

A Fixed Point Theorem In 2-Banach Space For Banach Contraction Principle

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ABSTRACT

In This Paper we prove An Extension of Banach contraction principle through rational expression in 2-Banach space satisfying Three continuous mappings . Some result with S. banach (1922). And discuss about fixed point theory in 2-Banach space also established a fixed point theorem in 2-Banach space which generalized the result of many mathematician.

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Preliminaries :

1.1 Definition:

Let X be a real linear space and $\|.,.\|$ be a non negative real valued function defined on X satisfying the following condition:

(1) $\|x, y\|$ = 0 iff x and y are linearly dependent.
(2) $\|x, y\| = \|y, x\|$ for all x, y∈ X.
(3) $\|x, ay\| = |a|\|x, y\|$,a being real, for all x, y∈ X.
(4) $\|x, y + z\| = \|x, y\| + \|y, z\|$ for all x, y, $z \in$ X.

Then $\|.,.\|$ is called a 2-norm and the pair (X, $\|.,.\|$) is called a linear 2-normed space.

So a 2-norm $\|x, y\|$ always satisfies $\|x, y + ax\| = \|x, y\|$ for all x, y∈ X and all scalars a.

1.2 Definition:

A sequence $\{x_n\}$ in a 2-normed space (X, $\|.,.\|$) is said to be a cauchy sequence if $\lim_{n\to\infty}\|x_m - x_n, a\| = 0$ for all a in X .

1.3 Definition:

A sequence $\{x_n\}$ in a 2-normed space (X, $\|.,.\|$) is said to be convergent if there is a point x in X such that $\lim_{n\to\infty}\|x_n - x, y\| = 0$ for all y in X . If x_n converges to x , we write $x_n \to x \; as \; n \to \infty$.

1.4 Definition:

A linear 2-normed space is said to be complete if every Cauchy sequence is convergent to an element of X. A complete 2-normed space X is called 2- Banach spaces.

1.5 Definition:

Let X be a 2-Banach space and T be a self mapping of X. T is said to continuous at x if for every sequence $\{x_n\}$ in X , $\{x_n\} \to x \; as \; n \to \infty$ implies $\{\mathrm{T}(x_n)\} \to T(x) \; as \; n \to \infty$.

Main Result

Theorem 2.1: Let E, F and T are three continuous mappings of a 2-Banach space satisfying the following conditions:

ET = TE , FT = TF, E(X) ⊂ T(X) and F(X) ⊂ T(X). **(1.1)**

$$\|(Ex - Fy), a\| \le \alpha \frac{\|Ty - Fy, a\| \, [1 + \|Tx - Ex, a\|]}{1 + \|Tx - Ty, a\|} + \beta\, [\|Tx - Ex, a\| + \|Ty - Fy, a\|] +$$

$$\gamma\, [\|Tx - Fy, a\| + \|Ty - Ex, a\|] + \delta \|Tx - Ty, a\| \quad \textbf{(1.2)}$$

For all x, y in X where $\alpha, \beta, \gamma, \delta \ge 0, \alpha + 2\beta + 2\gamma + \delta < 1.$ then E,F and T have a common fixed point in X.

Proof: Let x_0 be an arbitrary element of X and let $\{Tx_n\}$ be defined as

$Tx_{2n+1} = Ex_{2n}$, $Tx_{2n+2} = Fx_{2n+1}$ for n- 1,2,3,4…… **(1.3)**

We can do this since E(X) ⊂ T(X) and F(X) ⊂ T(X).

From (1.2) we have

$$\|Tx_{2n+1} - Tx_{2n+2}, a\| = \|Ex_{2n} - Fx_{2n+1}, a\|$$

$$\le \alpha \frac{\|Tx_{2n+1} - Tx_{2n+2}, a\| [1 + \|Tx_{2n} - Tx_{2n+1}, a\|]}{[1 + \|Tx_{2n} - Tx_{2n+1}, a\|]} + \beta [\|Tx_{2n} - Tx_{2n+1}, a\| + \|Tx_{2n+1} - Tx_{2n+2}, a\|] + \gamma [\|Tx_{2n} - Tx_{2n+2}, a\| + \|Tx_{2n+1} - Tx_{2n+1}, a\|] + \delta \|Tx_{2n} - Tx_{2n+1}, a\|$$

$$\le \alpha \frac{\|Tx_{2n+1} - Tx_{2n+2}, a\| [1 + \|Tx_{2n} - Tx_{2n+1}, a\|]}{[1 + \|Tx_{2n} - Tx_{2n+1}, a\|]} + \beta [\|Tx_{2n} - Tx_{2n+1}, a\| + \|Tx_{2n+1} - Tx_{2n+2}, a\|] + \gamma [\|Tx_{2n} - Tx_{2n+2}, a\| + 0] + \delta \|Tx_{2n} - Tx_{2n+1}, a\|$$

$$\le \alpha \frac{\|Tx_{2n+1} - Tx_{2n+2}, a\| [1 + \|Tx_{2n} - Tx_{2n+1}, a\|]}{[1 + \|Tx_{2n} - Tx_{2n+1}, a\|]} + \beta [\|Tx_{2n} - Tx_{2n+1}, a\| + \|Tx_{2n+1} - Tx_{2n+2}, a\|] + \gamma [\|Tx_{2n} - Tx_{2n+2}, a\| + \|Tx_{2n+1} - Tx_{2n+1}, a\|] + \delta \|Tx_{2n} - Tx_{2n+1}, a\|$$

$$\le \alpha \frac{\|Tx_{2n+1} - Tx_{2n+2}, a\| [1 + \|Tx_{2n} - Tx_{2n+1}, a\|]}{[1 + \|Tx_{2n} - Tx_{2n+1}, a\|]} + \beta [\|Tx_{2n} - Tx_{2n+1}, a\| + \|Tx_{2n+1} - Tx_{2n+2}, a\|] + \gamma + \delta \|Tx_{2n} - Tx_{2n+1}, a\|$$ [using triangle inequality]

$$\|Tx_{2n+1} - Tx_{2n+2}, a\| \le \alpha \|Tx_{2n+1} - Tx_{2n+2}, a\| + \beta \|Tx_{2n} - Tx_{2n+1}, a\| + \beta \|Tx_{2n+1} - Tx_{2n+2}, a\| + \gamma \|Tx_{2n} - Tx_{2n+1}, a\| + \gamma \|Tx_{2n+1} - Tx_{2n+2}, a\| + \delta \|Tx_{2n} - Tx_{2n+1}, a\|$$

$$\|Tx_{2n+1} - Tx_{2n+2}, a\| (1 - \alpha - \beta - \gamma) \le (\beta + \gamma + \delta) \|Tx_{2n} - Tx_{2n+1}, a\|$$

$$\|Tx_{2n+1} - Tx_{2n+2}, a\| \le \frac{(\beta + \gamma + \delta)}{(1 - \alpha - \beta - \gamma)} \|Tx_{2n} - Tx_{2n+1}, a\|$$

$$\|Tx_{2n+1} - Tx_{2n+2}, a\| \le h \, \|Tx_{2n} - Tx_{2n+1}, a\|$$

Where $h = \frac{(\beta + \gamma + \delta)}{(1 - \alpha - \beta - \gamma)} < 1$

Similarly we can see

$$\|Tx_{2n} - Tx_{2n+1}, a\| \le h \, \|Tx_{2n-1} - Tx_{2n}, a\|$$

Proceeding in t is way ,we have

$$\|Tx_{2n+1} - Tx_{2n+2}, a\| \le h \, \|Tx_{2n} - Tx_{2n+1}, a\|$$

$$\le h^2 \, \|Tx_{2n-1} - Tx_{2n}, a\|$$

$$\leq h^{2n+1} \|Tx_0 - Tx_1, a\|$$

By routine calculation the following inequalities hold for k>n

$$\|Tx_n - Tx_{n+k}, a\| \leq \sum_{i=1}^{k} \|Tx_{n+1-1} - Tx_{n+1}, a\|$$

$$\leq \sum_{i=1}^{k} h^{n+1-1} \|Tx_0 - Tx_1, a\|$$

$$\leq \frac{h^n}{1-h} \|Tx_0 - Tx_1, a\| \qquad \to 0 \; as \; n \to \infty.$$

Hence $\{Tx_n\}$ is a Cauchy sequence by the completeness of X. $\{Tx_n\}$ converges to a point u in X. It follows from (1.3) that $\{Ex_{2n}\}$and $\{Fx_{2n+1}\}$ also converges to u. since E,F and T are continuous we have

$$E(Tx_{2n}) \to Eu \;, F(Tx_{2n+1}) \to Fu \qquad \textbf{(1.4)}$$

From (1.1) t commutes with E and F therefore

$E(Tx_{2n}) = T(Ex_{2n})$, $F(Tx_{2n+1}) = T(Fx_{2n+1})$ for all n= 0,1,2,3…………

Taking $n \to \infty$ we have

Eu=Tu=Fu and **(1.5)**

T(Tu= T(Eu)) = E(Tu) = E(Fu) = F(Eu) =T(Fu) = F(Tu) = F(Eu) = F(Fu). **(1.6)**

By (1.2) , (1.5) and (1.6). if Eu $\neq$ F(Eu) we have

$$\|Eu - F(Eu), a\| \leq \alpha \frac{\|T(Eu)-F(Eu),a\| \, [1+\|Tu-Eu,a\|]}{[1+ \|Tu-T(Eu),a\|]} + \beta[\|Tu - Eu, a\| + \|T(Eu) - F(Eu), a\|] + \gamma[\|Tu - F(Eu), a\| + \|T(Eu) - Eu, a\|] + \delta\|Tu - T(Eu), a\|$$

$$\leq (2\gamma + \delta)\|Eu - F(Eu), a\|$$

$$< \|Eu - F(Eu), a\| \qquad [\because (2\gamma + \delta) < 1] \qquad \textbf{(1.7)}$$

Leading to a contradiction . Hence

Eu = F(Eu). Using (1.6) and (1.7) we get

Eu = F(Eu) = T(Eu) =E (Eu)

Which shows that Eu is the common fixed of E, F and T .

Let z and w ($z \neq w$) be two points in X such that

Ez = Fz =Tz= z and Ew = Fw= Tw = w .Then by (1.2) we have

$$\|z - w, a\| = \|Ez - Fw, a\|$$

$$\leq \alpha \frac{\|Tw-Fw,a\|[1+\|Tz-Ez,a\|]}{[1+\|Tz-Fw,a\|]} + \beta[\|Tz - Ez, a\| + \|Tw - Fw, a\|] + \gamma[\|Tz - Fw, a\| + \|Tw - Ez, a\|] + \delta\|Tz - Tw, a\|$$

$$\leq \alpha.0 + \beta.0 + \gamma[\|Tz - Fw, a\| + \|Tw - Ez, a\|] + \delta\|Tz - Tw, a\|$$

$$\leq (2\gamma + \delta) \|Ez - Fw, a\|$$

$$< \|Ez - Fw, a\| \quad [\because (2\gamma + \delta) < 1]$$

Leading to a contraction. Hence z = w. This implies the uniqueness of common fixed point for E,F and T. This completes the proof of the theorem.

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