}$ is a broken circuit. Since H_2 is the second hyperplane of $\mathcal{A}$ via $\trianglelefteq$, hence $\{H_1, H_2, H\}$ is a circuit of $\mathcal{A}$ related to $\{H_2, H\}$, i.e. $rk\,\{H_1, H_2, H\} = 2$, for each $H \in \pi_2 \setminus \{H_2\}$. Therefore, $rk(\pi_1 \cup \pi_2) = 2$. Indeed, if $|\pi_2| = d_2$, then π_2 will contain the second hyperplane, the third hyperplane and the $(d_k + 1)^{th}$ hyperplane of $\mathcal{A}$ via $\trianglelefteq$.

By continuing the above process we will induce a factorization $\pi = (\pi_1, \dots, \pi_\ell)$ satisfied; If $H \in \pi_m$ and $H' \in \pi_k$ for some $1 \le m < k \le \ell$, then $H \trianglelefteq H'$.

The following results highlight the properties of π:

2.2. Lemma:

Let $\mathcal{A}$ be a factored ℓ-arrangement with a factorization $\pi = (\pi_1, \dots, \pi_\ell)$. For $X \in L(\mathcal{A})$, if $B = \mathcal{A}_X = \{H \in \mathcal{A} | X \subseteq H\} \subseteq \mathcal{A}$ such that $rk(B) = 2$, then there exists $1 \leq m < k \leq \ell$ such that $B \subseteq \pi_m \cup \pi_k$ and either $|\pi_m \cap B| = 1$ or $|\pi_k \cap B| = 1$.

Proof: We need to prove the following:

- ***i.*** There exists $1 \leq m < k \leq \ell$ such that, $B \subseteq \pi_m \cup \pi_k$
- ***ii.*** Either $|\pi_m \cap B| = 1 \; or \; |\pi_k \cap B| = 1$

For i: By contrary assume, either (there is $1 \leq m \leq \ell$ with $B \subseteq \pi_m$) or (there is $1 \leq m < k < n \leq \ell$ such that $B \subseteq \pi_m \cup \pi_k \cup \pi_n$).
In fact, $rk\, B = 2$, so $|B| > 1$ and from definition (1.1) item 6, the hyperplanes of B must be distributed between at most two different blocks of π and cannot be contained in just one block. Therefore, our first assumption above contradicts this fact.
Secondly, if there are $1 \leq \mathrm{m} < k < n < \ell$ such that $\mathrm{B} \subseteq \pi_m \cup \pi_k \cup \pi_n$, then $B \cap \pi_i \neq \varphi$ for $i = m, k, n$. If $H \in \pi_m, H' \in \pi_k$ and $H'' \in \pi_n$, such that $H, H', H'' \in B$, then $rk\,\{H, H', H''\} = rk\,(B) = 2$. This is a contradiction, since $\{H, H', H''\}$ is a section of an independent partition π, i.e. $rk\,\{H, H', H''\} = 3$. Thus, B cannot be distributed among three blocks (or more) of π. So, B must be distributed between just two blocks of π and our claim is hold.
For ii: It is a direct result of definition (1.1) item 6. ∎

2.3. Corollary:

Let $\mathcal{A}$ be a completely factored ℓ-arrangement and $\pi = (\pi_1, \dots, \pi_\ell)$ be its induced factorization due construction (2.1.). For $X \in L(\mathcal{A})$, if $B = \mathcal{A}_X$ such that $rk(B) = 2$, then there exists $1 \leq m < k \leq \ell$ such that $B \subseteq \pi_m \cup \pi_k$ and $|B \cap \pi_m| = 1$.

Proof: By applying Lemma (2.2.), there exists $1 \leq m < k \leq \ell$ such that $B \subseteq \pi_m \cup \pi_k$ and either $|B \cap \pi_m| = 1$ or $|B \cap \pi_k| = 1$. By contrary assume $|B \cap \pi_m| > 1$, $|B \cap \pi_k| = 1$ and $B \cap \pi_k = \{H\}$. Now, let $H', H'' \in B \cap \pi_m$ satisfied $H' \trianglelefteq H''$. Then $rk\,\{H', H''. H\} = rk\,\{B\} = 2$. That is, $\{H'', H\}$ is a 2-broken circuit of $\mathcal{A}$. But $H'' \in \pi_m$ and $H \in \pi_k$, , so $\{H', H\}$ is a 2-section of π and this contradicts our assumption that $NBC_2\,(\mathcal{A}) = S_2\,(\pi)$. Therefore, $|\pi_m \cap B| = 1$. ∎

We can summarize our goal in construction (2.1.) and corollary (2.3.), by the following results:

2.4. Theorem:

Let $\pi = (\pi^1, \dots, \pi^\ell)$ be a factorization of a factored ℓ −arrangement $\mathcal{A}$ that has an ordering $\trianglelefteq$ on its hyperplanes such that $NBC_k(\mathcal{A}) = S_k\,(\pi)$, for all $1 \leq k \leq \ell$. Then, every subarrangement which contains no 2-broken circuit forms an NBC base of $\mathcal{A}$ via $\trianglelefteq$.

Proof: Firstly, reorder the blocks of π as given in construction (2.1.) to obtain the induced factorization $\pi = (\pi_1, \dots, \pi_\ell)$. Secondly, assume B be a subarrangement of $\mathcal{A}$ which contains no 2-broken circuit of $\mathcal{A}$ via $\trianglelefteq$. We need to show that B is an NBC base of $\mathcal{A}$, i.e. we need B is a k-section of π. By contrary, suppose B is a k-section of π, i.e. there exists $2 \leq m \leq \ell$, such that $|\pi_m \cap B| > 1$. Let $H, H' \in \pi_m \cap B$. Definitely, $\{H, H'\}$ is a 2-broken circuit, since $rk\,\{H, H'\} = 2$ and it is not a 2-section. This is a contradiction since our assumption states that B contained no 2-broken circuit. Therefore, every subarrangement which contains no 2-broken circuit from an NBC base of $\mathcal{A}$. ∎

2.5. Corollary:

If $\mathcal{A}$ is a completely factored ℓ −arrangement with a factorization $\pi = (\pi^1, \dots, \pi^\ell)$ and an ordering $\trianglelefteq$ defined on its hyperplanes such that $NBC_k(\mathcal{A}) = S_k(\pi)$, for all $1 \le k \le \ell$, then $\mathcal{A}$ is quadratic via $\trianglelefteq$.

Proof: Let $\pi = (\pi_1, \dots, \pi_\ell)$ be the induced factorization that given in construction (2.1.). Suppose $B \in BC_k(\mathcal{A})$, $k > 2$. Wanted, $B' \in BC_2(\mathcal{A})$ such that $B' \subseteq B$. By contrary, assume there is no $B' \in BC_2(\mathcal{A})$ such that $B' \subseteq B$; i.e., B contains no 2-broken circuit via $\trianglelefteq$. According to the theorem (2.4.), $B \in NBC_k(\mathcal{A})$ and this is a contradiction. Therefore, then $\mathcal{A}$ is quadratic. ■

The following result is motivated to classify the class of completely factored arrangements into two subclasses by the structure of its NBC bases:

2.6. Corollary:

A factored ℓ −arrangement $\mathcal{A}$ is completely factored arrangement via an ordering $\trianglelefteq$ if, and only if, $\mathcal{A}$ is supersolvable.

Proof: This is a direct result to construction (2.1.), theorem (1.5.) and corollary (1.6.). ■

2.7. Construction:

Let $\mathcal{A}$ be a completely factored ℓ −arrangement via an ordering $\trianglelefteq$ and let $\pi = (\pi_1, \dots, \pi_\ell)$ be the induced factorization due construction (2.1.). So, the identity map $I_k: S_k(\pi) \to NBC_k(\mathcal{A})$, define a one to one correspondence between the sections basis of the tensor partition module $(\pi)_k$ and the NBC monomial basis of the O-S algebra as free module;

$$\mathcal{I}_k: \{q_C | C \in S_k(\pi)\} \to \{e_B | B \in NBC_k(\mathcal{A})\};$$

as; $\mathcal{I}_k(q_C) = e_C$, for $C \in S_k(\pi)$. This induces a unique K-linear isomorphism $\mathcal{I}_k: (\pi)_k \to \boldsymbol{NBC}_k(\mathcal{A})$ between the k^{th} partition module and k^{th} broken circuit module that extend this assignment as follows:

$$\begin{array}{ccc} \{q_C | C \in S_k(\pi)\} & \xrightarrow{i_k} & (\pi)_k \\ \mathcal{I}_k \searrow & & \\ \{e_B | B \in NBC_k(\mathcal{A})\} & & \downarrow \exists!\, \mathcal{I}_k \\ & i'_k \searrow & \\ & & \boldsymbol{NBC}_k(\mathcal{A}) \end{array}$$

2.8. Theorem:

Assume we have the conclusions of construction (2.7.). Then $\mathcal{I}_*: (\pi)_* \to \boldsymbol{NBC}_*(\mathcal{A})$ forms a K-chain isomorphism between acyclic chain complexes, achieved by the fact that NBC monomial basis and sections monomial basis are equal.

Proof: For a fixed $1 \le k \le \ell$, we need to show the following diagram is commutative:

$$\begin{array}{ccc} (\pi)_k & \xrightarrow{\partial_k^\pi} & (\pi)_{k-1} \\ \mathcal{I}_k \downarrow & & \downarrow \mathcal{I}_{k-1} \\ \boldsymbol{NBC}_k(\mathcal{A}) & \xrightarrow{\partial_k^{NBC}} & \boldsymbol{NBC}_{k-1}(\mathcal{A}) \end{array}$$

Let $C = \{H_{i_1}, \dots H_{i_k}\} \in S_k(\pi)$. Then;

$\partial_k^{NBC} \circ \mathcal{I}_k(q_C) = \partial_k^{NBC}(\mathcal{I}_k(q_C)) = \partial_k^{NBC}(e_C)$

$$= \sum_{j=1}^{k} (-1)^{k-1} e_{H_{i_1}} \dots \widehat{e_{H_{i_j}}} \dots e_{H_{i_k}} = \sum_{j=1}^{k} (-1)^{k-1} \mathcal{I}_{k-1} \left(q_{H_{i_1}} \dots \widehat{q_{H_{i_j}}} \dots q_{H_{i_k}} \right)$$
$$= \mathcal{I}_{k-1} \left(\sum_{j=1}^{k} (-1)^{k-1} q_{H_{i_1}} \dots \widehat{q_{H_{i_j}}} \dots q_{H_{i_k}} \right) = \mathcal{I}_{k-1} \circ \partial_k^{\pi}(q_C)$$

Thus, $\partial_k^{NBC} \circ \mathcal{I}_k = \mathcal{I}_{k-1} \circ \partial_k^{\pi}$ and our claim is hold. ■

The important point to note here is, not every factored arrangement is completely factored (supersolvable, quadratic) arrangement. We will consider a construction of a factorization of a factored arrangement that not completely factored as follows:

2.9. Construction:

Let $\mathcal{A}$ be a factored ℓ-ararrangement such that there is no factorization $\pi = (\pi_1, \dots, \pi_\ell)$ and no ordering $\trianglelefteq$ can be defined on the hyperplanes of $\mathcal{A}$ satisfied, $NBC_k(\mathcal{A}) = S_k(\pi)$, for all $0 \leq k \leq \ell$. Thus, for any factorization $\pi = (\pi_1, \dots, \pi_\ell)$ and any ordering $\trianglelefteq$ defined on the hyperplanes of $\mathcal{A}$ we have, $NBC(\mathcal{A}) \neq S(\pi)$, i.e. $1 < p_{\trianglelefteq}(\mathcal{A}) + 1 < \ell$. However, for any given factorization $\pi = (\pi_1, \dots, \pi_\ell)$ on $\mathcal{A}$ we can define an ordering $\trianglelefteq$ on the hyperplanes of $\mathcal{A}$ that reserve the structure of π as follows:

i. If $H, H' \in \mathcal{A}$ such that $H \in \pi_m$ and $H' \in \pi_k$ for some $1 \leq m < k \leq \ell$, put $H \trianglelefteq H'$.

ii. For $1 \leq k \leq \ell$, order the hyperplanes of π_k by arbitrary order.

Now, compute $NBC(\mathcal{A})$ via $\trianglelefteq$. Under our assumption $NBC(\mathcal{A}) \neq S(\pi)$, i.e. there is a k-section of π but it is not a k-NBC base of $\mathcal{A}$ via $\trianglelefteq$ and there is a k-NBC base of $\mathcal{A}$ via $\trianglelefteq$ that it is not a k-section of π for $p_{\trianglelefteq}(\mathcal{A}) + 1 \leq k \leq \ell$. We can emphasis the properties of π by the following lemmas:

2.10. Lemma:

Let $\mathcal{A}$ be a factored ℓ-ararrangement and $\pi = (\pi_1, \dots, \pi_\ell)$ be its factorization. Then, there is $1 \leq k \leq \ell$ such that $\pi_k \cap \mathcal{A}_T = \pi_k \cap \mathcal{A}$ is a singleton, where $T = \bigcap_{H \in \mathcal{A}} H \in L(\mathcal{A})$.

Proof: This is a direct result to definition (1.1), item 6. ■

2.11. Lemma:

Let $\mathcal{A}$ be a factored ℓ-arrangement and its factorization is $\pi = (\pi_1, \dots, \pi_\ell)$. If $X \in L_k(\mathcal{A})$, for some $1 \leq k \leq \ell$, then π admits the induced partition π_X a structure as a factorization of $\mathcal{A}_X$ and $\ell(\pi_X) = k$.

Proof: According to definition (1.1.), π_X is independent and for each $X' \in L(\mathcal{A}_X)$, there is a block of π_X contains just one hyperplanes from $\mathcal{A}_{X'}$. Therefore, we need to show only $\ell(\pi_X) = k$. We will prove $\ell(\pi_X) = k$, inductively as follows:

For $k = 2$: Recall $\mathcal{A}_X = \{H \in \mathcal{A} | X \subseteq H\} \subseteq \mathcal{A}$. Since $rk(\mathcal{A}_X) = 2$, hence $|\mathcal{A}_X| \geq 2$. According to the properties of π, $\ell(\pi_X) = 2$.

For $k = 3$: By contrary assume that, $\ell(\pi_X) = 2$, (i.e. $\pi_X = (\pi_1^X, \pi_2^X)$), and without loss of generality, assume $|\pi_1^X| = 1$. Thus, for every two hyperplanes $H', H'' \in \mathcal{A}_X$, there is $X' \in L_2(\mathcal{A}_X)$ such that $H', H'' \in \mathcal{A}_{X'}$ and if $H', H'' \in \pi_2^X$, then $H \in \mathcal{A}_{X'}$ and this contradicts that $rk(\mathcal{A}_X) = 3$. That is, under our assumption $|\mathcal{A}_X| > 3$. Suppose $X_1, X_2 \in L_2(\mathcal{A}_X)$. Thus, $\mathcal{A}_{X_1}$ and $\mathcal{A}_{X_2}$ have with π_X the same singleton block $\pi_1^X = \{H\}$. It is clear, $rk\left(\bigcap_{\widetilde{H} \in \mathcal{A}_{X_1} \cup \mathcal{A}_{X_2}} \widetilde{H}\right) = rk(\mathcal{A}_X)$, i.e. $\bigcap_{\widetilde{H} \in \mathcal{A}_{X_1} \cup \mathcal{A}_{X_2}} \widetilde{H} = X_1 \cap X_2 = X$. As well as, if each of $\mathcal{A}_{X_1}$and $\mathcal{A}_{X_2}$ has just two hyperplanes, then $X_1 = X_2$. Thus, one of $\mathcal{A}_{X_1}$or $\mathcal{A}_{X_2}$contains more than two hyperplanes say $H_1, H_2 \in \mathcal{A}_{X_1}$ and let $H_3 \in \mathcal{A}_{X_2}$. Actually, $X_1 = H \cap H_1 \cap H_2$, $X_2 = H \cap H_3$ and;

$$X = H \cap H_1 \cap H_2 \cap H_3 = H_1 \cap H_2 \cap H_3 = H \cap H_1 \cap H_3 = H \cap H_2 \cap H_3.$$

Consequently, $X_3 = H \cap H_1 \cap H_3 \in L_2(\mathcal{A}_X)$, $X_4 = H \cap H_2 \cap H_3 \in L_2(\mathcal{A}_X)$ and this is a contradiction.

Therefore, $\ell(\pi_X) = 3$.

Suppose, our claim is hold **for** $k = s - 1$ and we will prove it **for** $k = s$ as follows:

By contrary assume that, $\ell(\pi_X) = s - 1$, (i.e. $\pi_X = (\pi_1^X, \dots, \pi_{s-1}^X)$) and without loss of generality, assume $|\pi_1^X| = 1$. Since $|\mathcal{A}_X| \geq s$, hence there is a block say π_m^X, of π_X with $|\pi_m^X| \geq 2$, $2 \leq m \leq s-1$ and $\mathcal{A}_X$ contains s independent hyperplanes. Accordingly, we can choose them as; $\{H_1, H_2, \dots, H_m^1, H_m^2, \dots, H_{s-1}\} \subseteq \mathcal{A}_X$ is independent,where $H_i \in \pi_i^X$, $1 \leq i \leq s-1$ and $H_m^1, H_m^2 \in \pi_m^X$. Let $X_1, X_2 \in L_{s-1}(\mathcal{A}_X)$ such that, $X_1 = H_1 \cap H_2 \cap \dots \cap H_m^1 \cap H_m^2 \cap \dots \cap H_{s-2}$ and $X_2 = H_2 \cap \dots \cap H_m^1 \cap H_m^2 \cap \dots \cap H_{s-1}$. It is clear that, $X_1 \cap X_2 = X$ and $H_1 \notin \mathcal{A}_{X_2} \subseteq \mathcal{A}_X$. Thus, the induced partition π_{X_2} has length $\ell(\pi_{X_2}) = s - 2$ and this contradicts induction hypothesis. Therefore, our claim is true. ■

2.12. Lemma:

Suppose we have the assumptions of construction (2.9.). For $p_{\trianglelefteq}(\mathcal{A}) + 1 \leq k \leq \ell$, if B is a k-section of π and it is not $k - NBC$ base of $\mathcal{A}$ such that, $B \subseteq \pi_{i_1} \cup \dots \cup \pi_{i_k}$, $1 \leq i_1 < \dots < i_k \leq \ell$. Then there is a $k - NBC$ base $C \subseteq \pi_{i_1} \cup \dots \cup \pi_{i_k}$ of $\mathcal{A}$ which is not a k-section of π that is satisfied, if $X = \bigcap_{H \in B} H \in L_k(\mathcal{A})$, then C is a $k - NBC$ of $\mathcal{A}_X$.

Proof: We will prove our conjecture inductively as follows:

For $\underline{\boldsymbol{k = p_{\trianglelefteq}(\mathcal{A}) + 1 = 2}}$: If B is a 2-section of π which is not a $2 - NBC$ base of $\mathcal{A}$, then $B = \{H_1, H_2\} \subseteq \pi_m \cup \pi_p$, for some $1 \leq m < p \leq \ell$ and there exists $H \in \mathcal{A}$ be the minimal hyperplane of $\mathcal{A}$ via $\trianglelefteq$ satisfied $H \trianglelefteq H_1$, $H \trianglelefteq H_2$ and $\{H\} \cup B$ is a circuit with B its broken circuit. Clearly, $H \in \pi_m$ and $C = \{H, H_1\} \subseteq \pi_m$ is a $2 - NBC$ of $\mathcal{A}$ that is not a 2-section of π. Moreover, if $X = H_1 \cap H_2 = H \cap H_1 \cap H_2 = H \cap H_2$, then C is a $2 - NBC$ of $\mathcal{A}_X$.

Suppose, our claim is hold ***for*** $k = s - 1$ and we will prove it ***for*** $k = s$ as follows:

If B is an s-section of π which not an $s - NBC$ base of $\mathcal{A}$, then $B \subseteq \pi_{i_1} \cup \dots \cup \pi_{i_s}$, $1 \leq i_1 < \dots < i_s \leq \ell$. Deduce that, either ($B$ contain broken circuits of $\mathcal{A}$ have ranks less than s) or (B is an s-broken circuit of $\mathcal{A}$).

Firstly, if $B_1, \dots, B_t \subseteq B$ be is the broken circuits of $\mathcal{A}$ that have ranks $m_1, \dots, m_t$ respectively, with $m_j < s$, $j = 1, \dots, t$. For a given $1 \leq j \leq t$, B_j is an m_j-section of π and m_j-broken circuit base of $\mathcal{A}$. Let $H^j \in \mathcal{A}$ be the minimal hyperplane of $\mathcal{A}$ via $\trianglelefteq$ satisfied $H^j \trianglelefteq H$, for all $H \in B_j$ and $\{H^j\} \cup B_j$ is a circuit with B_j its broken circuit. If $p = \min\{l \mid \pi_{i_l} \cap B_j \neq \varphi, 1 \leq l \leq s\}$, clearly, $H^j \in \pi_p$ and $C_j = \{H^j\} \cup (B_j \setminus \{H_j'\})$ is an m_j-NBC base of $\mathcal{A}$ and it is not an m_j-section of π, where H_j' is the maximal hyperplane of B_j via $\trianglelefteq$ where H_j' is the maximal hyperplane of B_j via $\trianglelefteq$ that contains the singleton block of the induced partition π_{X_j}, where $X_j = \bigcap_{H \in B_j} H$. Consequently, if $C' = \{H^1, \dots, H^t\} \cup (B \setminus \{H_1', \dots, H_t'\})$ is an s-NBC base of $\mathcal{A}$, then put $C = C'$, otherwise put $B' = C'$ and repeated the method above for B' by computing the broken circuits of it and add the minimal hyperplanes of $\mathcal{A}$ that make them circuits and removing the maximal hyperplane of them that contained of the singleton block of the induced partition of them flat of the intersection lattice. Continue this procedure unless C is an s-NBC base of $\mathcal{A}$ that is not an s-section of π. Furthermore, if $X = \bigcap_{H \in B} H = \bigcap_{H \in C} H$, then C is an s-NBC base of $\mathcal{A}_X$.

Secondly, if B is an s-broken circuit of $\mathcal{A}$ and H be the minimal hyperplane of $\mathcal{A}$ via $\trianglelefteq$, that satisfied $\{H\} \cup B$ is a circuit with B its broken circuit, then $C = \{H\} \cup (B \setminus \{H'\})$ is an s-NBC base of $\mathcal{A}$ and it is not an s-section of π, where H' is the maximal hyperplane of B via $\trianglelefteq$ that contained of the singleton block of the induced partition π_X, where $X = \bigcap_{H \in B} H = \bigcap_{H \in C} H$. Then, C is an s-NBC base of $\mathcal{A}_X$. ■

2.13. Lemma:

Suppose we have the assumptions of construction (2.9.). For $p_{\trianglelefteq}(\mathcal{A}) + 1 \leq k \leq \ell$, if C is a $k - NBC$ base of $\mathcal{A}$ and it is not a k-section of π such that, $C \subseteq \pi_{i_1} \cup \dots \cup \pi_{i_m}$, $1 \leq i_1 < \dots < i_m \leq \ell$ and $m < k$, then there is a k-section B of π which is not a $k-NBC$ base of $\mathcal{A}$ that satisfied, if $X=\bigcap_{H\in C} H\in L_k(\mathcal{A})$, then B is a k-section of π_X.

Proof: We will use our assumption to show the lemma inductively as follows:

If $p_{\trianglelefteq}(\mathcal{A})+1=2$ and $k=2$: If C is a $2-NBC$ base of $\mathcal{A}$ which is not a 2-section of π, then $C=\{H_1,H_2\}\subseteq\pi_m$ for some $1\le m\le\ell$. Since C is a $2-NBC$ base and we assume $\trianglelefteq$ respect the structure of π, hence there is no $H'\in\pi_1\cup\ldots\cup\pi_m$ such that $\{H',H_1,H_2\}$ is a circuit. Indeed, if $X=H_1\cap H_2\in L_2(\mathcal{A})$, then by applying lemma (2.1.8), $\mathcal{A}_X\subseteq\pi_m\cup\pi_t$ for some $m+1\le t\le\ell$ and $\mathcal{A}_X\cap\pi_t$ is a singleton. However, there exists a unique $H\in\pi_{m+1}\cup\ldots\cup\pi_\ell$ satisfied, $(C\cup\{H\})=2$. Thus, $B=\{H_2,H\}\subseteq\pi_m\cup\pi_t$ is a 2-section of π_X that is not a $2-NBC$ base of $\mathcal{A}_X$. Notice that, $\{H_1,H\}$ is $2-NBC$ base of $\mathcal{A}_X$ and a 2-section of π_X.

If $p_{\trianglelefteq}(\mathcal{A})+1\le k$ and $k=3$: Let $C=\{H_1,H_2,H_3\}$ is a $3-NBC$ base of $\mathcal{A}$ which is not a 3-section of π. Let $X=H_1\cap H_2\cap H_3\in L_3(\mathcal{A})$, then by applying lemma (2.10.), $\mathcal{A}_X\subseteq\pi_{m_1}\cup\pi_{m_2}\cup\pi_{m_3}$ for some $1\le m_1<m_2<m_3\le\ell$ and without loss of generality assume that, $\mathcal{A}_X\cap\pi_{m_3}$ is a singleton. Since C is not a 3-section of π, hence there are three ways to distribute the hyperplanes of π as, either $(C\subseteq\pi_{m_1})$, or $(C\subseteq\pi_{m_1}\cup\pi_{m_2})$, or $(C\subseteq\pi_{m_1}\cup\pi_{m_3})$. Since C is a $3-NBC$ base via $\trianglelefteq$, hence there is no $H'\in\pi_1\cup\ldots\cup\pi_{m_1}$ such that $\{H',H_1,H_2,H_3\}$ is a circuit. Thus, the guess $C\subseteq\pi_{m_2}\cup\pi_{m_3}$ is not agree the structure of C as $3-NBC$ base of $\mathcal{A}_X$. Therefore, we have just three possible cases:

Firstly, if $C\subseteq\pi_{m_1}$, then each of $\{H_1,H_2\}$, $\{H_1,H_3\}$ and $\{H_2,H_3\}$ is a $2-NBC$ of $\mathcal{A}$ and according to lemma (2.1.8), there are unique $H_{1,2},H_{1,3},H_{2,3}\in\pi_{m_2}\cup\pi_{m_3}$ such that $\{H_1,H_2,H_{1,2}\}$, $\{H_1,H_3,H_{1,3}\}$ and $\{H_2,H_3,H_{2,3}\}$ are circuits since $H_{1,2},H_{1,3},H_{2,3}\in\mathcal{A}_X$. If $\{H_{1,2},H_{1,3},H_{2,3}\}\cap\pi_{m_3}\neq\emptyset$, put $B=\{H_3,H',H\}$ where H' be the maximal hyperplane via $\trianglelefteq$ of $\{H_{1,2},H_{1,3},H_{2,3}\}\cap\pi_{m_2}$ and $\{H\}=\mathcal{A}_X\cap\pi_{m_3}$. Certainly, B is a 3-section of π_X that is not a $3-NBC$ base of $\mathcal{A}_X$. On the other hand, if $\{H_{1,2},H_{1,3},H_{2,3}\}\subseteq\pi_{m_2}$, put $B=\{H_3,H_{2,3},H\}$ where $\{H\}=\mathcal{A}_X\cap\pi_{m_3}$. Indeed, B is a 3-section of π_X, that is not a $3-NBC$ base of $\mathcal{A}_X$.

Secondly, suppose $C\subseteq\pi_{m_1}\cup\pi_{m_2}$ and without loss of generality, assume $|C\cap\pi_{m_1}|=1$ and $|C\cap\pi_{m_2}|=2$. Since $\{H_2,H_3\}$ is a $2-NBC$ of $\mathcal{A}$, hence $\{H_2,H_3,H\}$ is a circuit, where $\{H\}=\mathcal{A}_X\cap\pi_{m_3}$. Put $B=\{H_1,H_3,H\}$. Indeed, B is a 3-section of π_X that is not a $3-NBC$ base of $\mathcal{A}_X$.

Thirdly, suppose $C\subseteq\pi_{m_1}\cup\pi_{m_3}$. Then $H_1,H_2\in C\cap\pi_{m_1}$ and $H_3\in C\cap\pi_{m_3}$. Put $B=\{H_2,H_{1,2},H_3\}$, where $H_{1,2}\in\pi_{m_2}$ be the unique hyperplane of $\mathcal{A}_X$ such that $\{H_1,H_2,H_{1,2}\}$. Indeed, B is a 3-section of π_X that is not a $3-NBC$ base of $\mathcal{A}_X$.

Suppose, our statement is hold <u>for $k=s-1$</u> and we will prove it <u>for $k=s$</u> as follows:

Let $C=\{H_1,\ldots,H_s\}$ is an $s-NBC$ base of $\mathcal{A}$ and it is not an s-section of π. As the technique given in the case $k=3$ above we can assume that, $C\subseteq\pi_{m_1}\cup\ldots\cup\pi_{m_{s-1}}$, $1\le m_1<\cdots<m_{s-1}\le\ell$. Accordingly, there is $1\le p\le s-1$, such that $|C\cap\pi_{m_p}|=2$. So, $rk(C\cap\pi_{m_p})=2$ and $C\cap\pi_{m_p}$ is a $2-NBC$ base. Then there is $H\in\pi_l$, $m_p+1\le l\le\ell$, such that $(C\cap\pi_{m_p})\cup\{H\}$ is a circuit. If $l\neq m_{p+1},\ldots,m_{s-1}$, then $B=(C\setminus\{H'\})\cup\{H\}$ is an s-section of π that is not an $s-NBC$ base of $\mathcal{A}$. Else, apply lemma (2.2.3) as, let $X=\bigcap_{H\in C}H\in L_s(\mathcal{A})$, then the hyperplanes of $\mathcal{A}_X$ distributed among s blocks of π say, $\mathcal{A}_X\subseteq\pi_{m_1}\cup\ldots\cup\pi_{m_{s-1}}\cup\pi_{m_s}$, for some $1\le m_s\le\ell$. Let $C'=C\setminus\{H'''\}$, where H''' be the maximal hyperplane of C such that, $H'''\notin C\cap\pi_{m_p}$, i.e. H''' be the maximal hyperplane of C that contained in the singleton block of the induced partition π_X. Observe that, H''' is not collinear with the hyperplanes of $C\cap\pi_{m_p}$. It is clear that, C' is an $(s-1)-NBC$ base of $\mathcal{A}$ and it is not an $(s-1)$-section. Inductively, there exists an $(s-1)$-section of π_X say B' which is not an $(s-1)-NBC$ base of $\mathcal{A}$. Thus, if $B'\cup H'''$

is an s-section of π_X that is not a $s-NBC$ base of $\mathcal{A}_X$, put $B = B' \cup H'''$, otherwise put $C' = B' \cup H'''$ and repeated the method above for C'. Continue this procedure unless B is an s-section of π that is not s-NBC base of $\mathcal{A}$. ■

In spite of, a factored ℓ-arrangement $\mathcal{A}$ that has no factorization $\pi = (\pi_1, \ldots, \pi_\ell)$ and no ordering $\trianglelefteq$ such that $NBC_k(\mathcal{A}) = S_k(\pi)$, for all $1 \le k \le \ell$, is not quadratic under any ordering can be defined on its hyperplanes, the equality, $|NBC_k(\mathcal{A})| = \sum_{i_1=1}^{\ell-k} \sum_{i_2=i_1+1}^{\ell-k+1} \ldots \sum_{i_k=i_{k-1}+1}^{\ell} d_{i_1} d_{i_2} \ldots d_{i_k} = |S_k(\pi)|$, for $1 \le k \le \ell$, affords a one to one correspondence between $NBC(\mathcal{A})$ and $S(\mathcal{A})$ as shown in the following result:

2.14. Theorem:

Suppose we have the assumptions of construction (2.9.). Then, for all $0 \le k \le \ell$, there are one to one correspondences $f_k: NBC_k(\mathcal{A}) \to S_k(\pi)$ and $g_k = f_k^{-1}: S_k(\pi) \to NBC_k(\mathcal{A})$.

Proof:

First, for $0 \le k \le p_{\trianglelefteq}(\mathcal{A})$, define $f_k = g_k$ to be the identity mapping and it is one to one correspondences as we claimed.

Secondly, for $p_{\trianglelefteq}(\mathcal{A}) + 1 \le k \le \ell$, we can partition $S_k(\pi)$ and $NBC_k(\mathcal{A})$ into two parts as;

$S_k(\pi) = S_k^1(\pi) \cup S_k^2(\pi)$ and $NBC_k(\mathcal{A}) = NBC_k^1(\mathcal{A}) \cup NBC_k^2(\mathcal{A})$, where;

$S_k^1(\pi)$ is the set of all k-sections of π that are $k-NBC$ bases of $\mathcal{A}$;

$S_k^2(\pi)$ is the set of all k-section of π that are not $k-NBC$ bases of $\mathcal{A}$;

$NBC_k^1(\mathcal{A})$ is the set of all $k-NBC$ bases of $\mathcal{A}$ that are k-sections of π, and;

$NBC_k^2(\mathcal{A})$ is the set of all $k-NBC$ bases of $\mathcal{A}$ that are not a k-sections of π.

Clearly, $NBC_k^1(\mathcal{A}) = S_k^1(\pi)$. Moreover, $NBC_k^2(\mathcal{A}) \ne S_k^2(\pi)$ within $|NBC_k^2(\mathcal{A})| = |S_k^2(\pi)|$. Accordingly, we can define $f_k: NBC_k(\mathcal{A}) \to S_k(\pi)$ as;

$$f_k(C) = \begin{cases} C & \text{if } C \in NBC_k^1(\mathcal{A}) \\ B' & \text{if } C \in NBC_k^2(\mathcal{A}) \end{cases};$$

where B' is the k-section of π that is not a $k-NBC$ base of $\mathcal{A}$ given in lemma (2.13.), and we can define $g_k: S_k(\pi) \to NBC_k(\mathcal{A})$ as;

$$g_k(B) = \begin{cases} B & \text{if } B \in S_k^1(\pi) \\ C' & \text{if } B \in S_k^2(\pi) \end{cases};$$

where C' is the $k-NBC$ base of $\mathcal{A}$ which is not a k-section of π given in lemma (2.12.). We emphasize that, when we choose each of B' and C' in the lemma (2.13.) and lemma (2.12.), we used the concepts "unique', "minimal hyperplane" and "maximal hyperplane" via a total ordering $\trianglelefteq$, that are unique and that admits each of f_k and g_k, a structure as well-defined maps.

If $C \in NBC_k^1(\mathcal{A})$, it is clear that, $g_k \circ f_k(C) = C = I_{NBC_k(\mathcal{A})}(C)$ and since $S_k^1(\pi) = NBC_k^1(\mathcal{A})$, $f_k \circ g_k(C) = C = I_{S_k(\mathcal{A})}(C)$. Therefore, we need only to show $g_k \circ f_k(C) = C = I_{NBC_k(\mathcal{A})}(C)$, for $C \in NBC_k^2(\mathcal{A})$ and $f_k \circ g_k(B) = B = I_{S_k(\mathcal{A})}(B)$, for $B \in S_k^2(\pi)$. However, we will prove that by induction as follows:

<u>For $k = 2$</u>: Let $C = \{H_1, H_2\} \in NBC_2^2(\mathcal{A})$. By applying lemma (2.13.), let H be the hyperplane contained in the singleton block of the induced partition π_X, where $X = H_1 \cap H_2$. Then, $f_2(C) = B' = \{H_2, H\}$. Clearly, H_1 is the minimal hyperplane of $\mathcal{A}$ satisfied $\{H_1, H_2, H\}$ is a circuit. As have shown in lemma (2.12.), $g_2(B') = g_2 \circ f_2(C) = \{H_1, H_2\} = C = I_{NBC_2(\mathcal{A})}(C)$. As well as, $B = \{H_1', H_2'\} \in S_2^2(\pi)$, we will apply lemma (2.2.4) and let H' be the minimal hyperplane of $\mathcal{A}$ satisfied $\{H', H_1', H_2'\}$ is a circuit and $g_2(B) = C' = \{H', H_1'\}$. Since H' and H_1' are contained of the same block, hence H_2' will be the hyperplane contained in the singleton block of the induced partition $\pi_{X'}$, where $X' = H' \cap H_1'$. Thus, $f_2(C') = f_2 \circ g_2(B) = \{H_1', H_2'\} = B = I_{S_2(\mathcal{A})}(B)$. Therefore, $g_2 = f_2^{-1}$.

For $k = 3$: Let $C = \{H_1, H_2, H_3\} \in NBC_3^2(\mathcal{A})$. Let $X = H_1 \cap H_2 \cap H_3 \in L_3(\mathcal{A})$, then by applying lemma (2.2.3), $\mathcal{A}_X \subseteq \pi_{m_1} \cup \pi_{m_2} \cup \pi_{m_3}$ for some $1 \le m_1 < m_2 < m_3 \le \ell$ and without loss of generality assume that, $\mathcal{A}_X \cap \pi_{m_3}$ is a singleton. Lemma (2.13.) discussed three cases, so we will separate our discussion for each one of them as follows:

Case 1: $(C \subseteq \pi_{m_1})$

Recall the unique hyperplanes $H_{1,2}, H_{1,3}, H_{2,3} \in \pi_{m_2} \cup \pi_{m_3}$ such that such that $\{H_1, H_2, H_{1,2}\}$, $\{H_1, H_3, H_{1,3}\}$ and $\{H_2, H_3, H_{2,3}\}$ are circuits.

Case (1.1):

If $\{H_{1,2}, H_{1,3}, H_{2,3}\} \cap \pi_{m_3} \neq \emptyset$, then $f_3(C) = B' = \{H_3, H', H\}$, where H' is the maximal hyperplane via $\trianglelefteq$ of $\{H_{1,2}, H_{1,3}, H_{2,3}\} \cap \pi_{m_2}$ and $\{H\} = \mathcal{A}_X \cap \pi_{m_3}$. If $H' = H_{1,3}$ and $H = H_{2,3}$, then B'contains two broken circuit, $\{H_3, H_{1,3}\}$ and $\{H_3, H_{2,3}\}$. Therefore,

$$g_3(B') = g_3 \circ f_3(C) = \left(B' \setminus \{H_{1,3}, H_{2,3}\}\right) \cup \{H_1, H_2\} = C = I_{NBC_3(\mathcal{A})}(C).$$

If $H' = H_{1,2}$ and $H = H_{2,3}$. Then B'contains unique broken circuit $\{H_3, H_{2,3}\}$ and $\left(B' \setminus \{H_{2,3}\}\right) \cup \{H_2\}$ also contains unique broken circuit $\{H_2, H_{1,2}\}$. So,

$$g_3(B') = g_3 \circ f_3(C) = (((B' \setminus \{H_{2,3}\}) \cup \{H_2\}) \setminus \{H_{2,3}\}) \cup \{H_1\} = C = I_{NBC_3(\mathcal{A})}(C).$$

Similarly, for any other ordering of the hyperplanes $H_{1,2}, H_{1,3}$ and $H_{2,3}$.

Case (1.2):

If $\{H_{1,2}, H_{1,3}, H_{2,3}\} \subseteq \pi_{m_2}$, then $f_3(C) = B' = \{H_3, H_{2,3}, H\}$, where $\{H\} = \mathcal{A}_X \cap \pi_{m_3}$. Then $\left(B' \setminus \{H_{2,3}\}\right) \cup \{H_2\})$ is a 3-broken circuit since $\{H_1\} \cup \left(B' \setminus \{H_{2,3}\}\right) \cup \{H_2\})$ is a circuit. Therefore;

$$g_3(B') = g_3 \circ f_3(C) = (((B' \setminus \{H_{2,3}\}) \cup \{H_2\}) \setminus \{H\}) \cup \{H_1\} = C = I_{NBC_3(\mathcal{A})}(C).$$

Case 2: $(C \subseteq \pi_{m_1} \cup \pi_{m_2})$

Case (2.1):

If $|C \cap \pi_{m_1}| = 1$ and $|C \cap \pi_{m_2}| = 2$. Since $\{H_2, H_3\}$ is a $2 - NBC$ of $\mathcal{A}$, hence $\{H_2, H_3, H\}$ is a circuit, where $\{H\} = \mathcal{A}_X \cap \pi_{m_3}$. Then, $f_3(C) = B' = \{H_1, H_3, H\}$ is a 3-section of π contains broken circuit $\{H_3, H\}$. Therefore, $g_3(B') = g_3 \circ f_3(C) = (B' \setminus \{H\}) \cup \{H_2\} = C = I_{NBC_3(\mathcal{A})}(C)$.

Case (2.1):

If $|C \cap \pi_{m_1}| = 1$ and $|C \cap \pi_{m_2}| = 2$. Since $\{H_2, H_3\}$ is a $2 - NBC$ of $\mathcal{A}$, hence $\{H_2, H_3, H\}$ is a circuit, where $\{H\} = \mathcal{A}_X \cap \pi_{m_3}$. Then, $f_3(C) = B' = \{H_1, H_3, H\}$ is a 3-section of π contains broken circuit $\{H_3, H\}$. Therefore, $g_3(B') = g_3 \circ f_3(C) = (B' \setminus \{H\}) \cup \{H_2\} = C = I_{NBC_3(\mathcal{A})}(C)$.

Case (2.2):

If $|C \cap \pi_{m_1}| = 2$ and $|C \cap \pi_{m_2}| = 1$. Then, $f_3(C) = B' = \{H_2, H_3, H\}$ is a 3-section of π and it is a broken circuit of $\mathcal{A}$, where $\{H\} = \mathcal{A}_X \cap \pi_{m_3}$. Therefore,

$$g_3(B') = g_3 \circ f_3(C) = (B' \setminus \{H\}) \cup \{H_1\} = C = I_{NBC_3(\mathcal{A})}(C).$$

Case 3: $(C \subseteq \pi_{m_1} \cup \pi_{m_3})$:

In this case, $f_3(C) = B' = \{H_2, H_{1,2}, H_3\}$, where $H_{1,2} \in \pi_{m_2}$be the unique hyperplane of $\mathcal{A}_X$ such that $\{H_1, H_2, H_{1,2}\}$. Thus, $g_3(B') = g_3 \circ f_3(C) = \left(B' \setminus \{H_{1,2}\}\right) \cup \{H_1\} = C = I_{NBC_3(\mathcal{A})}(C)$.

Let $B = \{H_1', H_2', H_3'\} \in S_3^2(\mathcal{A})$ and B is a broken circuit, we will apply lemma (2.12.) and let H' be the minimal hyperplane of $\mathcal{A}$ satisfied $\{H', H_1', H_2', H_3'\}$ is a circuit. So, $g_3(B) = C' = (B \setminus \{H''\}) \cup \{H'\}$, where H'' will be the maximal hyperplane of B that contained in the singleton block of the induced partition $\pi_{X'}$, where $X' = H_1' \cap H_2' \cap H_3'$. Apply lemma (2.13.) for $C' \in NBC_3(\mathcal{A})$. Suppose $H'' = H_3'$, so $C' = \{H', H_1', H_2'\}$.

Since H' and H'_1 are contained in the same block, similarly as case (2.2) above we choose $H'' = H'_3$ and $g_3(B') = g_3 \circ f_3(C) = (B' \setminus \{H'\}) \cup \{H'_3\} = C = I_{NBC_3(\mathcal{A})}(C)$. Therefore, $g_3 = f_3^{-1}$.

Suppose, the statement is hold ***for*** $k = s - 1$ and we will prove it ***for*** $k = s$ as follows:

Let $C = \{H_1, \ldots, H_s\} \in NBC_s^2(\mathcal{A})$. Assume that, $C \subseteq \pi_{m_1} \cup \ldots \cup \pi_{m_{s-1}}$, $1 \le m_1 < \cdots < m_{s-1} \le \ell$. Accordingly, there is $1 \le j \le s-1$. Without loss of generality, assume $|C \cap \pi_{m_1}| = 2$. So, $rk(C \cap \pi_{m_1}) = 2$ and $C \cap \pi_{m_1} = \{H'_1, H'_2\}$ is a $2 - NBC$ base. Then there is $H \in \pi_l$, $m_1 + 1 \le l \le \ell$, such that, $\{H'_1, H'_2, H\}$ is a circuit. If $\neq m_2, \ldots, m_{s-1}$, then $f_s(C) = B' = (C \setminus \{H_1\}) \cup \{H\}$ and by applying lemma (2.12.), since $\{H'_2, H\}$ is the broken circuit of B', hence;

$$g_s(B') = g_s \circ f_s(C) = (B' \setminus \{H\}) \cup \{H'_1\} = C = I_{NBC_s(\mathcal{A})}(C).$$

Else, continue as we have discussed in case $k = 3$.

Let $B = \{H'_1, \ldots, H'_s\} \in S_s^2(\pi)$ and B is a broken circuit, we will apply lemma (2.12.) and let H' is the minimal hyperplane of $\mathcal{A}$ satisfied $\{H', H'_1, \ldots, H'_s\}$ is a circuit. So, $g_s(B) = C' = (B \setminus \{H''\}) \cup \{H'\}$, where H'' will be the maximal hyperplane of B contained in the singleton block of the induced partition $\pi_{X'}$, where $X' = H'_1 \cap \ldots \cap H'_s$. Apply lemma (2.13.) for $C' \in NBC_s(\mathcal{A})$. Suppose $H'' = H'_s$, so $C' = \{H', H'_1, \ldots, H'_{s-1}\}$. Since H' and H'_1 are contained in the same block, so;

$$g_s(B') = g_s \circ f_s(C) = (B' \setminus \{H'\}) \cup \{H'_s\} = C = I_{NBC_s(\mathcal{A})}(C).$$

Therefore, $g_s = f_s^{-1}$.■

2.15. Remark:

In view of theorem (2.14.), we notice that:

1. $|NBC_k^2(\mathcal{A})| = |NBC_k(\mathcal{A})| - |S_k^1(\pi)|$, for $0 \le k \le \ell$.
2. $p_{\trianglelefteq}(\mathcal{A}) + 1 = \min\{k \mid |NBC_k^2(\mathcal{A})| \neq 0\}$.
3. f_k is identity map, for $0 \le k \le p(\mathcal{A})$.

Moreover, theorem (2.14.) create a connection between two fashions of the O-S algebra of factored arrangement that not completely factored, a fashion as free submodule of the exterior algebra, and a fashion as a tensor factorization module. We provide this goal as follows:

2.16. Construction:

Let $\mathcal{A}$ be a factored $\ell -$arrangement with a factorization $\pi = (\pi_1, \ldots, \pi_\ell)$ such that $\mathcal{A}$ is not completely factored via any ordering can be defined on its hyperplanes, i.e. $NBC_k(\mathcal{A}) \neq S_k(\pi)$, for $p_{\trianglelefteq}(\mathcal{A}) + 1 \le k \le \ell$. Recall the one to one correspondences $f_k: NBC_k(\mathcal{A}) \to S_k(\pi)$ and $g_k: S_k(\pi) \to NBC_k(\mathcal{A})$, that given in theorem (2.14.). Accordingly there are one to one correspondences, $\mathfrak{f}_k: \{e_B | B \in NBC_k(\mathcal{A})\} \to \{q_C | C \in S_k(\pi)\}$ and $\mathfrak{g}_k: \{q_C | C \in S_k(\pi)\} \to \{e_B | B \in NBC_k(\mathcal{A})\}$, defined as;

$$\mathfrak{f}_k(e_B) = \begin{cases} q_B & \text{if } B \in NBC_k^1(\mathcal{A}) \\ q_{C'} & \text{if } B \in NBC_k^2(\mathcal{A}) \end{cases};$$

where C' is the k-section of π that is not a $k - NBC$ base of $\mathcal{A}$ given in lemma (2.13.), and;

$$\mathfrak{g}_k(q_C) = \begin{cases} e_C & \text{if } B \in S_k^1(\pi) \\ e_{B'} & \text{if } B \in S_k^2(\pi) \end{cases};$$

where B' is the $k - NBC$ base of $\mathcal{A}$ which is not a k-section of π given in lemma (2.12.). That induces unique K- linear isomorphisms, $\mathfrak{f}_k: \boldsymbol{NBC}_k(\mathcal{A}) \to (\pi)_k$ and $\mathfrak{g}_k: (\pi)_k \to \boldsymbol{NBC}_k(\mathcal{A})$ between the k^{th} partition module and k^{th} broken circuit module that extend this assignments as follows:

$$\{e_B | B \in NBC_k(\mathcal{A})\} \xrightarrow{i_k^{NBC}} \boldsymbol{NBC}_k(\mathcal{A})$$

$\mathfrak{f}_k$

$\{q_C | C \in S_k(\pi)$ $\quad \exists!\, \mathfrak{f}_k$

i_k^S

(π)

$$\{q_C | B \in S_k(\pi)\} \xrightarrow{i_k^S} (\pi)_k$$

$\mathfrak{g}_k$

$\{e_B | B \in NBC_k(\mathcal{A})\}$ $\quad \exists!\, \mathfrak{g}_k$

i_k^{NBC}

$\boldsymbol{NBC}_k(\mathcal{A})$

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2.17. Theorem:

Construction (3.16.) produces K -chain isomorphisms, $\mathscr{f}_*: \boldsymbol{NBC}_*(\mathcal{A}) \rightarrow (\pi)_*$ and $\mathscr{g}_*: (\pi)_* \rightarrow \boldsymbol{NBC}_*(\mathcal{A})$ between acyclic chain complexes and this creates a joining between two fashions of the O-S algebra as shown in the following commutative diagram;

$$\begin{array}{ccccccccccc}
0 \rightarrow & \boldsymbol{A}_\ell(\mathcal{A}) & \xrightarrow{\partial_\ell^A} & \boldsymbol{A}_{\ell-1}(\mathcal{A}) & \xrightarrow{\partial_{\ell-1}^A} \cdots \xrightarrow{\partial_2^A} & \boldsymbol{A}_1(\mathcal{A}) & \xrightarrow{\partial_1^A} & \boldsymbol{A}_0(\mathcal{A}) & \xrightarrow{\partial_0^A} 0 \\
& \psi_\ell^{-1} \downarrow & & \psi_{\ell-1}^{-1} \downarrow & & \psi_1^{-1} \downarrow & & \psi_0^{-1} \downarrow & \\
0 \rightarrow & \boldsymbol{NBC}_\ell(\mathcal{A}) & \xrightarrow{\partial_\ell^{NBC}} & \boldsymbol{NBC}_{\ell-1}(\mathcal{A}) & \xrightarrow{\partial_{\ell-1}^{NBC}} \cdots \xrightarrow{\partial_2^{NBC}} & \boldsymbol{NBC}_1(\mathcal{A}) & \xrightarrow{\partial_1^{NBC}} & \boldsymbol{NBC}_0(\mathcal{A}) & \xrightarrow{\partial_0^{NBC}} 0 \\
& \mathscr{f}_\ell \downarrow & & \mathscr{f}_{\ell-1} \downarrow & & \mathscr{f}_1 \downarrow & & \mathscr{f}_0 \downarrow & \\
0 \rightarrow & (\pi)_\ell & \xrightarrow{\partial_\ell^\pi} & (\pi)_{\ell-1} & \xrightarrow{\partial_{\ell-1}^\pi} \cdots \xrightarrow{\partial_2^\pi} & (\pi)_1 & \xrightarrow{\partial_1^\pi} & (\pi)_0 & \xrightarrow{\partial_0^\pi} 0 \\
& \varphi_\ell \downarrow & & \varphi_{\ell-1} \downarrow & & \varphi_1 \downarrow & & \varphi_0 \downarrow & \\
0 \rightarrow & \boldsymbol{A}_\ell(\mathcal{A}) & \xrightarrow{\partial_\ell^A} & \boldsymbol{A}_{\ell-1}(\mathcal{A}) & \xrightarrow{\partial_{\ell-1}^A} \cdots \xrightarrow{\partial_2^A} & \boldsymbol{A}_1(\mathcal{A}) & \xrightarrow{\partial_1^A} & \boldsymbol{A}_0(\mathcal{A}) & \xrightarrow{\partial_0^A} 0
\end{array}$$

Proof: For a fixed $1 \leq k \leq \ell$, we need to show that the following two diagrams are commutative:

$$\begin{array}{ccc}
(\pi)_k & \xrightarrow{\partial_k^\pi} & (\pi)_{k-1} \\
\mathscr{g}_k \downarrow & & \downarrow \mathscr{g}_{k-1} \\
\boldsymbol{NBC}_k(\mathcal{A}) & \xrightarrow{\partial_k^{NBC}} & \boldsymbol{NBC}_{k-1}(\mathcal{A})
\end{array}
\qquad\qquad
\begin{array}{ccc}
\boldsymbol{NBC}_k(\mathcal{A}) & \xrightarrow{\partial_k^{NBC}} & \boldsymbol{NBC}_{k-1}(\mathcal{A}) \\
\mathscr{f}_k \downarrow & & \downarrow \mathscr{f}_{k-1} \\
(\pi)_k & \xrightarrow{\partial_k^\pi} & (\pi)_{k-1}
\end{array}$$

(i) *(ii)*

For i: Let $C \in S_k(\pi)$. Then;

$$\partial_k^{NBC} \circ \mathscr{g}_k(q_C) = \partial_k^{NBC}\big(\mathscr{g}_k(q_C)\big)$$

$$= \partial_k^{NBC} \begin{cases} e_C & \text{if } C \in S_k^1(\pi) \\ e_{B'} & \text{if } C \in S_k^2(\pi) \end{cases}$$

$$= \begin{cases} \partial_k^{NBC}(e_C) & \text{if } C \in S_k^1(\pi) \\ \partial_k^{NBC}(e_{B'}) & \text{if } C \in S_k^2(\pi) \end{cases} = \begin{cases} \sum_{j=1}^{k} (-1)^{k-1} e_{C_j} & \text{if } C \in S_k^1(\pi) \\ \sum_{j=1}^{k} (-1)^{k-1} e_{B'_j} & \text{if } C \in S_k^2(\pi) \end{cases}$$

$$= \sum_{j=1}^{k} (-1)^{k-1} \mathscr{g}_{k-1}\left(q_{C_j}\right) = \mathscr{g}_{k-1}\left(\sum_{j=1}^{k} (-1)^{k-1} q_{C_j}\right) = \mathscr{g}_{k-1} \circ \partial_k^\pi(q_C)$$

Thus, $\partial_k^{NBC} \circ \mathscr{g}_k = \mathscr{g}_{k-1} \circ \partial_k^\pi$.

For ii: Let $B \in NBC_k(\mathcal{A})$. Then;

$$\partial_k^\pi \circ \mathscr{f}_k(e_B) = \partial_k^\pi\big(\mathscr{f}_k(e_B)\big)$$

$$= \partial_k^{\pi} \begin{cases} q_B & \text{if } B \in NBC_k^1(\mathcal{A}) \\ q_{C'} & \text{if } B \in NBC_k^2(\mathcal{A}) \end{cases}$$

$$= \begin{cases} \partial_k^{\pi}(q_B) & \text{if } B \in NBC_k^1(\mathcal{A}) \\ \partial_k^{\pi}(q_{C'}) & \text{if } B \in NBC_k^2(\mathcal{A}) \end{cases} = \begin{cases} \sum_{j=1}^{k} (-1)^{k-1} q_{B_j} & \text{if } B \in NBC_k^1(\mathcal{A}) \\ \sum_{j=1}^{k} (-1)^{k-1} q_{C_j'} & \text{if } B \in NBC_k^2(\mathcal{A}) \end{cases}$$

$$= \sum_{j=1}^{k} (-1)^{k-1} \mathscr{f}_{k-1}\left(e_{B_j}\right) = \mathscr{f}_{k-1}\left(\sum_{j=1}^{k} (-1)^{k-1} e_{B_j}\right) = \mathscr{f}_{k-1} \circ \partial_k^{NBC}(e_B)$$

Thus, $\partial_k^{\pi} \circ \mathscr{f}_k = \mathscr{f}_{k-1} \circ \partial_k^{NBC}$. Therefore, each of $\mathscr{f}_*: \boldsymbol{NBC}_k(\mathcal{A}) \to (\pi)_*$ and $\mathscr{g}_*: (\pi)_* \to \boldsymbol{NBC}_*(\mathcal{A})$ are K-chain isomorphisms between acyclic chain complexes. It is clear that, $\varphi_*^{-1} = \mathscr{f}_* \circ \psi_*^{-1}$, $\varphi_* = \psi_* \circ \mathscr{g}_*$ and the K-chain isomorphisms $\mathscr{k} = \varphi_* \circ \mathscr{f}_* \circ \psi_*^{-1}$ and $\mathscr{h}^{-1} = \psi_* \circ \mathscr{g}_* \circ \varphi_*^{-1}$ produce a connection between two fashions of the O-S algebra and this is precisely the assertion of the theorem. ■

3. On the Product of factored arrangement

This section is motivated to prove that section two classification of the class of factored arrangement is compatible with the product construction.

3.1. Construction:

For $i = 1,2$, let $\mathcal{A}_i = \{H_1^i, \dots, H_{n_i}^i\}$ be factored ℓ_i-arrangement with a factorization $\pi^i = (\pi_1^i, \dots, \pi_\ell^i)$. The product arrangement is defined as;

$$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 = (\mathcal{A}_1 \oplus \mathbb{C}^{\ell_2}) \cup (\mathbb{C}^{\ell_1} \oplus \mathcal{A}_2), \text{ where;}$$

$$\mathcal{A}_1 \oplus \mathbb{C}^{\ell_2} = \{H_1^1 \oplus \mathbb{C}^{\ell_2}, \dots, H_{n_1}^1 \oplus \mathbb{C}^{\ell_2}\} \text{ and } \mathbb{C}^{\ell_1} \oplus \mathcal{A}_2 = \{\mathbb{C}^{\ell_1} \oplus H_1^2, \dots, \mathbb{C}^{\ell_1} \oplus H_{n_2}^2\}.$$

We will partition $\mathcal{A}_1 \times \mathcal{A}_2$ by the partition $\pi = (\pi^1 \oplus \mathbb{C}^{\ell_2}) \cup (\mathbb{C}^{\ell_1} \oplus \pi^2)$, where;

$$\pi^1 \oplus \mathbb{C}^{\ell_2} = (\pi_1^1 \oplus \mathbb{C}^{\ell_2}, \dots, \pi_{\ell_1}^1 \oplus \mathbb{C}^{\ell_2}) \text{ and } \mathbb{C}^{\ell_1} \oplus \pi^2 = (\mathbb{C}^{\ell_1} \oplus \pi_1^2, \dots, \mathbb{C}^{\ell_1} \oplus \pi_{\ell_2}^2);$$

$$\text{i.e. } \pi = \pi^1 \times \pi^2 = (\pi_1^1 \oplus \mathbb{C}^{\ell_2}, \dots, \pi_{\ell_1}^1 \oplus \mathbb{C}^{\ell_2}, \mathbb{C}^{\ell_1} \oplus \pi_1^2, \dots, \mathbb{C}^{\ell_1} \oplus \pi_{\ell_2}^2)$$

Assume $\trianglelefteq_i$ be the ordering defined in the hyperplanes of $\mathcal{A}_i$, for $i = 1,2$, as given in construction (2.1.) or construction (2.9.), depending on the kind of $\mathcal{A}_i$. Then, define an ordering $\trianglelefteq = \trianglelefteq_1 \times \trianglelefteq_2$ on the hyperplanes of $\mathcal{A}_1 \times \mathcal{A}_2$ as follows:

1. For $1 \le i, j \le n_1$, put $H_i^1 \oplus \mathbb{C}^{\ell_2} \trianglelefteq H_j^1 \oplus \mathbb{C}^{\ell_2}$ if, and only if, $H_i^1 \trianglelefteq_1 H_j^1$.
2. For $1 \le i, j \le n_2$, put $\mathbb{C}^{\ell_1} \oplus H_i^2 \trianglelefteq \mathbb{C}^{\ell_1} \oplus H_j^2$ if, and only if, $H_i^2 \trianglelefteq_2 H_j^2$.
3. For $1 \le i \le n_1$ and $1 \le j \le n_2$, put $H_i^1 \oplus \mathbb{C}^{\ell_2} \trianglelefteq \mathbb{C}^{\ell_1} \oplus H_j^2$.

3.2. Lemma:

Suppose we have the assumptions of construction (3.1.). Then:

1. For $1 \le k \le \ell_1$, if $B^1 \in NBC_k(\mathcal{A}_1)$, then $B^1 \oplus \mathbb{C}^{\ell_2} \in NBC_k(\mathcal{A}_1 \oplus \mathbb{C}^{\ell_2}) \subseteq NBC_k(\mathcal{A}_1 \times \mathcal{A}_2)$.
2. For $1 \le k \le \ell_2$, if $B^2 \in NBC_k(\mathcal{A}_2)$, then $\mathbb{C}^{\ell_1} \oplus B^2 \in NBC_k(\mathbb{C}^{\ell_1} \oplus \mathcal{A}_2) \subseteq NBC_k(\mathcal{A}_1 \times \mathcal{A}_2)$.
3. For $1 \le k \le \ell_1 + \ell_2$, if $B^1 \in NBC_{k_1}(\mathcal{A}_1)$ and $B^2 \in NBC_{k_2}(\mathcal{A}_2)$, for some $1 \le k_1 \le \ell_1$ and $1 \le k_2 \le \ell_2$ such that $k_1 + k_2 = k$, then $B^1 \times B^2 = (B^1 \oplus \mathbb{C}^{\ell_2}) \cup (\mathbb{C}^{\ell_1} \oplus B^2) \in NBC_k(\mathcal{A}_1 \times \mathcal{A}_2)$.

Furthermore, for $0 \le k \le \ell_1 + \ell_2$, $NBC_k(\mathcal{A}_1 \times \mathcal{A}_2) = \bigcup_{\substack{B^1 \in NBC_{k_1}(\mathcal{A}_1),\ 0 \le k_1 \le \ell_1 \\ B^2 \in NBC_{k_2}(\mathcal{A}_2),\ 0 \le k_2 \le \ell_2 \\ k_1 + k_2 = k}} \{B^1 \times B^2\}$.

Proof: Due to construction (3.1.), if B^1 and B^2 are independent subarrangement of $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively, then $B^1 \oplus \mathbb{C}^{\ell_2}$, $\mathbb{C}^{\ell_1} \oplus B^2$ and $B^1 \times B^2$ are independent subarrangement of $\mathcal{A}_1 \times \mathcal{A}_2$. Moreover, our definition of the ordering $\trianglelefteq = \trianglelefteq_1 \times \trianglelefteq_2$ that respect the ordering $\trianglelefteq_1$ and $\trianglelefteq_2$ on $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively, involves if B^1 and B^2 contain no broken circuit of $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively, then $B^1 \oplus \mathbb{C}^{\ell_2}$, $\mathbb{C}^{\ell_1} \oplus B^2$ and $B^1 \times B^2$ contain no broken circuit subarrangement of $\mathcal{A}_1 \times \mathcal{A}_2$ and our claim is down. ■

3.3. Lemma:

Suppose we have the assumptions of construction (3.1.). Then:

1. For $0 \le k \le \ell_1$, if $C^1 \in S_k(\pi^1)$, then $C^1 \oplus \mathbb{C}^{\ell_2} \in S_k(\pi^1 \oplus \mathbb{C}^{\ell_2}) \subseteq S_k(\pi^1 \times \pi^2)$.
2. For $0 \le k \le \ell_2$, if $C^2 \in S_k(\pi^2)$, then $\mathbb{C}^{\ell_1} \oplus C^2 \in S_k(\mathbb{C}^{\ell_1} \oplus \pi^2) \subseteq S_k(\pi^1 \times \pi^2)$.
3. For $0 \le k \le \ell_1 + \ell_2$, if $C^1 \in S_{k_1}(\pi^1)$ and $C^2 \in S_{k_2}(\pi^2)$, for some $0 \le k_1 \le \ell_1$ and $0 \le k_2 \le \ell_2$ such that $k_1 + k_2 = k$, then $C^1 \times C^2 = (C^1 \oplus \mathbb{C}^{\ell_2}) \cup (\mathbb{C}^{\ell_1} \oplus C^2) \in S_k(\pi^1 \times \pi^2)$.

Furthermore, for $0 \le k \le \ell_1 + \ell_2$, $S_k(\pi^1 \times \pi^2) = \bigcup_{\substack{C^1 \in S_{k_1}(\pi^1),\ 0 \le k_1 \le \ell_1 \\ C^2 \in S_{k_2}(\pi^2),\ 0 \le k_2 \le \ell_2 \\ k_1 + k_2 = k}} \{C^1 \times C^2\}$.

Proof: According to construction (3.1.), it is clear, if C^1 and C^2 is independent subarrangement of π_1 and π_2 respectively, then $C^1 \oplus \mathbb{C}^{\ell_2}$, $\mathbb{C}^{\ell_1} \oplus C^2$ and $C^1 \times C^2$ are sections of $\pi = \pi_1 \times \pi_2$. ■

3.4. Lemma:

Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ be a reducible $(\ell_1 + \ell_2)$-arrangementm into a product of ℓ_1-arrangement $\mathcal{A}_1$ and ℓ_2-arrangement $\mathcal{A}_2$. If $\mathcal{A}$ is a factored arrangement with a factorization $\pi = (\pi_1, \dots, \pi_\ell)$, then for each $1 \le k \le \ell_1 + \ell_2$, either $\pi_k \subseteq \mathcal{A}_1 \oplus \mathbb{C}^{\ell_2}$ or $\pi_k \subseteq \mathbb{C}^{\ell_1} \oplus \mathcal{A}_2$.

Proof: By contrary, assume $\pi_k \subseteq \mathcal{A}_1 \oplus \mathbb{C}^{\ell_2}$ and $\pi_k \subseteq \mathbb{C}^{\ell_1} \oplus \mathcal{A}_2$. Thus, there are $H^1 \in \mathcal{A}_1$ and $H^2 \in \mathcal{A}_2$ such that $H^1 \oplus \mathbb{C}^{\ell_2}, \mathbb{C}^{\ell_1} \oplus H^2 \in \pi_k$. Let $X = H^1 \oplus \mathbb{C}^{\ell_2} \cap \mathbb{C}^{\ell_1} \oplus H^2 \in L_2(\mathcal{A})$. Since π is a factorization, hence, the induced partition π_X has two blocks. Thus, there is a unique $H \in \pi_m$, for some $1 \le m \le \ell_1 + \ell_2$ and $m \ne k$ with $rk(\{H^1 \oplus \mathbb{C}^{\ell_2}, \mathbb{C}^{\ell_1} \oplus H^2, H\}) = 2$. Then, either $H = H' \oplus \mathbb{C}^{\ell_2} \in \mathcal{A}_1 \oplus \mathbb{C}^{\ell_2}$ or $H = \mathbb{C}^{\ell_1} \oplus H'' \in \mathbb{C}^{\ell_1} \oplus \mathcal{A}_2$ and both of these two cases contradict the fact that there are no collinear relations among any three hyperplanes of $\mathcal{A}_1 \oplus \mathbb{C}^{\ell_2}$ and $\mathbb{C}^{\ell_1} \oplus \mathcal{A}_2$; i.e. $rk(\{H^1 \oplus \mathbb{C}^{\ell_2}, \mathbb{C}^{\ell_1} \oplus H^2, H\}) = 3$ for all $H \in \mathcal{A}_1 \times \mathcal{A}_2 \setminus \{H^1 \oplus \mathbb{C}^{\ell_2}, \mathbb{C}^{\ell_1} \oplus H^2\}$. Therefore, either $\pi_k \subseteq \mathcal{A}_1 \oplus \mathbb{C}^{\ell_2}$ or $\pi_k \subseteq \mathbb{C}^{\ell_1} \oplus \mathcal{A}_2$. ■

3.5. Proposition:

Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ be a reducible $(\ell_1 + \ell_2)$-arrangement into a product of ℓ_1-arrangement $\mathcal{A}_1$ and ℓ_2-arrangement $\mathcal{A}_2$. If $\mathcal{A}$ is a factored arrangement with a factorization $\pi = (\pi_1, \dots, \pi_\ell)$ and if;

$$T_1 = \bigcap_{H \in \mathcal{A}_1} H \oplus \mathbb{C}^{\ell_2} \in L_{\ell_1}(\mathcal{A}_1 \times \mathcal{A}_2) \text{ and;} T_2 = \bigcap_{H \in \mathcal{A}_2} \mathbb{C}^{\ell_1} \oplus H \in L_{\ell_2}(\mathcal{A}_1 \times \mathcal{A}_2);$$

then, $\mathcal{A}_{T_1} = \mathcal{A}_1 \oplus \mathbb{C}^{\ell_2}$, $\mathcal{A}_{T_2} = \mathbb{C}^{\ell_1} \oplus \mathcal{A}_2$ and for $i = 1,2$, the induced partition $\pi_{T_i} = (\pi_1^i, \dots, \pi_{\ell_i}^i)$ satisfied for $1 \le k \le \ell_i$, $\pi_k^i = \pi_m$, for some$1 \le m \le \ell_1 + \ell_2$. Moreover, $\pi_{k_1}^1 \ne \pi_{k_2}^2$, for each $1 \le k_1 \le \ell_1$ and $1 \le k_2 \le \ell_2$.

Proof: Firstly, we prove $\mathcal{A}_{T_1} = \mathcal{A}_1 \oplus \mathbb{C}^{\ell_2}$, $\mathcal{A}_{T_2} = \mathbb{C}^{\ell_1} \oplus \mathcal{A}_2$. It was known that, if $B_1 \subseteq \mathcal{A}_1$ and $B_2 \subseteq \mathcal{A}_2$ is linearly independent of $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively, then, for all $H \in \mathcal{A}_1$ and for all $H' \in \mathcal{A}_2$, each one of $B_1 \oplus \mathbb{C}^{\ell_2} \cup \{\mathbb{C}^{\ell_1} \oplus H'\}$ and $\mathbb{C}^{\ell_1} \oplus B_2 \cup \{H \oplus \mathbb{C}^{\ell_2}\}$ is linearly independent of $\mathcal{A}_1 \times \mathcal{A}_2$. As well as, if $X_1 \in L_{k_1}(\mathcal{A}_1 \oplus \mathbb{C}^{\ell_2})$, $1 \le k_1 \le \ell_1$, $X_2 \in L_{k_2}(\mathbb{C}^{\ell_1} \oplus \mathcal{A}_2)$, $1 \le k_2 \le \ell_2$, $H \in \mathcal{A}_1$ and $H' \in \mathcal{A}_2$, then; $X = X_1 \cap (\mathbb{C}^{\ell_1} \oplus H') \in L_{k_1+1}(\mathcal{A}_1 \times \mathcal{A}_2)$ and $X' = X_2 \cap (H \oplus \mathbb{C}^{\ell_2}) \in L_{k_2+1}(\mathcal{A}_1 \times \mathcal{A}_2$. Therefore, $T_1 \cap (\mathbb{C}^{\ell_1} \oplus H') \in L_{\ell_1+1}(\mathcal{A}_1 \times \mathcal{A}_2)$ and $T_2 \cap (H \oplus \mathbb{C}^{\ell_2}) \in L_{\ell_2+1}(\mathcal{A}_1 \times \mathcal{A}_2)$, for each $H \in \mathcal{A}_1$ and $H' \in \mathcal{A}_2$. Thus, $\mathcal{A}_{T_1} = \mathcal{A}_1 \oplus \mathbb{C}^{\ell_2}$, $\mathcal{A}_{T_2} = \mathbb{C}^{\ell_1} \oplus \mathcal{A}_2$.

Secondly, by applying lemma (3.4.), the factorization $\pi = (\pi_1, \dots, \pi_\ell)$ be split into two disjoint parts. The first one is the blocks that contain just hyperplanes from $\mathcal{A}_1 \oplus \mathbb{C}^{\ell_2}$and the second one includes blocks of π that contain just hyperplanes from $\mathbb{C}^{\ell_1}\oplus\mathcal{A}_2$. Straightly, one can deduce that for $i = 1,2$, the induced partition $\pi_{T_i} = (\pi_1^i, \dots, \pi_{\ell_i}^i)$ satisfied for $1 \le k \le \ell_i$, $\pi_k^i = \pi_m$, for some $1 \le m \le \ell_1 + \ell_2$ and for $1 \le k_1 \le \ell_1$ and $1 \le k_2 \le \ell_2$, $\pi_{k_1}^1 \ne \pi_{k_2}^2$. ■

3.6. ***Construction:***

Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ be a reducible $(\ell_1 + \ell_2)$-arrangement into a product of ℓ_1-arrangement $\mathcal{A}_1$ and ℓ_2-arrangement $\mathcal{A}_2$. Assume, $\mathcal{A}$ is a factored arrangement with a factorization $\pi = (\pi_1, \dots, \pi_\ell)$. As an application of proposition (3.5.), we can reorder the blocks of π as, $\pi = (\pi_{T_1}, \pi_{T_2}) = (\pi_1^1, \dots, \pi_{\ell_1}^1, \pi_1^2, \dots, \pi_{\ell_2}^2)$. Moreover, for $i = 1,2$, there is a partition $\pi^{\mathcal{A}_i} = (\pi_1^{\mathcal{A}_i}, \dots, \pi_{\ell_i}^{\mathcal{A}_i})$ given as; $H \in \pi_k^{\mathcal{A}_1}$ if, and only if, $H\oplus\mathbb{C}^{\ell_2} \in \pi_k^1$, $1 \le k \le \ell_1$ and $H' \in \pi_k^{\mathcal{A}_2}$ if, and only if, $\mathbb{C}^{\ell_1}\oplus H' \in \pi_k^2$, $1 \le k \le \ell_1$.

Assume $\trianglelefteq$ be the ordering defined on the hyperplanes of $\mathcal{A}_1 \times \mathcal{A}_2$ as given in construction (2.1.) or construction (2.9.), depending on the kind of $\mathcal{A}_1 \times \mathcal{A}_2$, i.e. (if $\mathcal{A}_1 \times \mathcal{A}_2$ is completely factored via $\trianglelefteq$, we will use construction (2.1.)) or (if $\mathcal{A}_1 \times \mathcal{A}_2$ is not completely factored via any ordering can be defined on the hyperplanes of $\mathcal{A}_1 \times \mathcal{A}_2$, we will use construction (2.9.)). Then, define an ordering $\trianglelefteq^{\mathcal{A}_1\times\mathcal{A}_2}$ on the hyperplanes of $\mathcal{A}_1 \times \mathcal{A}_2$ that respect the structure of $\pi = (\pi_{T_1}, \pi_{T_2})$ as follows:

1. For $1 \le i,j \le n_1$, put $H_i^1\oplus\mathbb{C}^{\ell_2} \trianglelefteq^{\mathcal{A}_1\times\mathcal{A}_2} H_j^1\oplus\mathbb{C}^{\ell_2}$ if, and only if, $H_i^1\oplus\mathbb{C}^{\ell_2} \trianglelefteq H_j^1\oplus\mathbb{C}^{\ell_2}$.

2. For $1 \le i,j \le n_2$, put $\mathbb{C}^{\ell_1}\oplus H_i^2 \trianglelefteq^{\mathcal{A}_1\times\mathcal{A}_2} \mathbb{C}^{\ell_1}\oplus H_j^2$ if, and only if, $\mathbb{C}^{\ell_1}\oplus H_i^2 \trianglelefteq \mathbb{C}^{\ell_1}\oplus H_j^2$.

3. For $1 \le i \le n_1$ and $1 \le j \le n_2$, put $H_i^1\oplus\mathbb{C}^{\ell_2} \trianglelefteq^{\mathcal{A}_1\times\mathcal{A}_2} \mathbb{C}^{\ell_1}\oplus H_j^2$.

Indeed, there are orderings; $\trianglelefteq_1$ on the hyperplanes of $\mathcal{A}_1$ that respect the structure of the partition $\pi^{\mathcal{A}_1}$ and $\trianglelefteq_2$ on the hyperplanes of $\mathcal{A}_2$ that respect the structure of the partition $\pi^{\mathcal{A}_2}$, satisfied $\trianglelefteq^{\mathcal{A}_1\times\mathcal{A}_2} = \trianglelefteq_1 \times \trianglelefteq_2$ and defined as follows:

1. For $1 \le i,j \le n_1$, put $H_i^1 \trianglelefteq_1 H_j^1$ if, and only if, $H_i^1\oplus\mathbb{C}^{\ell_2} \trianglelefteq^{\mathcal{A}_1\times\mathcal{A}_2} H_j^1\oplus\mathbb{C}^{\ell_2}$.

2. For $1 \le i,j \le n_2$, put $H_i^2 \trianglelefteq_2 H_j^2$ if, and only if, $\mathbb{C}^{\ell_1}\oplus H_i^2 \trianglelefteq^{\mathcal{A}_1\times\mathcal{A}_2} \mathbb{C}^{\ell_1}\oplus H_j^2$.

3.7. Theorem:

Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ be a reducible $(\ell_1 + \ell_2)$-arrangement into a product of ℓ_1-arrangement $\mathcal{A}_1$ and ℓ_2-arrangement $\mathcal{A}_2$. Then, $\mathcal{A}$ is a completely factored $(\ell_1 + \ell_2)$-arrangement if, and only if, each of $\mathcal{A}_1$ and $\mathcal{A}_2$ is a completely factored arrangement.

Proof: Firstly, assume $\mathcal{A}_1 \times \mathcal{A}_2$ is a completely factored $(\ell_1 + \ell_2)$-arrangement. We need to prove each of $\mathcal{A}_1$ and $\mathcal{A}_2$ is completely factored arrangement. Motivated by our purpose, recall construction (3.6.). So we need to prove, for $i = 1,2$, $\pi^{\mathcal{A}_i}$ is a factorization of $\mathcal{A}_i = \{H_1^i, \dots, H_{n_i}^i\}$ and $\mathcal{A}_i$ is quadratic via $\trianglelefteq_i$. By contrary, assume either $\pi^{\mathcal{A}_1}$, or $\pi^{\mathcal{A}_2}$ is not nice. If $\pi^{\mathcal{A}_1}$ is not factorization of $\mathcal{A}_1$. Then, either $\pi^{\mathcal{A}_1}$ is not independent, or there is $X \in L(\mathcal{A}_1)$ such that the induced partition $\pi_X^{\mathcal{A}_1}$ has no singleton block. As we know, if $B \in S(\pi^{\mathcal{A}_1})$, then $B\oplus\mathbb{C}^{\ell_2} \in S(\pi)$. Assuming that, section B is dependent implies $B\oplus\mathbb{C}^{\ell_2}$ is dependent which contradicts our assumption π is independent. As well as, if $X \in L(\mathcal{A}_1)$ such that the induced partition $\pi_X^{\mathcal{A}_1}$ has no singleton block, then the induced partition $\pi_{X\oplus\mathbb{C}^{\ell_2}} \subseteq \pi_{T_1}$, has no singleton block and that contradicts the nice structure of π. Similarly, assuming $\pi^{\mathcal{A}_2}$ is not nice partition on $\mathcal{A}_2$, leads to a contradiction. Therefore, $\pi^{\mathcal{A}_1}$ and $\pi^{\mathcal{A}_2}$ are nice partitions.

In addition to that, by contrary we will prove $\mathcal{A}_i = \{H_1^i, \dots, H_{n_i}^i\}$ is a completely factored via $\trianglelefteq_i$, for $i = 1,2$. Suppose $\mathcal{A}_1$ is not completely factored via $\trianglelefteq_1$. By applying theorem (1.5.) and corollary (1.6.), there is a subarrangement B of $\mathcal{A}_1$ that contains no rank 2 broken circuit and it is not an NBC base of $\mathcal{A}_1$. According to lemma (2.3.2), $B\oplus\mathbb{C}^{\ell_2}$ is not an NBC base of $\mathcal{A}_1 \times \mathcal{A}_2$. This is a contradiction since $\mathcal{A}_1 \times \mathcal{A}_2$ is quadratic and contains a subarrangement $B\oplus\mathbb{C}^{\ell_2}$ that contains no rank 2 broken circuit and it is not an NBC base of $\mathcal{A}_1 \times \mathcal{A}_2$. Therefore, $\mathcal{A}_1$ is completely factored via $\trianglelefteq_1$. Similarly, deduce $\mathcal{A}_2$ is completely factored via $\trianglelefteq_2$.

Conversely, for $i = 1,2$, assume $\mathcal{A}_i = \{H_1^i, \dots, H_{n_i}^i\}$ be a completely factored ℓ_i-arrangement and recall construction (2.1.) in order to construct a factorization $\pi^i = (\pi_1^i, \dots, \pi_\ell^i)$ of $\mathcal{A}_i$ and an ordering $\trianglelefteq_i$ that emphasize the quadratic property of $\mathcal{A}_i$, i.e. $NBC_k(\mathcal{A}_i) = S_k(\pi^i)$, for $1 \le k \le \ell_i$. Moreover, we recall construction (3.1.) to create the partition $\pi = (\pi^1 \oplus \mathbb{C}^{\ell_2}) \cup (\mathbb{C}^{\ell_1} \oplus \pi^2)$ and the ordering $\trianglelefteq = \trianglelefteq_1 \times \trianglelefteq_2$. So, we wanted π is a facorization and $\mathcal{A}_1 \times \mathcal{A}_2$ is completely factored via $\trianglelefteq$.

By contrary, assume π is not a factorization of $\mathcal{A}_1 \times \mathcal{A}_2$. Then, either π is not independent, or there is $X \in L(\mathcal{A}_1 \times \mathcal{A}_2)$ such that the induced partition π_X has no singleton block.

If $\pi = (\pi^1 \oplus \mathbb{C}^{\ell_2}) \cup (\mathbb{C}^{\ell_1} \oplus \pi^2)$ is not independent, then there is a dependent section $C \in S(\pi)$. So, $C = (C^1 \oplus \mathbb{C}^{\ell_2}) \cup (\mathbb{C}^{\ell_1} \oplus C^2)$, for some $C^1 \in S(\pi^1)$ and $C^2 \in S(\pi^2)$. Thus, either C^1 or C^2 is dependent and that contradicts our assumption that each of π^1 and π^2 is independent. Therefore, π is independent.

In fact, if $X \in L(\mathcal{A}_1 \times \mathcal{A}_2)$ such that the induced partition π_X has no singleton block. Then $X = (X^1 \oplus \mathbb{C}^{\ell_2}) \cap (\mathbb{C}^{\ell_1} \oplus X^2)$, for some $X^1 \in L(\mathcal{A}_1)$ and $X^2 \in L(\mathcal{A}_2)$. From construction (3.1.), $\pi_X = (\pi_{X^1}^1 \oplus \mathbb{C}^{\ell_2}) \cup (\mathbb{C}^{\ell_1} \oplus \pi_{X^2}^2)$, where $\pi_{X^i}^i$is the induced partition of X^i via π^i, $i = 1,2$. Consequently, either $\pi_{X^1}^1$ or $\pi_{X^2}^2$ has no singleton block and that contradicts the fact each of π^1 and π^2 is nice. Therefore, π is a factorization of $\mathcal{A}_1 \times \mathcal{A}_2$.

Now, suppose $\mathcal{A}_1 \times \mathcal{A}_2$ is not completely factored via $\trianglelefteq$. Thus, there is a subarrangement B of $\mathcal{A}_1 \times \mathcal{A}_2$ that contains no rank 2 broken circuit and it is not an NBC base of $\mathcal{A}_1 \times \mathcal{A}_2$. According to lemma (3.2.), $B = (B^1 \oplus \mathbb{C}^{\ell_2}) \cup (\mathbb{C}^{\ell_1} \oplus B^2)$, where $B^1 \subseteq \mathcal{A}_1$, $B^2 \subseteq \mathcal{A}_2$ and either B^1 contains no rank 2 broken circuit of $\mathcal{A}_1$, or B^2 contains no rank 2 broken circuit of $\mathcal{A}_2$, and it is not an NBC base of $\mathcal{A}_2$. This is a contradiction since both of $\mathcal{A}_1$ and $\mathcal{A}_2$ are completely factored. Therefore, $\mathcal{A}_1 \times \mathcal{A}_2$ is a completely factored arrangement via $\trianglelefteq = \trianglelefteq_1 \times \trianglelefteq_2$. ■

3.8. Corollary:

Assume we have the conclusions of theorem (3.7.) and for $i = 1,2$, let $\pi^i = (\pi_1^i, \dots, \pi_\ell^i)$ be the factorization on $\mathcal{A}_i = \{H_1^i, \dots, H_{n_i}^i\}$ with exponent vector $d^i = (d_1^i, \dots, d_{\ell_i}^i)$. Then, for $1 \le k \le \ell = \ell_1 + \ell_2$, $NBC_k(\mathcal{A}_1 \times \mathcal{A}_2) = S_k(\pi)$ and;

$$|NBC_k(\mathcal{A}_1 \times \mathcal{A}_2)| = \sum_{i_1=1}^{\ell-k} \sum_{i_2=i_1+1}^{\ell-k+1} \dots \sum_{i_k=i_{k-1}+1}^{\ell} d_{i_1} d_{i_2} \dots d_{i_k} = |S_k(\pi)|;$$

where, for $1 \le j \le \ell_1$, $d_j = d_j^1$ and $\ell_1 + 1 \le j \le \ell_1 + \ell_2$, $d_j = d_{j-\ell_2}^2$.

Proof: This is a direct result to theorem (3.7.). ■

3.9. Corollary :

Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ be a reducible $(\ell_1 + \ell_2)$-arrangement into a product of ℓ_1-arrangement $\mathcal{A}_1$ and ℓ_2-arrangement $\mathcal{A}_2$. Then, $\mathcal{A}$ is a factored $(\ell_1 + \ell_2)$-arrangement that is not completely factored via any ordering can be defined on the hyperplanes of $\mathcal{A}$ if, and only if, either $\mathcal{A}_1$ or $\mathcal{A}_2$ is not completely factored arrangement via any ordering on its hyperplanes.

Proof: Firstly, assume $\mathcal{A}_1 \times \mathcal{A}_2$ is not completely factored $(\ell_1 + \ell_2)$-arrangement via any ordering can be defined on its hyperplanes. Recall construction (3.6.). So we need to prove, for $i = 1,2$, $\pi^{\mathcal{A}_i}$ is a factorization and either $\mathcal{A}_1$ or $\mathcal{A}_2$ is not completely factored via any ordering on its hyperplanes. Similar to our proof in theorem (3.7.), π exhibits a factorization analogue of $\pi^{\mathcal{A}_1}$ and $\pi^{\mathcal{A}_2}$ on $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively. By contrary, if we assume each of $\mathcal{A}_1$ and $\mathcal{A}_2$ is completely factored via ordering $\trianglelefteq_1$ and $\trianglelefteq_2$, respectively, then due to theorem (3.7.), $\mathcal{A}_1 \times \mathcal{A}_2$ is completely factored via $\trianglelefteq = \trianglelefteq_1 \times \trianglelefteq_2$ which contradicts our assumption.

Conversely, assume either $\mathcal{A}_1$ or $\mathcal{A}_2$ is a not completely factored arrangement via any ordering on its hyperplanes and without loss of generality, suppose that $\mathcal{A}_1$ is not completely factored via any ordering. Recall construction (2.9.) in order to construct a nice partition $\pi^1 = (\pi_1^1, \dots, \pi_\ell^1)$ and an ordering $\trianglelefteq_1$ that emphasize the not completely factored fashion of $\mathcal{A}_1$, i.e. $NBC_k(\mathcal{A}_1) \ne S_k(\pi^1)$, for some $1 \le k \le \ell_1$. Moreover, we

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recall construction (3.1.) to obtain the partition $\pi = (\pi^1 \oplus \mathbb{C}^{\ell_2}) \cup (\mathbb{C}^{\ell_1} \oplus \pi^2)$ and the ordering $\trianglelefteq = \trianglelefteq_1 \times \trianglelefteq_2$. So, we wanted π is nice and neither $\trianglelefteq$, nor any other ordering on the hyperplanes of $\mathcal{A}_1 \times \mathcal{A}_2$ can produce a structure on $\mathcal{A}_1 \times \mathcal{A}_2$ as a completely factored arrangement. From theorem (3.7.), π is nice. Furthermore, if we assume by contrary that $\mathcal{A}_1 \times \mathcal{A}_2$ is completely factored via $\trianglelefteq$, then due to theorem (3.7.), each of $\mathcal{A}_1$ and $\mathcal{A}_2$ is completely factored via the ordering $\trianglelefteq_1$ and $\trianglelefteq_2$ respectively, which contradicts our assumption that $\mathcal{A}_1$ is not completely factored via any ordering. Therefore, our claim is hold. ■

We mentioned that, the following result was firstly given in [7]

3.10. Corollary:

For $i = 1,2$, let $\mathcal{A}_i = \{H_1^i, \dots, H_{n_i}^i\}$ be an ℓ_i-arrangement. Then, $\mathcal{A}_1 \times \mathcal{A}_2$ is a factored $\ell_1 + \ell_2$-arrangement if, and only if, each of $\mathcal{A}_1$ and $\mathcal{A}_2$ is a factored arrangement.

Proof: This is a direct result to construction (3.6.) and theorem (3.7.). ■

3.11. Proposition:

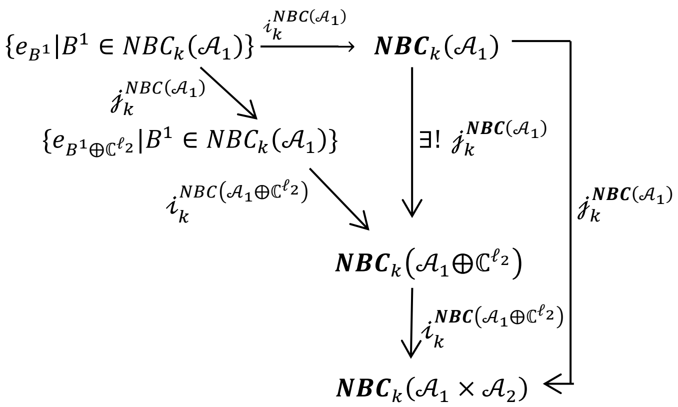
Suppose we have the assumptions of the constructions (3.1.) and (3.6.). Then the acyclic broken circuit complexes $(\boldsymbol{NBC}_*(\mathcal{A}_1), \partial_*^{NBC(\mathcal{A}_1)})$ and $(\boldsymbol{NBC}_*(\mathcal{A}_2), \partial_*^{NBC(\mathcal{A}_2)})$ can be embedded of the O-S complex $(\boldsymbol{A}_*(\mathcal{A}_1 \times \mathcal{A}_2), \partial_*^{A(\mathcal{A}_1 \times \mathcal{A}_2)})$.

Proof: Due construction (3.1.), we can follow our claim as:

1. For $1 \le k \le \ell_1$, the one to one mapping (inclusion);

$$j_k^{NBC(\mathcal{A}_1)} : NBC_k(\mathcal{A}_1) \to NBC_k(\mathcal{A}_1 \oplus \mathbb{C}^{\ell_2});$$

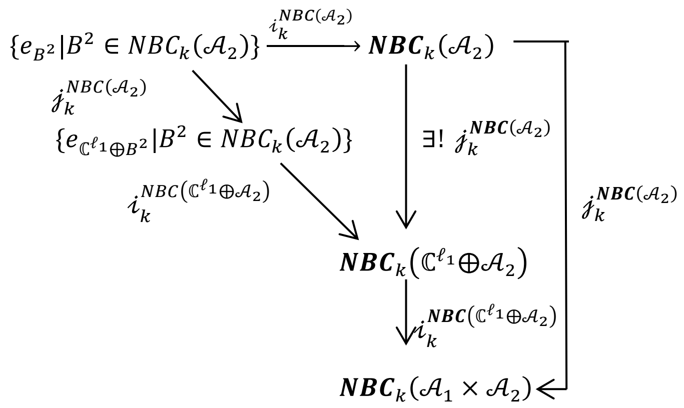
that defined as, $j_k^{NBC(\mathcal{A}_1)}(B^1) = B^1 \oplus \mathbb{C}^{\ell_2}$, for $B^1 \in NBC_k(\mathcal{A}_1)$, embedding the broken circuit module $\boldsymbol{NBC}_k(\mathcal{A}_1)$ of $\boldsymbol{NBC}_k(\mathcal{A}_1 \times \mathcal{A}_2)$ by unique K-monomorphism $\jmath_k^{\boldsymbol{NBC}(\mathcal{A}_1)} : \boldsymbol{NBC}_k(\mathcal{A}_1) \to \boldsymbol{NBC}_k(\mathcal{A}_1 \times \mathcal{A}_2)$ that extends the following assignment:



Therefore, the acyclic broken circuit complex $(\boldsymbol{NBC}_*(\mathcal{A}_1), \partial_*^{NBC(\mathcal{A}_1)})$ can be embedded of the O-S complex $(\boldsymbol{A}_*(\mathcal{A}_1 \times \mathcal{A}_2), \partial_*^{A(\mathcal{A}_1 \times \mathcal{A}_2)})$ by injective K-chain mapping;

$$\psi_*^{\mathcal{A}_1 \times \mathcal{A}_2} \circ \jmath_*^{\boldsymbol{NBC}(\mathcal{A}_1)} : (\boldsymbol{NBC}_*(\mathcal{A}_1), \partial_*^{NBC(\mathcal{A}_1)}) \to (\boldsymbol{A}_*(\mathcal{A}_1 \times \mathcal{A}_2), \partial_*^{A(\mathcal{A}_1 \times \mathcal{A}_2)}).$$

2. For $1 \le k \le \ell_2$, the one to one mapping (inclusion), $j_k^{NBC(\mathcal{A}_2)}: NBC_k(\mathcal{A}_2) \to NBC_k(\mathbb{C}^{\ell_1} \oplus \mathcal{A}_2)$, defined as, $j_k^{NBC(\mathcal{A}_2)}(B^2) = \mathbb{C}^{\ell_1} \oplus B^2$, for $B^2 \in NBC_k(\mathcal{A}_2)$, embedding the broken circuit module $\boldsymbol{NBC}_k(\mathcal{A}_2)$ of $\boldsymbol{NBC}_k(\mathcal{A}_1 \times \mathcal{A}_2)$ by unique K-monomorphism $\jmath_k^{\boldsymbol{NBC}(\mathcal{A}_2)}: \boldsymbol{NBC}_k(\mathcal{A}_2) \to \boldsymbol{NBC}_k(\mathcal{A}_1 \times \mathcal{A}_2)$ that extends the following assignment:



Therefore, the acyclic broken circuit complex $(\boldsymbol{NBC}_*(\mathcal{A}_2), \partial_*^{NBC(\mathcal{A}_2)})$ can be embedded of the O-S complex $(\boldsymbol{A}_*(\mathcal{A}_1 \times \mathcal{A}_2), \partial_*^{A(\mathcal{A}_1 \times \mathcal{A}_2)})$ by injective K-chain mapping;

$$\psi_*^{\mathcal{A}_1 \times \mathcal{A}_2} \circ \jmath_*^{\boldsymbol{NBC}(\mathcal{A}_2)}: (\boldsymbol{NBC}_*(\mathcal{A}_2), \partial_*^{NBC(\mathcal{A}_2)}) \to (\boldsymbol{A}_*(\mathcal{A}_1 \times \mathcal{A}_2), \partial_*^{A(\mathcal{A}_1 \times \mathcal{A}_2)}). \; \blacksquare$$

3.12. Theorem:

Suppose we have the assumptions of the constructions (3.1.) and (3.6.). Then, the complexes;

$$\left(\boldsymbol{A}_*(\mathcal{A}_1 \times \mathcal{A}_2), \partial_*^{A(\mathcal{A}_1 \times \mathcal{A}_2)}\right), \left(\boldsymbol{NBC}_*(\mathcal{A}_1 \times \mathcal{A}_2), \partial_*^{NBC(\mathcal{A}_1 \times \mathcal{A}_2)}\right) \text{ and}$$

$$\left(\boldsymbol{NBC}_*(\mathcal{A}_1) \oplus \boldsymbol{NBC}_*(\mathcal{A}_2), \partial_*^{NBC(\mathcal{A}_1) \oplus NBC(\mathcal{A}_2)}\right) \text{ are isomorphic, i.e.;}$$

$$\boldsymbol{A}_*(\mathcal{A}_1 \times \mathcal{A}_2) \cong \boldsymbol{NBC}_*(\mathcal{A}_1 \times \mathcal{A}_2) \cong \boldsymbol{NBC}_*(\mathcal{A}_1) \oplus \boldsymbol{NBC}_*(\mathcal{A}_2) = \underbrace{\sum_{k_1=0}^{\ell_1} \sum_{k_2=0}^{\ell_2}}_{0 \le k_1 + k_2 = k \le \ell_1 + \ell_2} \boldsymbol{NBC}_{k_1}(\mathcal{A}_1) \oplus \boldsymbol{NBC}_{k_2}(\mathcal{A}_2)$$

where $\boldsymbol{NBC}_*(\mathcal{A}_1) \oplus \boldsymbol{NBC}_*(\mathcal{A}_2)$ is the external direct sum of $\boldsymbol{NBC}_*(\mathcal{A}_1)$ and $\boldsymbol{NBC}_*(\mathcal{A}_2)$.

Proof: Indeed, for $1 \le k \le \ell_1 + \ell_2$, if $B^1 \in NBC_{k_1}(\mathcal{A}_1)$ and $B^2 \in NBC_{k_2}(\mathcal{A}_2)$, for some $1 \le k_1 \le \ell_1$ and $1 \le k_2 \le \ell_2$ such that $k_1 + k_2 = k$, then, $e_{B^1 \times B^2} = e_{B^1 \oplus \mathbb{C}^{\ell_2}} e_{\mathbb{C}^{\ell_1} \oplus B^2} \in E_k(\mathcal{A}_1 \times \mathcal{A}_2)$ is a homogeneous monomial of degree k. Furthermore, due to construction (3.1.), for $0 < k \le \ell_1 + \ell_2$;

$$\boldsymbol{NBC}_k(\mathcal{A}_1 \times \mathcal{A}_2) = \left\langle e_{B^1 \times B^2}, \quad \begin{matrix} B^1 \in NBC_{k_1}(\mathcal{A}_1), \; 0 \le k_1 \le \ell_1 \\ B^2 \in NBC_{k_2}(\mathcal{A}_2), \; 0 \le k_2 \le \ell_2 \\ k_1 + k_2 = k \end{matrix} \right\rangle.$$

Accordingly, for $1 \le k \le \ell_1 + \ell_2$, the external direct sum;

$$\left(\boldsymbol{NBC}(\mathcal{A}_1) \oplus \boldsymbol{NBC}(\mathcal{A}_2)\right)_{\boldsymbol{k}} = \sum_{\substack{0 \le k_1 \le \ell_1 \\ 0 \le k_2 \le \ell_2 \\ k_1 + k_2 = k}} \boldsymbol{NBC}_{k_1}(\mathcal{A}_1) \oplus \boldsymbol{NBC}_{k_2}(\mathcal{A}_2);$$

can be embedded of $\boldsymbol{NBC}_k(\mathcal{A}_1 \times \mathcal{A}_2)$ as the internal direct sum;

$$\sum_{\substack{0\le k_1\le \ell_1\\ 0\le k_2\le \ell_2\\ k_1+k_2=k}} \boldsymbol{NBC}_{k_1}(\mathcal{A}_1 \oplus \mathbb{C}^{\ell_2}) \oplus \boldsymbol{NBC}_{k_2}(\mathbb{C}^{\ell_1} \oplus \mathcal{A}_2) \text{ by a } K\text{-isomorphism;}$$

$$j_k^{\boldsymbol{NBC}(\mathcal{A}_1)\oplus \boldsymbol{NBC}(\mathcal{A}_2)} : \left(\boldsymbol{NBC}(\mathcal{A}_1) \oplus \boldsymbol{NBC}(\mathcal{A}_2)\right)_{\boldsymbol{k}} \to \boldsymbol{NBC}_k(\mathcal{A}_1 \times \mathcal{A}_2);$$

defined as follows;

$$j_k^{\boldsymbol{NBC}(\mathcal{A}_1)\oplus \boldsymbol{NBC}(\mathcal{A}_2)}\left(e_{B^1}, e_{\emptyset_{\ell_2}}\right) = e_{B^1\oplus \mathbb{C}^{\ell_2}}, \text{ for } B^1 \in NBC_k(\mathcal{A}_1);$$

$$j_k^{\boldsymbol{NBC}(\mathcal{A}_1)\oplus \boldsymbol{NBC}(\mathcal{A}_2)}\left(e_{\emptyset_{\ell_1}}, e_{B^2}\right) = e_{\mathbb{C}^{\ell_1}\oplus B^2}, \text{ for } B^2 \in NBC_k(\mathcal{A}_2) \text{ and;}$$

$$j_k^{\boldsymbol{NBC}(\mathcal{A}_1)\oplus \boldsymbol{NBC}(\mathcal{A}_2)}(e_{B^1}, e_{B^2}) = e_{B^1\oplus \mathbb{C}^{\ell_2}} e_{\mathbb{C}^{\ell_1}\oplus B^2} = e_{B^1 \times B^2}, \text{ for } B^1 \in NBC_{k_1}(\mathcal{A}_1),\ B^2 \in NBC_{k_2}(\mathcal{A}_2) \text{ and } k_1 + k_2 = k.$$

Therefore, the complex $(\boldsymbol{NBC}_*(\mathcal{A}_1) \oplus \boldsymbol{NBC}_*(\mathcal{A}_2), \partial_*^{NBC(\mathcal{A}_1)\oplus NBC(\mathcal{A}_2)})$ is isomorphic to the O-S complex $(\boldsymbol{A}_*(\mathcal{A}_1 \times \mathcal{A}_2), \partial_*^{A(\mathcal{A}_1\times\mathcal{A}_2)})$ by the K-chain isomorphism mapping;

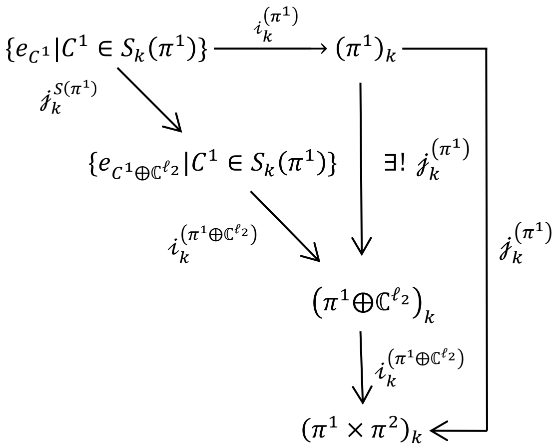
$$\psi_*^{\mathcal{A}_1\times\mathcal{A}_2} \circ j_*^{\boldsymbol{NBC}_*(\mathcal{A}_1)\oplus \boldsymbol{NBC}_*(\mathcal{A}_2)} : \boldsymbol{NBC}_*(\mathcal{A}_1) \oplus \boldsymbol{NBC}_*(\mathcal{A}_2) \to \boldsymbol{A}_*(\mathcal{A}_1 \times \mathcal{A}_2). \ \blacksquare$$

3.13. Proposition:

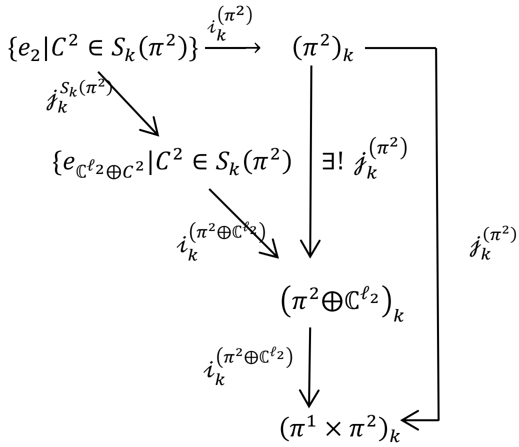
Suppose we have the assumptions of the constructions (3.1.) and (3.6.). Then the acyclic partition complexes $((\pi^1)_*, \partial_*^{\pi^1})$ and $((\pi^2)_*, \partial_*^{\pi^2})$ can be embedded of the partition complex $((\pi^1 \times \pi^2)_*, \partial_*^{\pi^1\times\pi^2})$.

Proof: Suppose we have the assumptions of construction (3.1.). Then:

1. For $1 \le k \le \ell_1$, the one to one mapping (inclusion) $j_k^{S(\pi^1)} : S_k(\pi^1) \to S_k(\pi^1 \times \pi^2)$, that defined as, $j_k^{S(\pi^1)}(C^1) = C^1 \oplus \mathbb{C}^{\ell_2}$, for $C^1 \in S_k(\pi^1)$, embedding the partition module $(\pi^1)_k$ of $(\pi^1 \times \pi^2)_k$ by the unique K- linear monomorphism $j_k^{(\pi^1)}$: $(\pi^1)_k \to (\pi^1 \times \pi^2)_k$ that extends the following assignments:



2. For $1 \le k \le \ell_2$, the one to one correspondence, $j_k^{S(\pi^2)}: S_k(\pi^2) \to S_k(\pi^1 \times \pi^2)$ that defined as, $j_k^{S(\pi^2)}(C^2) = \mathbb{C}^{\ell_2} \oplus C^2$, for $C^2 \in S_k(\pi^2)$, embedding the partition module $(\pi^2)_k$ of $(\pi^1 \times \pi^2)_k$ by the unique K- linear monomorphism $\jmath_k^{(\pi^2)}: (\pi^2)_k \to (\pi^1 \times \pi^2)_k$ that extends the following assignments:



■

3.14. Theorem:

Suppose we have the assumptions of construction (3.1.). Then;

$$(\pi^1 \times \pi^2)_* \cong (\pi^1)_* \otimes (\pi^2)_* = \sum_{0 \le k \le \ell_1 + \ell_2} \left((\pi^1) \otimes (\pi^2)\right)_k = \underbrace{\sum_{k_1=0}^{\ell_1} \sum_{k_2=0}^{\ell_2}}_{0 \le k_1 + k_2 = k \le \ell_1 + \ell_2} (\pi^1)_{k_1} \otimes (\pi^2)_{k_2};$$

i.e. the chain complexes $((\pi^1)_* \otimes (\pi^2)_*, \partial_*^{(\pi^1) \otimes (\pi^2)})$ and $((\pi^1 \times \pi^2)_*, \partial_*^{\pi^1 \times \pi^2})$ are isomorphic.

Proof: We are emphasizing that, for $1 \le k \le \ell_1 + \ell_2$, if $C^1 \in S_{k_1}(\pi^1)$ and $C^2 \in S_{k_2}(\pi^2)$, for some $1 \le k_1 \le \ell_1$ and $1 \le k_2 \le \ell_2$ such that $k_1 + k_2 = k$, then;

1. $q_{C^1 \oplus \mathbb{C}^{\ell_2}}$ and $q_{C^1} \otimes q_{\emptyset_{\ell_2}}$ are homogeneous tensors with degree k_1 of $(\pi^1 \times \pi^2)_{k_1}$ and $\left((\pi^1) \otimes (\pi^2)\right)_{k_1}$ respectively.

2. $q_{\mathbb{C}^{\ell_1} \oplus C^2}$ and $q_{\emptyset_{\ell_1}} \otimes q_{C^2}$ are homogeneous tensors with degree k_2 of $(\pi^1 \times \pi^2)_{k_2}$ and $\left((\pi^1) \otimes (\pi^2)\right)_{k_2}$ respectively.

3. $q_{C^1 \times C^2}$ and $q_{C^1} \otimes q_{C^2}$ are homogeneous tensors with degree k of $(\pi^1 \times \pi^2)_k$ and $\left((\pi^1) \otimes (\pi^2)\right)_k$ respectively.

Furthermore, for $0 \le k \le \ell_1 + \ell_2$; $(\pi^1 \times \pi^2)_k = \left\langle q_{C^1 \times C^2}, \begin{array}{c} C^1 \in S_{k_1}(\pi^1),\ 0 \le k_1 \le \ell_1 \\ C^2 \in S_{k_2}(\pi^2),\ 0 \le k_2 \le \ell_2 \\ k_1 + k_2 = k \end{array} \right\rangle$.

Therefore, for $1 \le k \le \ell_1 + \ell_2$, the k^{th} tensor product $\left((\pi^1) \otimes (\pi^2)\right)_k$ is isomorphic to $(\pi^1 \times \pi^2)_k$ by a K-isomorphism, $\jmath_k^{(\pi^1) \otimes (\pi^2)}: \left((\pi^1) \otimes (\pi^2)\right)_k \to (\pi^1 \times \pi^2)_k$ defined as follows;

$$\jmath_k^{(\pi^1) \otimes (\pi^2)}\left(q_{C^1} \otimes q_{\emptyset_{\ell_2}}\right) = q_{C^1 \oplus \mathbb{C}^{\ell_2}}, \text{ for } C^1 \in S_k(\pi^1)\ ;$$

$$j_k^{(\pi^1)\otimes(\pi^2)}\left(q_{\emptyset_{\ell_1}}\otimes q_{C^2}\right) = q_{\mathbb{C}^{\ell_1}\oplus C^2}, \text{ for } C^2 \in S_k(\pi^2) \text{ and;}$$

$$j_k^{(\pi^1)\otimes(\pi^2)}(q_{C^1}\otimes q_{C^2}) = q_{C^1\times C^2}, \text{ for } C^1 \in S_{k_1}(\pi^1),\ \ C^2 \in S_{k_2}(\pi^2) \text{ and } k_1 + k_2 = k.$$

Therefore, the chain complexes $((\pi^1)_*\otimes(\pi^2)_*, \partial_*^{(\pi^1)\otimes(\pi^2)})$ and $((\pi^1 \times \pi^2)_*, \partial_*^{\pi^1\times\pi^2})$ are isomorphic by the bijective K-chain mapping $j_*^{(\pi^1)\otimes(\pi^2)}: (\pi^1)_*\otimes(\pi^2)_* \to (\pi^1 \times \pi^2)_*$.■

3.15. Construction:

Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ be a completely factored $(\ell_1 + \ell_2)$-arrangement of ℓ_1-arrangement $\mathcal{A}_1$ and ℓ_2-arrangement $\mathcal{A}_2$. According to theorem (3.7.), each of $\mathcal{A}_1$ and $\mathcal{A}_2$ are completely factored. So, for $i = 1,2$, let π^i be the factorization of $\mathcal{A}_i$ and let $\trianglelefteq_i$ be an ordering defined on its hyperplanes such that $\mathcal{A}_i$ is completely factored, i.e. $NBC_k(\mathcal{A}_i) = S_k(\pi^i)$, via the ordering $\trianglelefteq_i$ that given in construction (2.1.10), for all $1 \le k \le \ell_i$. Recall the constructions (2.7.) and (3.1.), so we have the following K-chain isomorphism's between acyclic chain complexes;

- $\mathcal{I}_*^1: (\pi^1)_* \to \boldsymbol{NBC}_*(\mathcal{A}_1)$.
- $\mathcal{I}_*^2: (\pi^2)_* \to \boldsymbol{NBC}_*(\mathcal{A}_2)$.
- $\mathcal{I}_*^{1\times2}: (\pi^1 \times \pi^2)_* \to \boldsymbol{NBC}_*(\mathcal{A}_1 \times \mathcal{A}_2)$.
- $\tau_* = j_*^{\boldsymbol{NBC}_*(\mathcal{A}_1)\oplus\boldsymbol{NBC}_*(\mathcal{A}_2)^{-1}} \circ \mathcal{I}_*^{1\times2} \circ j_*^{(\pi^1)\otimes(\pi^2)}: (\pi^1)_*\otimes(\pi^2)_* \to \boldsymbol{NBC}_*(\mathcal{A}_1)\oplus\boldsymbol{NBC}_*(\mathcal{A}_2)$.
- $\kappa_* = \psi_*^{\mathcal{A}_1\times\mathcal{A}_2} \circ j_*^{\boldsymbol{NBC}_*(\mathcal{A}_1)\oplus\boldsymbol{NBC}_*(\mathcal{A}_2)}: \boldsymbol{NBC}_*(\mathcal{A}_1)\oplus\boldsymbol{NBC}_*(\mathcal{A}_2) \to \boldsymbol{A}_*(\mathcal{A}_1 \times \mathcal{A}_2)$.
- $\rho_* = (\kappa_* \circ \tau_*)^{-1}: \boldsymbol{A}_*(\mathcal{A}_1 \times \mathcal{A}_2) \to (\pi^1)_*\otimes(\pi^2)_*$

3.16. Construction:

Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ be a factored $(\ell_1 + \ell_2)$-arrangement of ℓ_1-arrangement $\mathcal{A}_1$ and ℓ_2-arrangement $\mathcal{A}_2$ that is not completely factored. Without loss of generality assume that each of $\mathcal{A}_1$ and $\mathcal{A}_2$ are not completely factored arrangements via any ordering. So, for $i = 1,2$, let π^i be the factorization of $\mathcal{A}_i$ and let $\trianglelefteq_i$ be any ordering defined on the hyperplanes of $\mathcal{A}_i$, i.e. $NBC_k(\mathcal{A}_i) \neq S_k(\pi^i)$, via $\trianglelefteq_i$ for all $1 \le k \le \ell_i$. Recall the constructions (2.16) and (3.1.), so we have the following K-chain isomorphism's between acyclic chain complexes;

- $\mathcal{g}_*^1: (\pi^1)_* \to \boldsymbol{NBC}_*(\mathcal{A}_1)$ and $\mathcal{f}_*^1: \boldsymbol{NBC}_*(\mathcal{A}_1) \to (\pi^1)_*$.
- $\mathcal{g}_*^2: (\pi^2)_* \to \boldsymbol{NBC}_*(\mathcal{A}_2)$ and $\mathcal{f}_*^2: \boldsymbol{NBC}_*(\mathcal{A}_2) \to (\pi^2)_*$.
- $\mathcal{g}_*^{1\times2}: (\pi^1 \times \pi^2)_* \to \boldsymbol{NBC}_*(\mathcal{A}_1 \times \mathcal{A}_2)$ and $\mathcal{f}_*^{1\times2}: \boldsymbol{NBC}_*(\mathcal{A}_1 \times \mathcal{A}_2) \to (\pi^1 \times \pi^2)_*$.
- $\tau_* = j_*^{\boldsymbol{NBC}_*(\mathcal{A}_1)\oplus\boldsymbol{NBC}_*(\mathcal{A}_2)^{-1}} \circ \mathcal{g}_*^{1\times2} \circ j_*^{(\pi^1)\otimes(\pi^2)}: (\pi^1)_*\otimes(\pi^2)_* \to \boldsymbol{NBC}_*(\mathcal{A}_1)\oplus\boldsymbol{NBC}_*(\mathcal{A}_2)$.
- $\kappa_* = \psi_*^{\mathcal{A}_1\times\mathcal{A}_2} \circ j_*^{\boldsymbol{NBC}_*(\mathcal{A}_1)\oplus\boldsymbol{NBC}_*(\mathcal{A}_2)}: \boldsymbol{NBC}_*(\mathcal{A}_1)\oplus\boldsymbol{NBC}_*(\mathcal{A}_2) \to \boldsymbol{A}_*(\mathcal{A}_1 \times \mathcal{A}_2)$.
- $\rho_* = (\kappa_* \circ \tau_*)^{-1}: \boldsymbol{A}_*(\mathcal{A}_1 \times \mathcal{A}_2) \to (\pi^1)_*\otimes(\pi^2)_*$.

4. Illustrations

In this section, we will demonstrate our work by the following examples:

4.1. Corollary:

If $\mathcal{A} = \prod_{i=1}^{\ell} \mathcal{A}_i$ is a reducible 2ℓ-arrangement such that $rk(\mathcal{A}_i) = 2$ for each $1 \le i \le \ell$, then $\mathcal{A}$ is completely factored via an ordering defined on its hyperplanes and $NBC(\prod_{i=1}^{\ell} \mathcal{A}_i) = S(\prod_{i=1}^{\ell} \pi^i)$ and $\boldsymbol{A}_*(\mathcal{A}) = \boldsymbol{A}_*(\prod_{i=1}^{\ell} \mathcal{A}_i) \cong \sum_{i=0}^{\ell} \boldsymbol{NBC}_*(\mathcal{A}_i) \cong \bigotimes_{i=1}^{\ell} (\pi^i)_*$.

Proof: For $1 \le i \le \ell$, $\mathcal{A}_i = \{H_1^i, \dots, H_{n_i}^i\}$ is a 2-arrangement, hence it is a factored arrangement with factorization $\pi^i = (\{H_1^i\}, \{H_2^i, \dots, H_{n_i}^i\})$ with exponent vector $d^i = (1, n_i - 1)$. Via this ordering given on the hyperplanes of $\mathcal{A}_i$, the arrangement $\mathcal{A}_i$ will be completely factored arrangement since it is satisfied $NBC\ (\mathcal{A}_i) = S(\pi^i)$. By applying theorem (3.7.) inductively, we will be hold of $\mathcal{A} = \prod_{i=1}^{\ell} \mathcal{A}_i$ is a completely factored arrangement with factorization, $\pi = \prod_{i=1}^{\ell} \pi^i = (\pi^1 \oplus \mathbb{C}^{2\ell-2}, \mathbb{C}^2 \oplus \pi^2 \oplus \mathbb{C}^{2\ell-4}, \dots, \mathbb{C}^{2\ell-2} \oplus \pi^\ell)$, where for $1 \le i \le \ell$;

$$\mathbb{C}^{2i-2} \oplus \pi^i \oplus \mathbb{C}^{2\ell-2i} = \begin{pmatrix} \{\mathbb{C}^{2i-2} \oplus H_1^i \oplus \mathbb{C}^{2\ell-2i}\}, \\ \{\mathbb{C}^{2i-2} \oplus H_2^i \oplus \mathbb{C}^{2\ell-2i}, \dots, \mathbb{C}^{2i-2} \oplus H_{n_i}^i \oplus \mathbb{C}^{2\ell-2i}\} \end{pmatrix};$$

with exponent vector is $d = (1, n_1 - 1, 1, n_2 - 1, \dots, 1, n_\ell - 1)$ and it is satisfied $NBC\ (\mathcal{A}) = S(\pi)$ and due construction (3.15) our claim is hold. ■

4.2. Corollary:

If $\mathcal{A} = \prod_{i=1}^{n} \mathcal{A}_i$ is a reducible arrangement such that $rk(\mathcal{A}_i) = 1 \text{ or } 2$ for each $1 \le i \le n$, then $\mathcal{A}$ is a completely factored arrangement via an ordering defined on its hyperplanes and;

$$\boldsymbol{A}_*(\mathcal{A}) = \boldsymbol{A}_*\left(\prod_{i=1}^{\ell} \mathcal{A}_i\right) \cong \bigoplus_{i=1}^{\ell} \boldsymbol{NBC}_*(\mathcal{A}_i) \cong \bigotimes_{i=1}^{\ell} (\pi^i)_*.$$

Proof: We claim that if $\mathcal{A}_i$ is a 1-arrangement for some $1 \le i \le n$, then $\mathcal{A}_i$ can considered to be completely factored arrangement. Indeed, $Q(\mathcal{A}_i) = x$ and its factorization assumed to be $\pi^i = (\{H_1^i\})$ with exponent vector $d^i = (1)$, where $H_1^{+i} = ker(x)$. As well as if $\mathcal{A}_i = \{H_1^i, \dots, H_{n_i}^i\}$ is a 2-arrangement for some $1 \le i \le n$, then it is a completely factored arrangement with factorization $\pi^i = (\{H_1^i\}, \{H_2^i, \dots, H_{n_i}^i\})$ with exponent vector $d^i = (1, n_i - 1)$ as we explained in proof of corollary (4.1.). By applying theorem (3.7.) inductively, we will be hold of $\mathcal{A} = \prod_{i=1}^{\ell} \mathcal{A}_i$ is a completely factored arrangement. Due construction (3.15.), the proof is complete. ■

4.3. Corollary:

Every reducible 3-arrangement is a completely factored arrangement via an ordering defined on its hyperplanes.

Proof: It is clear that, if $\mathcal{A}$ is reducible 3-arrangement such that $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3$ is a product of three 1-arrangements $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$, then $\mathcal{A}$ is a Boolean 3-arrangement which is a completely factored arrangement as we claimed. On the other hand, if $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ is a product of two arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ of ranks such that $rk(\mathcal{A}_i) = 1 \text{ or } 2$, then as a direct application of corollary (4.2) $\mathcal{A}$ is a completely factored arrangement. This is our claim. ■

4.4. Corollary:

If $\mathcal{A} = \prod_{i=1}^{\ell} \mathcal{A}_k$ is a reducible arrangement such that each of $\mathcal{A}_k$ is the complexification Coxeter arrangement, either of type $(A_n\ , n \ge 3)$ or $(B_n\ ;\ n \ge 3)$ for each $1 \le k \le \ell$, then $\mathcal{A}$ is a completely factored arrangement via an ordering defined on its hyperplanes and;

$$\boldsymbol{A}_*(\mathcal{A}) = \boldsymbol{A}_*\left(\prod_{i=1}^{\ell} \mathcal{A}_i\right) \cong \bigoplus_{i=1}^{\ell} \boldsymbol{NBC}_*(\mathcal{A}_i) \cong \bigotimes_{i=1}^{\ell} (\pi^i)_*.$$

Proof: Recalling, the defining polynomial of a Coxeter arrangement of type $(A_n\ , n \ge 3)$ or $(B_n\ ;\ n \ge 3)$ and its factorization from [2];

- ❖ By the complexification of Coxeter arrangement of type A_n, we mean the Braid arrangement $\mathcal{A}_k = \mathcal{A}(A_n) = \{H_{i-j} | 1 \le i < j \le n+1\}$ of $\mathbb{C}^{n+1}$, where, $H_{i-j} = \{(x_1, \dots, x_{n+1}) | x_i = x_j\}$, $1 \le i < j \le n+1$, i.e. its defaning polynomial is; $Q(\mathcal{A}(A_n)) = \prod_{1 \le i < j \le n+1} (x_i - x_j)$. It is known that, $\mathcal{A}(A_n)$ is a non-essential supersolvable arrangement contain $\frac{n(n-1)}{2}$ hyperplanes and it has a factorization, $\pi^{\mathcal{A}(A_n)} = (\{H_{1-2}\}, \{H_{1-3}, H_{2-3}\}, \dots, \{H_{1-(n+1)}, \dots, H_{(n)-(n+1)}\})$ with exponent vector $d^k = (d_1, \dots, d_n), = (1, 2, \dots, n)$. Thus, we have $NBC(\mathcal{A}(A_n)) \equiv S_\pi(\mathcal{A}(A_n))$, via the ordering defined on the hyperplanes of $\mathcal{A}(A_n)$ as given in construction (2.1.), i.e.;

$$\pi^k = \pi^{\mathcal{A}(A_n)} = \left(\{H_1^k\},\{H_2^k,H_3^k\},\dots.\{H^k_{\left(\frac{n(n-1)}{2}-n\right)},\dots,H^k_{\left(\frac{n(n-1)}{2}\right)}\}\right);$$

where, H_j^k is the j^{th} hyperplane of $\mathcal{A}(A_n)$, $1 \le j \le \frac{n(n-1)}{2}$.

- By the complexification of Coxeter arrangement of type B_n, we mean the arrangement; $\mathcal{A}_k = \mathcal{A}(B_n) = \{H_i | 1 \le i \le n\} \cup \{H_{i-j} | 1 \le i < j \le n\} \cup \{H_{i+j} | 1 \le i < j \le n\}$ of $\mathbb{C}^n$, where;

$$H_i = \{(x_1,\dots,x_{n+1}) |\ x_i = 0\},\ 1 \le i \le n;$$
$$H_{i-j} = \{(x_1,\dots,x_{n+1}) |\ x_i = x_j\},\ 1 \le i < j \le n;$$
$$H_{i+j} = \{(x_1,\dots,x_{n+1}) |\ x_i = -x_j\},\ 1 \le i < j \le n;$$

i.e. its defaning polynomial is, $Q(\mathcal{A}(B_n)) = x_1x_2\dots x_n \prod_{1\le i<j\le n}(x_i - x_j)(x_i + x_j)$. Its known that, $\mathcal{A}(B_n)$ is an essential supersolvable arrangement contained n^2 hyperplanes and it has a factorization;

$$\pi^{\mathcal{A}(B_n)} = \begin{pmatrix} \{H_1\},\{H_2,H_{1-2},H_{1+2}\},\{H_3,H_{1-3},H_{1+3},H_{2-3},H_{2+3}\},\dots, \\ \{H_{(n)},H_{1-(n)},H_{1+(n)},H_{2-(n)},H_{2+(n)},\dots,H_{(n-1)-(n)},H_{(n-1)+(n)}\} \end{pmatrix};$$

with exponent vector $d^k = (d_1,\dots,d_n), = (1,3,5,\dots,2n-1)$. Thus, we have $NBC(\mathcal{A}(B_n)) \equiv S_\pi(\mathcal{A}(B_n))$, via the ordered that defined on the hyperplanes of $\mathcal{A}(B_n)$as given in construction (2.1.), i.e. $\pi^k = \pi^{\mathcal{A}(B_n)} = (\{H_1^k\},\{H_2^k,H_3^k,H_4^k\},\dots,\{H^k_{n^2-2n+1},\dots,H^k_{n^2}\})$, where, H_j^k is the j^{th} hyperplane of $\mathcal{A}(B_n)$, $1 \le j \le n^2$.

Now, without loss of generality assume $\ell = 2$, i.e. $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$. As a direct application of theorem (3.7.), we have the following:

i. If $\mathcal{A}_1$ and $\mathcal{A}_2$ of type A_{n_1} and A_{n_2} respectively, then $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ is a completely factored arrangement with factorization;

$$\pi = \pi^1 \times \pi^2 = (\pi^1 \oplus \mathbb{C}^{n_2}) \cup (\mathbb{C}^{n_1} \oplus \pi^2)$$
$$= \begin{pmatrix} \{H_1^1\oplus\mathbb{C}^{n_2}\},\{H_2^1\oplus\mathbb{C}^{n_2},H_3^1\oplus\mathbb{C}^{n_2}\},\dots,\left\{H^1_{\left(\frac{n_1(n_1-1)}{2}-n_1\right)}\oplus\mathbb{C}^{n_2},\dots,H^1_{\left(\frac{n_1(n_1-1)}{2}\right)}\oplus\mathbb{C}^{n_2}\right\}, \\ \{\mathbb{C}^{n_1}\oplus H_1^2\},\{\mathbb{C}^{n_1}\oplus H_2^2,\mathbb{C}^{n_1}\oplus H_3^2\},\dots,\left\{\mathbb{C}^{n_1}\oplus H^2_{\left(\frac{n_2(n_2-1)}{2}-n_2\right)},\dots,\mathbb{C}^{n_1}\oplus H^2_{\left(\frac{n_2(n_2-1)}{2}\right)}\right\} \end{pmatrix}$$

and exponent vector $d = (1,2,\dots,n_1,1,2,\dots,n_2)$ and length $\ell(\pi) = n_1 + n_2$.

ii. If $\mathcal{A}_1$ and $\mathcal{A}_2$ of type A_{n_1} and B_{n_2} respectively, then $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ is a completely factored arrangement with factorization;

$$\pi = \pi^1 \times \pi^2 = (\pi^1 \oplus \mathbb{C}^{n_2}) \cup (\mathbb{C}^{n_1} \oplus \pi^2)$$
$$= \begin{pmatrix} \{H_1^1\oplus\mathbb{C}^{n_2}\},\{H_2^1\oplus\mathbb{C}^{n_2},H_3^1\oplus\mathbb{C}^{n_2}\},\dots,\left\{H^1_{\left(\frac{n_1(n_1-1)}{2}-n_1\right)}\oplus\mathbb{C}^{n_2},\dots,H^1_{\left(\frac{n_1(n_1-1)}{2}\right)}\oplus\mathbb{C}^{n_2}\right\}, \\ \{\mathbb{C}^{n_1}\oplus H_1^2\},\{\mathbb{C}^{n_1}\oplus H_2^2,\mathbb{C}^{n_1}\oplus H_3^2,\mathbb{C}^{n_1}\oplus H_4^2\},\dots,\{\mathbb{C}^{n_1}\oplus H^2_{(n_2)^2-2n_2+1},\dots,\mathbb{C}^{n_1}\oplus H^2_{(n_2)^2}\} \end{pmatrix}$$

and exponent vector $d = (1,2,\dots,n_1,1,3,\dots,2n_2-1)$ and length $\ell(\pi) = n_1 + n_2$.

iii. If $\mathcal{A}_1$ and $\mathcal{A}_2$ of type B_{n_1} and B_{n_2} respectively, then $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ is a completely factored arrangement with factorization;

$$\pi = \pi^1 \times \pi^2 = (\pi^1 \oplus \mathbb{C}^{n_2}) \cup (\mathbb{C}^{n_1} \oplus \pi^2)$$
$$= \begin{pmatrix} \{H_1^1\oplus\mathbb{C}^{n_2}\},\{H_2^1\oplus\mathbb{C}^{n_2},H_3^1\oplus\mathbb{C}^{n_2},H_4^1\oplus\mathbb{C}^{n_2}\},\dots,\left\{H^1_{(n_1)^2-2n_1+1}\oplus\mathbb{C}^{n_2},\dots,H^1_{(n_1)^2}\oplus\mathbb{C}^{n_2}\right\}, \\ \{\mathbb{C}^{n_1}\oplus H_1^2\},\{\mathbb{C}^{n_1}\oplus H_2^2,\mathbb{C}^{n_1}\oplus H_3^2,\mathbb{C}^{n_1}\oplus H_4^2\},\dots,\{\mathbb{C}^{n_1}\oplus H^2_{(n_2)^2-2n_2+1},\dots,\mathbb{C}^{n_1}\oplus H^2_{(n_2)^2}\} \end{pmatrix}$$

and exponent vector $d^k = (1,3,\dots,2n_1-1,1,3,\dots,2n_2-1)$ and length $\ell(\pi) = n_1 + n_2$. ■

4.5. Example:

Let $\mathcal{A} = \{H_1,\dots,H_7\}$ be a 3-arrangemant that has the defining polynomial;

$$Q(\mathcal{A}) = x_1x_2x_3(x_1+x_2+x_3)(x_1+x_2-x_3)(x_1-x_2+x_3)(x_1-x_2-x_3).$$

Via the ordering that given on the one degree polynomials of $Q(\mathcal{A})$, put;

$$H'_i = Ker(\alpha_{H_i}) = \{(x_1,x_2,x_3) \in | \alpha_{H'_i}(x_1,x_2,x_3) = 0\}\ ,\ \ 1 \le i \le 7.$$

Deduce that the partition $\pi = (\pi_1,\pi_2,\pi_3) = (\{H'_3\},\{H'_1,H'_6,H'_7\},\{H'_2,H'_4,H'_5\})$ of $\mathcal{A}$ is a factorization that inherits $\mathcal{A}$ a structure as a factored arrangement. In fact, every 3-section of π is independent. Then π is independent and for each $X \in L(\mathcal{A})$, there is a singleton block of $\mathcal{A}_X$. Now, we reordered the hyperplanes of $\mathcal{A}$ by using the order given in construction (2.9.). Thus;

$$\pi = (\pi_1,\pi_2,\pi_3) = (\{H_1\},\{H_2,H_3,H_4\},\{H_5,H_6,H_7\});$$

such that, $H_1 = ker(x_3)$, $H_2 = ker(x_1)$, $H_3 = ker(x_1-x_2+x_3)$, $H_4 = ker(x_1-x_2-x_3)$, $H_5 = ker(x_2)$, $H_6 = ker(x_1+x_2+x_3)$ and $H_7 = ker(x_1+x_2-x_3)$ and via this order we have;

$$NBC_0(\mathcal{A}) = \{\mathbb{C}^3\},\ NBC_1(\mathcal{A}) = \{\{H_1\},\{H_2\},\{H_3\},\{H_4\},\{H_5\},\{H_6\},\{H_7\}\};$$

$$NBC_2(\mathcal{A}) = \{\{H_1,H_2\},\{H_1,H_3\},\{H_1,H_4\},\{H_1,H_5\},\{H_1,H_6\},\{H_1,H_7\},$$
$$\{H_2,H_3\},\{H_2,H_4\},\{H_2,H_5\},\{H_2,H_6\},\{H_2,H_7\},\{H_3,H_5\},\{H_3,H_6\},\{H_4,H_5\},\{H_4,H_7\}\};$$

$$NBC_3(\mathcal{A}) = \{\{H_1,H_2,H_3\},\{H_1,H_2,H_4\},\{H_1,H_2,H_5\},\{H_1,H_2,H_6\},$$
$$\{H_1,H_2,H_7\},\{H_1,H_3,H_5\},\{H_1,H_3,H_6\},\{H_1,H_4,H_5\},\{H_1,H_4,H_7\}\};$$

$$S_0(\pi) = \{\emptyset_3\},\ S_1(\pi) = NBC_1(\mathcal{A});$$

$$S_2(\pi) = \{\{H_1,H_2\},\{H_1,H_3\},\{H_1,H_4\},\{H_1,H_5\},\{H_1,H_6\},\{H_1,H_7\},\{H_2,H_5\},$$
$$\{H_2,H_6\},\{H_2,H_7\},\{H_3,H_5\},\{H_3,H_6\},\{H_3,H_7\},\{H_4,H_5\},\{H_4,H_6\},\{H_4,H_7\}\};$$

$$S_3(\pi) = \{\{H_1,H_2,H_5\},\{H_1,H_2,H_6\},\{H_1,H_2,H_7\},\{H_1,H_3,H_5\},\{H_1,H_3,H_6\},$$
$$\{H_1,H_3,H_7\},\{H_1,H_4,H_5\},\{H_1,H_4,H_6\},\{H_1,H_4,H_7\}\}.$$

The important points to note here are:

1- Each of $\{H_2,H_3\}$ and $\{H_2,H_4\}$ are $2-NBC$ bases of $\mathcal{A}$, but they are not 2-sections of π. As well as, $\{H_3,H_7\}$ and $\{H_4,H_6\}$ are 2-sections of π, but they are not $2-NBC$ bases of $\mathcal{A}$. By applying theorem (2.14.), we have; $f_2(\{H_2,H_3\}) = \{H_3,H_7\}$ and $f_2(\{H_2,H_4\}) = \{H_4,H_6\}$. Accordingly, $g_2(\{H_3,H_7\}) = \{H_2,H_3\}$ and $g_2(\{H_4,H_6\}) = \{H_2,H_4\}$

2- Each of $\{H_1,H_2,H_3\}$ and $\{H_1,H_2,H_4\}$ are $3-NBC$ bases of $\mathcal{A}$, but they are not 3-sections of π. As well as, $\{H_1,H_3,H_7\}$ and $\{H_1,H_4,H_6\}$ are 3-sections of π, but they are not $3-NBC$ bases of $\mathcal{A}$. By applying theorem (2.14.), we have $f_3(\{H_1,H_2,H_3\}) = \{H_1,H_3,H_7\}$ and $f_3(\{H_1,H_2,H_4\}) = \{H_1,H_4,H_6\}$. Accordingly, $g_3(\{H_1,H_3,H_7\}) = \{H_1,H_2,H_3\}$ and $g_3(\{H_1,H_4,H_6\}) = \{H_1,H_2,H_4\}$.

It is known that the factorization of $\mathcal{A}$ need not to be unique. Definitely, for any factorization of $\mathcal{A}$ and any ordering defined on its hyperplanes, $\mathcal{A}$ is not completely factored arrangement. So, in this example we illustrate a factored arrangement that is not completely factored arrangement. On the other hand,

$$\boldsymbol{NBC}_0(\mathcal{A}) = \langle e_{\mathbb{C}^3}\rangle,\ \boldsymbol{NBC}_1(\mathcal{A}) = \langle e_{H_1},e_{H_2},e_{H_3},e_{H_4},e_{H_5},e_{H_6},e_{H_7}\rangle$$

$$\boldsymbol{NBC}_2(\mathcal{A}) = \left\langle \begin{matrix} e_{\{H_1,H_2\}}, e_{\{H_1,H_3\}}, e_{\{H_1,H_4\}}, e_{\{H_1,H_5\}}, e_{\{H_1,H_6\}}, \\ e_{\{H_1,H_7\}}, e_{\{H_2,H_3\}}, e_{\{H_2,H_4\}}, e_{\{H_2,H_5\}}, e_{\{H_2,H_6\}}, \\ e_{\{H_2,H_7\}}, e_{\{H_3,H_5\}}, e_{\{H_3,H_6\}}, e_{\{H_4,H_5\}}, e_{\{H_4,H_7\}} \end{matrix}\right\rangle,\ \boldsymbol{NBC}_3(\mathcal{A}) = \left\langle \begin{matrix} e_{\{H_1,H_2,H_3\}}, e_{\{H_1,H_2,H_4\}}, e_{\{H_1,H_2,H_5\}}, \\ e_{\{H_1,H_2,H_6\}}, e_{\{H_1,H_2,H_7\}}, e_{\{H_1,H_3,H_5\}}, \\ e_{\{H_1,H_3,H_6\}}, e_{\{H_1,H_4,H_5\}}, e_{\{H_1,H_4,H_7\}} \end{matrix}\right\rangle;$$

$$(\pi)_0 = \langle q_{\{\}}\rangle,\ (\pi)_1 = \langle q_{H_1},q_{H_2},q_{H_3},q_{H_4},q_{H_5},q_{H_6},q_{H_7}\rangle;$$

$$(\pi)_2 = \left\langle \begin{matrix} q_{\{H_1,H_2\}}, q_{\{H_1,H_3\}}, q_{\{H_1,H_4\}}, q_{\{H_1,H_5\}}, q_{\{H_1,H_6\}}, \\ q_{\{H_1,H_7\}}, q_{\{H_2,H_5\}}, q_{\{H_2,H_6\}}, q_{\{H_2,H_7\}}, q_{\{H_3,H_5\}}, \\ q_{\{H_3,H_6\}}, q_{\{H_3,H_7\}}, q_{\{H_4,H_5\}}, q_{\{H_4,H_6\}}, q_{\{H_4,H_7\}} \end{matrix}\right\rangle \text{ and } (\pi)_3 = \left\langle \begin{matrix} q_{\{H_1,H_2,H_5\}}, q_{\{H_1,H_2,H_6\}}, q_{\{H_1,H_2,H_7\}}, \\ q_{\{H_1,H_3,H_5\}}, q_{\{H_1,H_3,H_6\}}, q_{\{H_1,H_3,H_7\}}, \\ q_{\{H_1,H_4,H_5\}}, q_{\{H_1,H_4,H_6\}}, q_{\{H_1,H_4,H_7\}} \end{matrix}\right\rangle.$$

By applying theorem (2.17), we have two K-chain isomorphism's, $\mathscr{f}_*: \boldsymbol{NBC}_*(\mathcal{A}) \to (\pi)_*$ and $\mathscr{g}_*: (\pi)_* \to \boldsymbol{NBC}_*(\mathcal{A})$ between acyclic chain complexes that create a connection between two fashions of the O-S algebra of $\mathcal{A}$, a fashion as free submodule of the exterior algebra, and a fashion as a tensor factorization module. In view of this; $\boldsymbol{A}_*(\mathcal{A}) \cong \boldsymbol{NBC}_*(\mathcal{A}) \cong \sum_{0\le k\le 3} \boldsymbol{NBC}_k(\mathcal{A}) \cong (\pi)_* \cong \sum_{0\le k\le 3} (\pi)_k$.

4.6. Example:

Let $\mathcal{A} = \{H_1, \dots, H_{16}\}$ be a 6-arrangemant that has the defining polynomial;

$$Q(\mathcal{A}) = x_1x_2x_3x_4x_5x_6(x_1 + x_2 + x_3)(x_1 + x_2 - x_3)(x_1 - x_2 + x_3)$$
$$(x_1 - x_2 - x_3)(x_4 - x_5)(x_4 + x_5)(x_4 - x_6)(x_4 + x_6)(x_5 - x_6)(x_5 + x_6).$$

It is clear that, $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ is reducible arrangement such that $\mathcal{A}_1$ is the 3-arrangement given in example (4.5.) above and $\mathcal{A}_2$ is the complexification of Coxeter arrangement of type B_3 (recall corollary (4.4.)). By applying corollary (3.9.), the arrangement $\mathcal{A}$ is a factored arrangement that not completely factored with factorization:

$$\pi = (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6) = (\{H_1\}, \{H_2, H_3, H_4\}, \{H_5, H_6, H_7\}, \{H_8\}, \{H_9, H_{10}, H_{11}\}, \{H_{12}, H_{13}, H_{14}, H_{15}, H_{16}\})$$

where;

$$H_1 = ker(x_3),\ H_2 = ker(x_1),\ H_3 = ker(x_1 - x_2 + x_3);$$
$$H_4 = ker(x_1 - x_2 - x_3),\ H_5 = ker(x_2),\ H_6 = ker(x_1 + x_2 + x_3);$$
$$H_7 = ker(x_1 + x_2 - x_3)\ ,\ H_8 = ker(x_4),\ H_9 = ker(x_5);$$
$$H_{10} = ker(x_4 - x_5),\ H_{11} = ker(x_4 + x_5),\ H_{12} = ker(x_6);$$
$$H_{13} = ker(x_4 - x_6), H_{14} = ker(x_4 + x_6),\ H_{15} = ker(x_5 - x_6) \text{ and;}$$
$$H_{16} = ker(x_5 + x_6)$$

and via this order we have;

$$NBC_0(\mathcal{A}) = \{\mathbb{C}^6\},\ NBC_1(\mathcal{A}) = \{\{H_1\}, \dots, \{H_{16}\}\} = S_1(\pi);$$
$$NBC_2(\mathcal{A}) = \left(S_2(\pi) - \{\{H_3, H_7\}, \{H_4, H_6\}\}\right) \cup \{\{H_2, H_3\}, \{H_2, H_4\}\};$$
and for $3 \le k \le 6$;
$$NBC_k(\mathcal{A}) = (S_k(\pi) - S'_k(\pi)) \cup NBC_k^1(\pi) \cup NBC_k^2(\pi);$$
where, $S'_k(\pi) = \{S \in S_k(\pi) |$ either $\{H_3, H_7\} \subseteq S$ or $\{H_4, H_6\} \subseteq S\}$;
$$NBC_k^1(\pi) = \{(S - \{H_7\}) \cup \{H_2\} | S \in S'_k(\pi) \text{ and } \{H_3, H_7\} \subseteq S\} \text{ and;}$$
$$NBC_k^2(\pi) = \{(S - \{H_6\}) \cup \{H_2\} | S \in S'_k(\pi) \text{ and } \{H_4, H_6\} \subseteq S\}.$$

The important points to note here are $|NBC_k^1(\pi) \cup NBC_k^2(\pi)| = |S'_k(\pi)|$ and for $3 \le k \le 6$, if $S \in S'_k(\pi)$ and $\{H_3, H_7\} \subseteq S$, then, $f_k\left((S - \{H_7\}) \cup \{H_2\}\right) = S$ and if $S \in S'_k(\pi)$ and $\{H_4, H_6\} \subseteq S$, then, $f_k\left((S - \{H_6\}) \cup \{H_2\}\right) = S$. Accordingly, if $S \in S'_k(\pi)$ and $\{H_3, H_7\} \subseteq S$, then, $g_k(S) = (S - \{H_7\}) \cup \{H_2\}$ and if $S \in S'_k(\pi)$ and $\{H_4, H_6\} \subseteq S$, then, $g_k(S) = (S - \{H_6\}) \cup \{H_2\}$. By applying theorem (3.12.) and theorem (3.14.) we have;

$$\boldsymbol{NBC}_0(\mathcal{A}) = \langle e_{\mathbb{C}^6}\rangle,\ \boldsymbol{NBC}_1(\mathcal{A}) = \langle e_{H_1}, \dots, e_{H_{16}}\rangle = \boldsymbol{NBC}_1(\mathcal{A}_1) \oplus \boldsymbol{NBC}_1(\mathcal{A}_2);$$
$$\boldsymbol{NBC}_2(\mathcal{A}) = \left\langle e_B |\ B \in \left(S_2(\pi) - \{\{H_3, H_7\}, \{H_4, H_6\}\}\right) \cup \{\{H_2, H_3\}, \{H_2, H_4\}\}\right\rangle$$
$$= \boldsymbol{NBC}_2(\mathcal{A}_1) \oplus \boldsymbol{NBC}_2(\mathcal{A}_2), \text{ for } 3 \le k \le 6;$$
$$\boldsymbol{NBC}_k(\mathcal{A}) = \langle q_B,\ B \in (S_k(\pi) - S'_k(\pi)) \cup NBC_k^1(\pi) \cup NBC_k^2(\pi)\rangle = \boldsymbol{NBC}_k(\mathcal{A}_1) \oplus \boldsymbol{NBC}_k(\mathcal{A}_2);$$
$$(\pi)_0 = \langle q_{\{\}}\rangle,\ (\pi)_1 = \langle q_{H_1}, \dots, q_{H_{16}}\rangle = (\pi^1)_1 \otimes (\pi^2)_1;$$
and for $2 \le k \le 6$, $(\pi)_k = \langle q_B,\ B \in S_k(\pi)\rangle = (\pi^1)_k \otimes (\pi^2)_k$.

By applying theorem (3.17.), we have two K-chain isomorphisms, $f_*: \boldsymbol{NBC}_*(\mathcal{A}) \to (\pi)_*$ and $g_*: (\pi)_* \to \boldsymbol{NBC}_*(\mathcal{A})$ between acyclic chain complexes which produce a connection between two fashions of the O-S algebra of $\mathcal{A}$, a fashion as free module, and a fashion as a tensor factorization module. So;

$$\boldsymbol{A}_*(\mathcal{A}) \cong \boldsymbol{NBC}_*(\mathcal{A}) \cong \textstyle\sum_{0 \le k \le 6} \boldsymbol{NBC}_k(\mathcal{A}) \cong (\pi)_* \cong \sum_{0 \le k \le 6} (\pi)_k.$$

4.7. Example:

Let $\mathcal{A} = \{H_1, \dots, H_{14}\}$ be a 6-arrangemant that has the defining polynomial;

$$Q(\mathcal{A}) = x_1x_2x_3x_4x_5x_6(x_1+x_2+x_3)(x_1+x_2-x_3)(x_1-x_2+x_3)$$
$$(x_1-x_2-x_3)(x_4+x_5+x_6)(x_4+x_5-x_6)(x_4-x_5+x_6)(x_4-x_5-x_6).$$

It is clear that, $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ is a reducible arrangement such that each one of $\mathcal{A}_1$ and $\mathcal{A}_2$ is a 3-arrangement given in example (4.4.). By applying corollary (3.9.), the arrangement $\mathcal{A}$ is a factored arrangement that not completely factored with factorization:

$$\pi = (\pi_1,\pi_2,\pi_3,\pi_4,\pi_5,\pi_6) = (\{H_1\},\{H_2,H_3,H_4\},\{H_5,H_6,H_7\},\{H_8\},\{H_9,H_{10},H_{11}\},\{H_{12},H_{13},H_{14}\});$$

where, $H_1 = ker(x_3)$, $H_2 = ker(x_1)$, $H_3 = ker(x_1-x_2+x_3)$, $H_4 = ker(x_1-x_2-x_3)$, $H_5 = ker(x_2)$, $H_6 = ker(x_1+x_2+x_3)$, $H_7 = ker(x_1+x_2-x_3)$, $H_8 = ker(x_6)$, $H_9 = ker(x_4)$, $H_{10} = ker(x_4-x_5+x_6)$, $H_{11} = ker(x_4-x_5-x_6)$, $H_{12} = ker(x_5)$, $H_{13} = ker(x_4+x_5+x_6)$ and $H_{14} = ker(x_4+x_5-x_6)$ and via this order we have;

$$NBC_0(\mathcal{A}) = \{\mathbb{C}^6\},\ NBC_1(\mathcal{A}) = \{\{H_1\},\dots,\{H_{14}\}\} = S_1(\pi);$$

$$NBC_2(\mathcal{A}) = \big(S_2(\pi) - \{\{H_3,H_7\},\{H_4,H_6\},\{H_{10},H_{14}\},\{H_{11},H_{13}\}\}\big) \cup \{\{H_2,H_3\},\{H_2,H_4\},\{H_9,H_{10}\},\{H_9,H_{11}\}\};$$

and for $3 \le k \le 6$, $NBC_k(\mathcal{A}) = (S_k(\pi) - S'_k(\pi)) \cup NBC_k^1(\pi) \cup NBC_k^2(\pi)) \cup NBC_k^3(\pi) \cup NBC_k^4(\pi)$, where,

$$S'_k(\pi) = \{S \in S_k(\pi) | \text{ either } \{H_3,H_7\} \subseteq S \text{ or } \{H_4,H_6\} \subseteq S\} \cup \{S \in S_k(\pi) | \text{ either } \{H_{10},H_{14}\} \subseteq S \text{ or } \{H_{11},H_{13}\} \subseteq S\};$$

$$NBC_k^1(\pi) = \{(S - \{H_7\}) \cup \{H_2\} | S \in S'_k(\pi) \text{ and } \{H_3,H_7\} \subseteq S\};$$

$$NBC_k^2(\pi) = \{(S - \{H_6\}) \cup \{H_2\} | S \in S'_k(\pi) \text{ and } \{H_4,H_6\} \subseteq S\};$$

$$NBC_k^3(\pi) = \{(S - \{H_{14}\}) \cup \{H_9\} | S \in S'_k(\pi) \text{ and } \{H_{10},H_{14}\} \subseteq S\} \text{ and;}$$

$$NBC_k^4(\pi) = \{(S - \{H_{13}\}) \cup \{H_9\} | S \in S'_k(\pi) \text{ and } \{H_{11},H_{13}\} \subseteq S\}.$$

The important points to note here are $|NBC_k^1(\pi) \cup NBC_k^2(\pi)| = |S'_k(\pi)|$ and for $3 \le k \le 6$, if $S \in S'_k(\pi)$ and if $\{H_3,H_7\} \subseteq S$, then, $f_k\big((S-\{H_7\}) \cup \{H_2\}\big) = S$, if $S \in S'_k(\pi)$ and $\{H_4,H_6\} \subseteq S$, then, $f_k\big((S-\{H_6\}) \cup \{H_2\}\big) = S$, if $S \in S'_k(\pi)$ and $\{H_{10},H_{14}\} \subseteq S$, then, $f_k\big((S-\{H_{14}\}) \cup \{H_9\}\big) = S$ and if $S \in S'_k(\pi)$ and $\{H_{11},H_{13}\} \subseteq S$, then, $f_k\big((S-\{H_{13}\}) \cup \{H_9\}\big) = S$. Accordingly, if $S \in S'_k(\pi)$ and $\{H_3,H_7\} \subseteq S$, then, $g_k(S) = (S-\{H_7\}) \cup \{H_2\}$, if $S \in S'_k(\pi)$ and $\{H_4,H_6\} \subseteq S$, then, $g_k(S) = (S-\{H_6\}) \cup \{H_2\}$, if $S \in S'_k(\pi)$ and $\{H_{10},H_{14}\} \subseteq S$, then, $g_k(S) = (S-\{H_{14}\}) \cup \{H_9\}$, if $S \in S'_k(\pi)$ and $\{H_{11},H_{13}\} \subseteq S$, then, $g_k(S) = (S-\{H_{13}\}) \cup \{H_9\}$. By applying theorem (3.12.) and theorem (3.14.) we have;

$$\boldsymbol{NBC}_0(\mathcal{A}) = \langle e_{\mathbb{C}^6}\rangle,\ \boldsymbol{NBC}_1(\mathcal{A}) = \langle e_{H_1},\dots,e_{H_{14}}\rangle;$$

$$\boldsymbol{NBC}_2(\mathcal{A}) = \left\langle e_B \middle| \begin{matrix} B \in \big(S_2(\pi) - \{\{H_3,H_7\},\{H_4,H_6\},\{H_{10},H_{14}\},\{H_{11},H_{13}\}\}\big) \cup \\ \{\{H_2,H_3\},\{H_2,H_4\},\{H_9,H_{10}\},\{H_9,H_{11}\}\} \end{matrix} \right\rangle \text{ and for } 3 \le k \le 6;$$

$$\boldsymbol{NBC}_k(\mathcal{A}) = \left\langle e_B \middle| \begin{matrix} B \in \big(S_k(\pi) - S'_k(\pi)\big) \cup NBC_k^1(\pi) \cup \\ NBC_k^2(\pi)) \cup NBC_k^3(\pi) \cup NBC_k^4(\pi) \end{matrix} \right\rangle;$$

$$(\pi)_0 = \langle q_{\{\}}\rangle,\ (\pi)_1 = \langle q_{H_1},\dots,q_{H_{14}}\rangle \text{ and for } 3 \le k \le 6,\ (\pi)_k = \langle q_B | B \in S_k(\pi)\rangle.$$

By applying theorem (2.17), we have two K-chain isomorphisms, $\mathscr{f}_*: \boldsymbol{NBC}_*(\mathcal{A}) \to (\pi)_*$ and $\mathscr{g}_*: (\pi)_* \to \boldsymbol{NBC}_*(\mathcal{A})$ between acyclic chain complexes that induced a connection between two fashions of the O-S algebra of $\mathcal{A}$, a fashion as free module and a fashion as a tensor factorization module. Accordingly,

$$\boldsymbol{A}_*(\mathcal{A}) \cong \boldsymbol{NBC}_*(\mathcal{A}) \cong \textstyle\sum_{0\le k\le 6} \boldsymbol{NBC}_k(\mathcal{A}) \cong (\pi)_* \cong \sum_{0\le k\le 6} (\pi)_k.$$

6. Concluding remarks:

In this paper, we studied several fashions of the O-S algebra of Terao class of factored arrangements in order to examine, how Terao generalization of the class of supersolvable arrangements preserved the tensor factorization of the O-S algebra. It is found that;

1. We used the properties of the NBC monomial basis of O-S algebra of factored arrangement to establish a partition on the class of factored arrangements into two subclasses. The first one is the class of supersolvable

arrangements, which their arrangements have very interesting topological properties produced from the structure of their factorizations.

2. For the second subclass, we consider a connection between two fashions of the O-S algebra of $\mathcal{A}$, a fashion as free submodule of the Exterior algebra and a fashion as a tensor factorization module, in order to ensure our conjecture that there are a relation between the factorizations properties and the NBC bases properties.
3. As an application, we proved that: "A reducible arrangement $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ is completely factored if, and only if, each of $\mathcal{A}_1$ and $\mathcal{A}_2$ is completely factored". Moreover, we showed that, "A reducible arrangement $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ is factored arrangement that not completely factored if, and only if, $\mathcal{A}_1$ and $\mathcal{A}_2$ are factored arrangements such that either $\mathcal{A}_1$ or $\mathcal{A}_2$ is not completely factored".
4. As an illustration, we proved: "every reducible 3-arrangement is supersolvable", "every reducible arrangement into product of rank 2 arrangements is supersolvable", "every reducible arrangement into product of rank 2 arrangements and rank 1-arrangements is supersolvable" and "every reducible arrangement into product of the complexification Coxeter arrangement, either of type A or B is supersolvable".

As a future work, we are looking to study the topological properties of the complement of a factored arrangement that not supersolvable.

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