On The CW Complex of the Complement of A Hypersolvable Graphic Arrangement

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Abstract:

This paper interested in studying a CW complex for the complement $M(\mathcal{A}_G)$ of a hypersolvable graphic arrangement $\mathcal{A}_G$ that related to a hypersolvable graph G, by comparing it with the minimal CW complex for the complement of Jambu's-Papadima's deformed supersolvable arrangement $\tilde{\mathcal{A}}$. Motivated by our aim, a dimension of the first non-vanishing higher homotopy group for $M(\mathcal{A}_G)$ was calculated and a fashion of the cohomological ring $\boldsymbol{H}^*(M(\mathcal{A}_G))$ of the complement $M(\mathcal{A}_G)$ was considered, just by using the hypersolvable partition analogue on G. Moreover, an algorithm to deform any hypersolvable graph into a supersolvable graph was stated.

Key-words: connected simple graph, graphic arrangement, hypersolvable (supersolvable) graph, Orlik-Solomon algebra, no broken circuit module, CW complex.

Introduction:

One of powerful mathematical tool for a wide range of applications is the graph theory. Our work are specialized on a very interesting class of graphs is the “hypersolvable class of graphs” which is firstly introduced by Papadima and Suciu in (2002, [12]) as a generalization of Stanly Supersolvable (triangulated graphs or rigid circuit graphs or chordal) class of graphs (1972, [1]).

In (2012, [5]) Fadhil introduced a partition to a graph G, called a hypersolvable partition. In her M.Sc. thesis under my supervision, Fadhil produced the existence of a hypersolvable partition as a sufficient and necessary condition to a graph to be hypersolvable. The advantage of studying the hypersolvable partition analogue lies in the fact, it makes the computations of the cycles of G more easer and by using the duality; every graph G, defined a graphic arrangement $\mathcal{A}_G$, the analogue of the induced partition of $\mathcal{A}_G$ makes the computation of the NBC (no broken circuits) bases of $\mathcal{A}_G$ more easer. So, in this work the duality between the notions "cycle of G " and "circuit of $\mathcal{A}_G$" had been used to introduce a fashion of the cohomological ring of complement of a hypersolvable arrangement, $\boldsymbol{H}^*\big(M(\mathcal{A}_G)\big)$ as a tensor module, since the set of all the NBC bases of $\mathcal{A}_G$ forms an explicit bases of the cohomological ring of $M(\mathcal{A}_G)$, (we refer the reader to [11] as a general reference). This was achieved in section (3) by two parts. First, we recall the isomorphism between the chomological group of the complement $M(\mathcal{A}_G)$ and The Orlik-Solomon algebra $A_*(\mathcal{A}_G)$ of $\mathcal{A}_G$ that had been studied firstly by Orlik and Solomon in (1980, [10]). Secondly, we used analogue defined in (2010, [3]) to embedding $\boldsymbol{H}^*\big(M(\mathcal{A}_G)\big)$ as a submodule of the partition tensor module that related to the induced hypersolvable partition of $\mathcal{A}_G$, duo to [1].

Randell in (2002, [14]), showed that $M(\mathcal{A})$ of any complex hyperplane arrangement has homotopy type of a minimal (finite type) CW-complex, i.e. the number of the k-cells is equal to the k^{th}-Betti number $b_k\big(M(\mathcal{A})\big) = rk\big(\boldsymbol{H}^*\big(M(\mathcal{A})\big)\big)$. Accordingly, $M(\mathcal{A}_G)$ has a minimal (finite type) CW-complex and we concern to study its structure by using the well known structure of its higher homotopy group due Papadima and Suciu in (2002, [16]).

The hypersolvable class of arrangements was intoduced firstly by Jambu and Papadima in (1998, [8]) and

(2002, [9]), as a generalization of the supersolvable (fiber-type) class. They defined a vertical deformation method which deformed the hypersolvable arrangement $\mathcal{A}$ with s-singular blocks into supersolvable arrangement $\tilde{\mathcal{A}} = \tilde{\mathcal{A}}_1$ by one-parameter family of arrangements $\{\tilde{\mathcal{A}}_t\}_{t\in\mathbb{C}}$ in $\mathbb{C}^r \times \mathbb{C}^s = \mathbb{C}^\ell$, with preserving the lattice intersection pattern up to codimension two, $\ell_2(\mathcal{A}) = \{B \subseteq \mathcal{A} \big| \, |B| \leq 3\} \approx \ell_2(\tilde{\mathcal{A}})$ and they proved $\mathcal{A}$ and $\tilde{\mathcal{A}}$ have isomorphic fundamental groups. The class of hypersolvable arrangements contains supersolvable class of arrangements and the generic class of arrangements and many others. For a supersolvable arrangement (fiber-type) all the higher homotopy groups of $M(\mathcal{A})$ are vanished (1985, [6]). The first computation of non trivial higher homotopy groups of $M(\mathcal{A})$ of a generic arrangement was made by Hattori (1975, [7]). Papadima and Suciu in (2002, [12]), generalized Hattori's result to a hypersolvable arrangement and compute the first non vanishing higher homotopy group of $M(\mathcal{A})$. They showed that the first non vanishing higher homotopy group of $M(\mathcal{A})$ has dimension;

$$p\big(M(\mathcal{A})\big) = \sup\{k|P\big(\boldsymbol{H}^*\big(M(\mathcal{A})\big),s\big) \equiv_{mod\ k} P\left(\boldsymbol{H}^*\left(M(\tilde{\mathcal{A}})\right),s\right)\}$$

where $P\big(\boldsymbol{H}^*\big(M(\mathcal{A})\big),s\big)$ and $P\left(\boldsymbol{H}^*\left(M(\tilde{\mathcal{A}})\right),s\right)$ are the Poincaré polynomials of the cohomological rings $M(\mathcal{A})$ and $M(\tilde{\mathcal{A}})$ respectively. Ali in [1], showed a conjecture of $p\big(M(\mathcal{A})\big)$ as; $p\big(M(\mathcal{A})\big) = max\{k||NBC_k(\mathcal{A})|=|S_k(\Pi)|\}$, where $NBC_k(\mathcal{A})$ be the set of all k-NBC bases of $\mathcal{A}$ via the hypersolvable ordering and $S_k(\Pi)$ is the set of all k-sections of a hypersolvable partition Π.

In section (1), some basic facts that we needed in this work was stated. Section (2), is devoted to compute the dimension $p\big(M(\mathcal{A}_G)\big)$ of the first non vanishing higher homotopy group of $M(\mathcal{A}_G)$ for any hypersolvable graphic arrangement by using the properties of the hypersolvable partition on the graph G due [5]. Finally, the structure of the cohomological ring that given in section (3), $\boldsymbol{H}^*\big(M(\mathcal{A}_G)\big)$ had been used to construct the second skeleton of the minimal CW complex of $M(\mathcal{A}_G)$ in section (4) and to study the $p\big(M(\mathcal{A}_G)\big)^{th}$ skeleton of minimal CW complex of $M(\mathcal{A}_G)$ in section (5).

We mentioned that, the structure of the higher homotopy groups of $M(\mathcal{A}_G)$ is due to [12] and the technique of constructing the skeletons of Minimal CW complex of $M(\mathcal{A}_G)$ is due to [16], so it is to be expected these constructions without proof and for evedance see [12, 16].

1. Basic Facts:

This section briefly sketch the notion of a hypersolvable partition of a graph G due ([5], 2012), in order to use its structure to embedding the cohomological group of the complement of a graphic arrangement as a submodule of the partition tensor module. For this motivation, we will review some of the standard facts on the notions O-S algebra, NBC module, Partition module.

1.1. Definition: [5]

Let $G = (V, \mathcal{E})$ be a connected simple graph with a finite set of vertices, i.e. $V = \{v_1, \ldots, v_m\}$. A pair of partitions, $\Pi^G = (\Pi^V, \Pi^{\mathcal{E}})$ is said to be a **hypersolvable partition** of G and denoted by Hp Π^G, if $\Pi^V = (\Pi_1^V, \ldots, \Pi_{m-1}^V)$ and $\Pi^{\mathcal{E}} = (\Pi_1^{\mathcal{E}}, \ldots, \Pi_\ell^{\mathcal{E}})$ are partitions of V and $\mathcal{E}$ respectively, such that the following properties are satisfied:

HP$_1$: $\Pi_1^V = \{v_1, v_2\}$ and $\Pi_1^{\mathcal{E}} = \{e_1\}$, such that $e_1 = [v_1, v_2]$, i.e. $\Pi_1^{\mathcal{E}}$ is a singleton.

HP$_2$: For each $2 \leq j \leq m-1$, the block Π_j^V is a singleton.

HP$_3$: For each $2 \leq k \leq \ell$, the block $\Pi_k^{\mathcal{E}}$ satisfying the following properties:

HP$_3$$i$: For each $e_{i_1}, e_{i_2} \in \Pi_1^{\mathcal{E}} \cup \ldots \cup \Pi_k^{\mathcal{E}}$, there is no edge $e \in \Pi_{k+1}^{\mathcal{E}} \cup \ldots \cup \Pi_\ell^{\mathcal{E}}$ such that $\{e_{i_1}, e_{i_2}, e\}$ forms a set of edges of a triangle.

HP$_3$*ii*: There exists a positive integer $1 < m_k \leq m-1$, such that $V_k = \Pi_1^V \cup \ldots \cup \Pi_{m_k}^V$ is a subset of V that contains all the end points of the edges in $\Pi_1^{\mathcal{E}} \cup \ldots \cup \Pi_k^{\mathcal{E}}$, i.e $G_k = (V_k, \Pi_1^{\mathcal{E}} \cup \ldots \cup \Pi_k^{\mathcal{E}})$ forms a subgraph of G. Then, either;

1. $\Pi_k^{\mathcal{E}} = \{e\}$ such that $V_k = V_{k-1}$,
   or;
2. $\Pi_k^{\mathcal{E}} = \{e_{i_1}, \ldots, e_{i_{d_k}}\}$, such that $V_k \backslash V_{k-1} = \Pi_{m_{k-1}+1}^V = \Pi_{m_k}^V = \{v\}$ and for $1 \leq j \leq d_k$, $e_{i_j} = [v_{i_j}, v]$, for some $v_{i_j} \in \Pi_1^V \cup \ldots \cup \Pi_{m_{k-1}}^V$, where $\{v_{i_1}, \ldots, v_{i_{d_k}}\} \subseteq V_{k-1} = \Pi_1 \cup \ldots \cup \Pi_{m_{k-1}}$ induces a complete subgraph of G.

$\ell(G) = \ell = |\Pi^{\mathcal{E}}|$ is called the **length** of. For $1 \leq k \leq \ell$, let $d_k = |\Pi_k^{\mathcal{E}}|$ and the vector $d = (d_1, \ldots, d_\ell)$ is called the **exponent vecto**r of Π. Define the **rank** of $\Pi_k^{\mathcal{E}}$ as; $rk\, \Pi_k^{\mathcal{E}} = |V_k| - 1$ and $rk(G) = rk\, \Pi_\ell^{\mathcal{E}} = m - 1$. We will call the block $\Pi_k^{\mathcal{E}}$ **singular block,** if $|V_{k-1}| = |V_k|$ and **non-singular** otherwise, i.e. $\Pi_k^{\mathcal{E}}$ is non-singular if $|V_k \backslash V_{k-1}| = 1$.

A hypersolvable partition Π is said to be **supersolvable** if, and only if, $\Pi^{\mathcal{E}}$ has no singular block.

We will call a hypersolvable partition Π^G, **generic** if $\ell \geq m$, the exponent vector $d = (1, \ldots, 1)$ and every k-eadges of $\mathcal{E}$ cannot be an k-cycle, $3 < k \leq m-1$.

It is worth pointing out that;

1. For $1 \leq k \leq \ell$, the positive integer m_k needs not to be equal to $k-1$ in general.
2. $\ell \geq m - 1 = rk(G)$.
3. $\ell = m - 1$ if, and only if, Π is supersolvable.
4. $\Pi_2^{\mathcal{E}}$ cannot be a singular block, since $|V_2| = 3$.
5. For $3 \leq k \leq \ell$, if $\Pi_k^{\mathcal{E}}$ is a singular block, then $\Pi_k^{\mathcal{E}}$ is a singleton.

1.2. Theorem: [5]

Let G be a connected graph. Then G is hypersolvable if, and only if, G has a hypersolvable partition. A connected hypersolvable graph G is supersolvable if, and only if, G has a supersolvable partition.

1.3. Lemma: (The complete property of $\Pi_k^{\mathcal{E}}$) [5]

Let G be a connected hypersolvable graph with a hypersolvable partition $\Pi^G = (\Pi^V, \Pi^{\mathcal{E}})$. For $2 \leq k \leq \ell$, if $e_1, e_2 \in \Pi_k^{\mathcal{E}}$, then there exists a unique $e \in \Pi_1^{\mathcal{E}} \cup \ldots \cup \Pi_{k-1}^{\mathcal{E}}$ such that $\{e_1, e_2, e\}$ forms a triangle..

1.4. Definition: [5]

Let G be a hypersolvable graph with hypersolvable partition. Define a **hypersolvable order** on G associated to an Hp $\Pi^G = (\Pi^V, \Pi^{\mathcal{E}})$ and denoted by $\trianglelefteq$, as follows:

1. Put an arbitrary order on the vertices of Π_1^V.
2. If $v_i \in \Pi_i^V$ and $v_j \in \Pi_j^V$ such that; $i < j$, put $v_i \trianglelefteq v_j$.
3. If $e \in \Pi_i^{\mathcal{E}}$ and $e' \in \Pi_j^{\mathcal{E}}$ such that; $i < j$, put $e \trianglelefteq e'$.
4. If $e, e', e'' \in \Pi_k^{\mathcal{E}}$, set$e_{i_1} \trianglelefteq e_{i_2} \trianglelefteq e_{i_3} \Leftrightarrow e_{i_1,i_2} \trianglelefteq e_{i_1,i_3} \trianglelefteq e_{i_2,i_3}$, where, $\{e_{i_1}, e_{i_2}, e_{i_3}\} = \{e, e', e''\}$.

1.5. Theorem: [15]

A graph $G = (V, \mathcal{E})$ is supersolvable if, and only if, there exists an ordering $v_1, v_2, \ldots, v_m$ of its vertices such that if $1 \leq i < j < k \leq m$, such that $[v_i, v_k] \in \mathcal{E}$ and $[v_j, v_k] \in \mathcal{E}$, then $[v_i, v_j] \in \mathcal{E}$. Equivalently, in the restriction of G to the vertices $v_1, \ldots, v_i$ the neighborhood of v_i is a clique.

1.6. Proposition: [5]

Let $G = (V, \mathcal{E})$ be a supersolvable graph with a supersolvable partition $\Pi^G = (\Pi^V, \Pi^{\mathcal{E}})$. Via a supersolvable ordering $\trianglelefteq$ on G, if $[v_i, v_k] \in \mathcal{E}$ and $[v_j, v_k] \in \mathcal{E}$, then $[v_i, v_j] \in \mathcal{E}$, where $1 \leq i < j < k \leq m$.

1.7. Definition: [11]

By a **hyperplane** H in a finite dimensional vector space $V \cong K^m$ over a field $K = \mathbb{R}$ or $\mathbb{C}$, we mean an affine subspace of dimension $(\dim V - 1 = m - 1)$ and **an arrangement** $\mathcal{A}$ is a finite collection of hyperplanes H in V. The **variety** of $\mathcal{A}$ is $N(\mathcal{A}) = \cup_{H\in\mathcal{A}} H$ and its **complement** is $M(\mathcal{A}) = V \setminus \cup_{H\in\mathcal{A}} H$. The **intersection lattice** is defined to be, $L = L(\mathcal{A}) = \{X \big| X = \cap_{H\in B} H \text{ and } B \subseteq \mathcal{A}\}$ that ordered by reverse the inclusion, (i.e. $X \leq Y \Leftrightarrow Y \subseteq X$, for $X, Y \in L(\mathcal{A})$), and ranked by $rk(X) = codim(X) = \dim(V) - \dim(X)$, for $X \in L(\mathcal{A})$.

An arrangement $\mathcal{A}_G$ is said to be **graphic arrangement** if, there is a graph $G = (V, \mathcal{E})$ such that the defining polynomial of $\mathcal{A}_G$ is, $Q(\mathcal{A}_G) = \prod_{[i,j]\in\mathcal{E}}(x_i - x_j)$.

We mention that;

1. $rk(\mathcal{A}_G) = rk(G) = |V| - 1$.
2. If $K = (V_K, \mathcal{E}_K) \subseteq G$, then $rk(\mathcal{A}_K) = 2 = |v| - 1$ if, and only if, either $|\mathcal{E}_K| = 2$ or K is a triangle of G.
3. Poincare polynomial of $\mathcal{A}_G$, $P(\mathcal{A}_G, t) = \chi(G, -t)$, where $\chi(G, -t)$ is the chromatic function of G. Thus, for $1 \leq j \leq \ell$, if b_j is a $\boldsymbol{j^{th}}$ **Betti number** of the Poincare polynomial $P(\mathcal{A}, t)$, then $b_j =$ The number of colorings of j vertices of G with t colors.

1.8. Proposition: [12]

A graph G is hypersolvable if, and only if, the graphic arrangement $\mathcal{A}_G$ is hypersolvable.

1.9. Proposition: [5]

A graph G is supersolvable if, and only if, the graphic arrangement $\mathcal{A}_G$ is supersolvable. A graph $G = (V, \mathcal{E})$ is a generic graph if, and only if, its graphic arrangement $\mathcal{A}_G$ is generic.

The important points to note here are that, if $G = (V, \mathcal{E})$ is a hypersolvable graph, then $\mathcal{A}_G$ has a hypersolvable partition $\Pi' = (\Pi_1, \dots, \Pi_\ell)$ induced from the hypersolvable partition $\Pi^G = (\Pi^V, \Pi^{\mathcal{E}})$, as for $1 \leq k \leq \ell$, $H_{ij} \in \Pi_k$ if, and only if, $[i, j] \in \Pi_k^{\mathcal{E}}$. Π' is called the **induced partition** of Π^G.

1.10. Definition: [2]

Let $\Pi = (\Pi_1, \dots, \Pi_\ell)$ be a partition of an ℓ-arrangement $\mathcal{A}$.

1. A **section** S of Π is a subarrangement of $\mathcal{A}$ satisfied for each $1 \leq k \leq \ell$, either $S \cap \Pi_k$ is empty or a singleton. By $S(\Pi)$ we denote the set of all sections of Π and the set $S_k(\Pi)$ denotes the set of all sections S of Π with $|S| = k$, we call such sections of Π, k-sections of Π. We will agree that the empty section $\emptyset_\ell$ is a 0-sections of Π.
2. The integer ℓ is called the **length** of Π and denoted by $\ell(\Pi)$.
3. $rk(\Pi_k) = rk\left(\cap_{H\in\Pi_1\cup\ldots\cup\Pi_k} H\right)$.
4. Π is called **independent** if for every choice of hyperplanes $H_k \in \Pi_k$ for $1 \leq k \leq \ell$, the resulting ℓ hyperplanes are independent, i.e. $rk(H_1 \cap \dots \cap H_\ell) = \ell$.
5. Let $X \in L$. Let $\Pi = (\Pi_1, \dots, \Pi_\ell)$ be a partition of $\mathcal{A}$. Then the **induced partition** Π_X is a partition of $\mathcal{A}_X$, its blocks are the nonempty subsets $\Pi_k \cap \mathcal{A}_X$, $1 \leq k \leq \ell$.
6. π is called **nice**, if Π is independent and if $X \in L \setminus \{V\}$, then the induced partition Π_X contains a block, which is a singleton.
7. $\mathcal{A}$ is called **nice** arrangement if, it has a nice partition $\Pi = (\Pi_1, \dots, \Pi_\ell)$. The vector of integers $d = (d_1, \ldots, d_\ell)$ is said to be the exponent vector of $\mathcal{A}$, if $d_k = |\pi_k|$, $1 \leq k \leq \ell$.

1.11. Definition: [2]

1. A subarrangement C of $\mathcal{A}$ is said to be a **circuit**, if it is a minimal dependent subarrangement of $\mathcal{A}$, i.e. $C \setminus \{H\}$ is linearly independent, for any $H \in C$, i.e. $rk(C) = |C| - 1$.

2. Via a total ordering $\trianglelefteq$ on the hyperplanes of $\mathcal{A}$, the corresponding **broken circuit** of a circuit C is $\bar{C} = C\backslash\{H\}$, where H is the smallest hyperplane in C. If $|\bar{C}| = k$, then $\bar{C}$ is said to be ***k*-broken circuit**. The set of all k-broken circuits of $\mathcal{A}$ will be denoted by $BC_k(\mathcal{A})$ and $BC(\mathcal{A}) = \bigcup_{k=2}^{\ell} BC_k(\mathcal{A})$.
3. We call $B \subseteq \mathcal{A}$, an ***NBC* base** of $\mathcal{A}$, if it contains no broken circuit. Note that, such a set must be independent and we will write ***k-NBC* base** for B if $|B| = k$ and we will agree that $\emptyset^{\ell}$ is the 0-NBC of $\mathcal{A}$. By $NBC_k(\mathcal{A})$ we denote the set of all k-NBC bases of $\mathcal{A}$ and $NBC(\mathcal{A}) = \bigcup_{k=0}^{\ell} NBC_k(\mathcal{A})$.
4. If $X \in L(\mathcal{A})$. Then the NBC base $B \subseteq \mathcal{A}_X$, (i.e. $\bigcap_{H\in B} H = X$) is said to be an NBC base of X.
5. If $\mathcal{A}$ is a factored arrangement with a factorization π. Due a total ordering $\trianglelefteq$ on the hyperplanes of $\mathcal{A}$, define, $p_{\trianglelefteq}(\mathcal{A}) = Max\{k| \, NBC_k(\mathcal{A}) = S_k(\pi)\}$. We remarked that, $1 \le p_{\trianglelefteq}(\mathcal{A}) \le \ell$.

In view of definitions (1.10.) and (1.11), we remarked the following:

1. If $d = (d_1, .., d_\ell)$ be the exponent vector of a nice partition Π, it is known that;

$$P(\mathcal{A}, t) = \prod_{k=1}^{\ell}(1 + d_k t) = 1 + (d_1 + .. + d_\ell)t + \left(\sum_{i_1=1}^{\ell-1}\sum_{i_2=i_1+1}^{\ell} d_{i_1} d_{i_2}\right)t^2 + \cdots + d_1 \ldots d_\ell \, t^\ell.$$

2. Independent of our choice of an ordering $\trianglelefteq$ on the hyperplanes of $\mathcal{A}$, It is known that, the k^{th} Betti number of the Poincare polynomial $P(\mathcal{A}, t) = b_k(\mathcal{A}) = |NBC_k(\mathcal{A})|$. According to [1], for a hypersolvable arrangement $\mathcal{A}$;

$$b_k(\mathcal{A}) = |NBC_k(\mathcal{A})| \le \sum_{i_1=1}^{\ell-k}\sum_{i_2=i_1+1}^{\ell-k+1} \ldots \sum_{i_k=i_{k-1}+1}^{\ell} d_{i_1} d_{i_2} \ldots d_{i_k} = |S_k(\Pi)|, \text{ for } 1 \le k \le \ell.$$

1.12. Definition: [11]

Let K be any commutative ring and Let $\trianglelefteq$ be an arbitrary total order that defined on the hyperplanes of an ℓ-arrangement $\mathcal{A}$. The Orlik-Solomon algebra (or for simplicity O-S algebra) $A_*(\mathcal{A})$ is defined to be the quotient of the exterior K-algebra $E_* = \bigwedge_{k\ge0}(\bigoplus_{H\in\mathcal{A}} Ke_H)$, by the homogeneous ideal $I_*(\mathcal{A})$ is generated by the relations, $\sum_{j=1}^{k}(-1)^{k-1} e_{H_{i_1}} \ldots \widehat{e_{H_{i_j}}} \ldots e_{H_{i_k}}$, for all $1 \le i_1 < \cdots < i_k \le n$ such that $\{H_{i_1}, \ldots H_{i_k}\}$ is dependent subarrangement of $\mathcal{A}$, i.e. $(rk(H_{i_1}, \ldots H_{i_k}) < k)$ and the circumflex ˆ means $e_{H_{i_j}}$ is deleted. Define a K-linear mapping $\partial_*^E : E_* \to E_*$ as; $\partial_0^E(e_{\emptyset_\ell}) = 0$, $\partial_1^E(e_H) = 1$, for all $H \in \mathcal{A}$ and for $2 \le k \le \ell$, $\partial_k^E(e_C) = \sum_{j=1}^{k}(-1)^{k-1} e_{H_{i_1}} \ldots \widehat{e_{H_{i_j}}} \ldots e_{H_{i_k}}$, $C = \{H_{i_1}, \ldots H_{i_k}\}$. ∂_*^E is a differentiation on E_* and the chain complex

(E_*, ∂_*^E): $\cdots \xrightarrow{\partial_{k+1}^E} E_k \xrightarrow{\partial_k^E} E_{k-1} \xrightarrow{\partial_{k-1}^E} \cdots \xrightarrow{\partial_2^E} E_1 \xrightarrow{\partial_1^E} E_0 \xrightarrow{\partial_0^E} 0$, is called the exterior complex.

1.13. Theorem: [11]

The complex $(\boldsymbol{A}_*(\mathcal{A}), \partial_*^A)$ inherits a structure as acyclic chain complex from the exterior complex (E_*, ∂_*^E), where $\partial_*^A = \psi_* \circ \partial_*^E$ and $\psi_* : E_* \to \boldsymbol{A}_*(\mathcal{A})$ is the canonical chain map. The acyclic chain complex $(\boldsymbol{A}_*(\mathcal{A}), \partial_*^A)$ is called the O-S complex.

1.14. Definition: [11]

Let K be any commutative ring. The broken circuit module $\boldsymbol{NBC}_*(\mathcal{A})$ of the exterior K-algebra $E_* = \bigwedge_{k\ge0}\left(\bigoplus_{H\in\mathcal{A}} Ke_H\right)$, is defined as; $\boldsymbol{NBC}_0(\mathcal{A}) = K$ and for $1 \le k \le \ell$, $\boldsymbol{NBC}_k(\mathcal{A})$ be the free K-module of E_k with NBC (no broken circuit) monomials basis $\{e_C | C \in NBC_k(\mathcal{A})\} \subseteq E_k$, i.e.;

$$\boldsymbol{NBC}_k(\mathcal{A}) = \bigoplus_{C\in NBC_k(\mathcal{A})} Ke_C \text{ and } \boldsymbol{NBC}_*(\mathcal{A}) = \bigoplus_{k=0}^{\ell} \boldsymbol{NBC}_k(\mathcal{A}).$$

1.15. Theorem: [11]

The broken circuit subcomplex $(\boldsymbol{NBC}_*(\mathcal{A}), \partial_*^{NBC})$ inherits a structure as acyclic chain complex from the exterior complex (E_*, ∂_*^E), where $\partial_*^{NBC} = \partial_*^E \circ i_*$ and $i_* : E_* \to \boldsymbol{NBC}_*(\mathcal{A})$ is the inclusion chain map.

Moreover, the restriction of the canonical chain map $\psi_*: E_* \to \boldsymbol{A}_*(\mathcal{A})$ of the broken circuit module $\boldsymbol{NBC}_*(\mathcal{A})$, is a chain isomorphism, defined as; for $1 \le k \le \ell$, $\psi_k(e_C) = e_C + I_k(\mathcal{A}) = a_C$, $C \in NBC_k(\mathcal{A})$. Thus, the O-S algebra has the following structure as a free K-submodule of the exterior algebra: $\boldsymbol{A}_*(\mathcal{A}) = \oplus_{k=0}^{\ell} (\oplus_{C \in NBC_k(\mathcal{A})} Ka_C)$.

1.16. Definition: [11]

Let $\Pi = (\Pi_1, \dots, \Pi_\ell)$ be a partition on an ℓ-arrangement $\mathcal{A}$ and let K be any commutative ring. A partition K-module is defined to be $(\Pi)_* = (\Pi_1)_* \otimes \dots \otimes (\Pi_\ell)_*$, where for $1 \le k \le \ell$, $(\Pi_k)_*$ is the free K-module with basis 1 and the elements of Π_k. For each $B = \{H_{i_1}, \dots H_{i_k}\} \in S_k(\Pi)$, i.e. $H_{i_m} \in \Pi_{i_m}$, $1 \le i_1 < \dots < i_k \le \ell$ and $1 \le m \le k$, define;$q_B = x_1 \otimes \dots \otimes x_\ell \in (\Pi)_*$ as;

$$x_j = \begin{cases} H_j & if\ j = i_m \text{ for some } 1 \le m \le k \\ 1 & if\ j \ne i_m \text{ for all } 1 \le m \le k \end{cases}$$

We agree that each of $q_{\emptyset_\ell} = 1 \otimes \dots \otimes 1$ and q_B is homogeneous of degree k. We denoting the k^{th}-homogeneous part of $(\Pi)_*$ by $(\Pi)_k$. Therefore, $(\Pi)_* = \oplus_{k=0}^{\ell} (\Pi)_k = \oplus_{k=0}^{\ell} (\oplus_{B \in S_k(\Pi)} Kq_B)$ and $\{q_B | B \in S_k(\Pi)\}$ forms a basis to the free K-module $(\Pi)_*$. Furthermore, $\{q_{\{H\}} | H \in \Pi_k\}$ forms a basis to the free K-module $(\Pi_k)_*$, $1 \le k \le \ell$. Define a K-linear mapping $\partial_*^\Pi: (\Pi)_* \to (\Pi)_*$ as; $\partial_0^\Pi\left(q_{\{\}}\right) = 0$, $\partial_1^\Pi(q_H) = 1$, for all $H \in \mathcal{A}$ and for $2 \le k \le \ell$, $\partial_k^\Pi(q_B) = \sum_{j=1}^{k} (-1)^{k-1} \widehat{q_{B_j}}$, where $B = \{H_{i_1}, \dots H_{i_k}\} \in S_k(\Pi)$, $q_B = x_1 \otimes \dots \otimes x_\ell$ as given in (1.8), and $\widehat{q_{B_j}} = x_1 \otimes \dots \otimes \widehat{H_{i_j}} \otimes \dots \otimes x_\ell$ by means of $\widehat{H_{i_j}} = 1$. ∂_*^π is a differentiation on $(\Pi)_*$ and the chain complex $((\Pi)_*, \partial_*^\pi)$ is called the partition complex;

$$0 \to (\Pi)_\ell \xrightarrow{\partial_\ell^\Pi} (\Pi)_{\ell-1} \xrightarrow{\partial_{\ell-1}^\Pi} \cdots \xrightarrow{\partial_2^\Pi} (\Pi)_1 \xrightarrow{\partial_1^\Pi} (\Pi)_0 \xrightarrow{\partial_0^\Pi} 0.$$

1.17. Definition: [11]

For $1 \le k \le \ell$, define the a map $\widetilde{\varphi}_k: \{q_B | B \in S_k(\Pi)\} \to \boldsymbol{A}_*(\mathcal{A})$, as $\varphi_k(q_B) = a_B = e_B + I_k(\mathcal{A})$, $B \in S_k(\Pi)$. Let $\varphi_k: (\Pi)_k \to \boldsymbol{A}_k(\mathcal{A})$ be the unique K-linear map that extend this assignment. Accordingly, there is a unique K-chain mapping $\varphi_*: (\Pi)_* \to \boldsymbol{A}_*(\mathcal{A})$ between acyclic chain complexes.

1.18. Theorem: [11]

The chain map $\varphi_*: (\Pi)_* \to \boldsymbol{A}_*(\mathcal{A})$ is a K-isomorphism between chain complexes if and only if the partition π is a Nice.

The theorems (1.16.) and (1.19), afford a K-isomorphism, $\chi_* = \psi_*^{-1} \circ \varphi_*: (\Pi)_* \to \boldsymbol{NBC}_*(\mathcal{A})$ between the partition complex and broken circuit complex.

1.19. Theorem: [11]

Let $\mathcal{A}$ be a complex ℓ-arrangement and let $\boldsymbol{A}_*(\mathcal{A})$ be its Orlik-Solomon algebra over the integer ring $\mathbb{Z}$. The map $e_H \mapsto (1/2\pi\sqrt{-1})\omega_H$ induces an isomorphism $\omega^*: \boldsymbol{A}_*(\mathcal{A}) \to H^*(M(\mathcal{A}), \mathbb{Z})$ of graded $\mathbb{Z}$-algebras, where $\omega_H = d\alpha_H/\alpha_H$ is the deferential 1-form for $H \in \mathcal{A}$ and $H = \ker(\alpha_H)$.

1.20. Theorem: [3]

For any commutative ring K and for $k \ge 0$;

$$H^k(M(\mathcal{A}), K) \cong H^k(M(\mathcal{A}), \mathbb{Z}) \otimes Tor(H^{k+1}(M(\mathcal{A}), \mathbb{Z}), K),$$

where $Tor(H^{k+1}(M(\mathcal{A}), \mathbb{Z}), K) = \ker(i^{k+1}, 1_K)$ from a free presentation;

$$0 \to R^{k+1} \xrightarrow{i^{k+1}} F^{k+1} \to H^{k+1}(M(\mathcal{A}), \mathbb{Z}) \to 0;$$

of $H^{k+1}(M(\mathcal{A}), \mathbb{Z})$ as generators F^{k+1} and relations R^{k+1}.

2. *An algorithm to compute the dimension of the first non vanishing higher homotopy group of the complement of hypersolvable graphic arrangement*

The advantage of studying the hypersolvable class of graphs lies in the fact it includes enormous applications, including the class of supersolvable (triangulated or rigid circuit) graphs, the class of graphs with no triangles and many others.

In view of definition (1.1.) and definition (1.4.), an algorithm to reorder the vertices and the edges of G by an order that preserve the hypersolvable structure of G was stated. So, we will used this algorithm to compute $p\big(M(\mathcal{A}_G)\big)$, the dimension of the first non vanishing higher homotopy group of $M(\mathcal{A}_G)$ for any hypersolvable graphic arrangement that not supersolvable as follows:

2.1. Construction:

Let G be a hypersolvable graph with hypersolvable partition $\Pi^G = (\Pi^V, \Pi^\mathcal{E})$ and a hypersolvable ordering $\trianglelefteq$. Assume, $\Pi^\mathcal{E}$ has s singular blocks say, $\Pi^\varepsilon_{l_1}, \dots, \Pi^\varepsilon_{l_s}$, $2 < l_1 \le \dots \le l_s \le \ell$. Due definition (1.1.) and definition (1.5.), we will reordering the vertices and the edges of G by the hypersolvable order that preserve Π^G structure. Since G is not supersolvable, hence it has a k-circuit (cycle) with no chord, $k \ge 4$. Every k- circuit, forms a k-Polygon, $k \ge 4$ and there is no mention about how many such circuit are there of G.

2.2. Theorem:

Suppose we have the conclusions of construction (2.1.). If;

$$D = \{C \subseteq G | C \text{ is a } j - \text{circuit with no chord}, j \ge 4\};$$

then $s = |D|$. In fact, $p(M(\mathcal{A}_G)) = c - 2$, where;

$$c(G) = c = Min\{|C| \big| C \in D\}.$$

Proof: First, we will prove $s = |D|$. So we need to verify that, the edges of a j-circuit with no chord, $j \ge 4$, must be distributed among j different blocks of $\Pi^\mathcal{E}$.

By contrary, assume there exists a j-circuit C with no chord and a block $\Pi^\mathcal{E}_i$ of $\Pi^\mathcal{E}$ contains two edges of C say e_1 and e_2. From the complete property of $\Pi^\mathcal{E}_i$ (lemma (1.3.)), there exists an edge $e \in \Pi^\mathcal{E}_1 \cup \dots \cup \Pi^\mathcal{E}_{i-1}$, such that $\{e_1, e_2, e\}$ is a triangle. This contradicts our assumption that C is a j-circuit with no chord. So, inductively the edges of C must be contained in j different blocks of $\Pi^\mathcal{E}$ and via the hypersolvable ordering the maximal edge e' satisfied that there is no vertex added to V_{j-1} , (i.e. $V_j = V_{j-1}$). Thus, the block that contains e' must be a singleton. Therefore, $s = |D|$.

Secondly, if $c(G) = c = Min\{|C| \big| C \in D\}$, we prove $p\big(M(\mathcal{A}_G)\big) = c - 2$. Recall Ali conjecture of $p\big(M(\mathcal{A}_G)\big)$ from [1] as, $p\big(M(\mathcal{A}_G)\big) = \{k| |NBC_k(\mathcal{A}_G)| = |S_k(\Pi')| \}$, where $NBC_k(\mathcal{A}_G)$ be the set of all k-NBC bases of the hypersolvable grphic arrangement $\mathcal{A}_G$ via the hypersolvable ordering and $S_k(\Pi)$ is the set of all k-sections of the induced hypersolvable partition Π' due Π^G. According our first part proof, $\mathcal{A}_C$ is a c-circuit of $\mathcal{A}_G$ and $\mathcal{A}_C/\{H_{i_1 i_2}\} \in S_{c-1}(\Pi')$ is its broken circuit. Thus, $|NBC_{c-1}(\mathcal{A}_G)| \neq |S_{c-1}(\Pi')|$. Therefore, $\big(M(\mathcal{A}_G)\big) = \{k| |NBC_k(\mathcal{A}_G)| = |S_k(\Pi')|\} = c - 2$. ■

2.3. Deformation method:

Suppose we have the conclusion of construction (2.1.). It is worth pointing out that, any hypersolvable graph can be deformed into a supersolvable graph either by adding edges or by deleting edges of every k- Polygon with no chord. So, we can easily use the hypersolvable partition $\Pi^\mathcal{E}$ and its exponent vector $d = (d_1, \dots, d_\ell)$ to

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complete the graph G either by just adding edges to deform G into a complete graph B_m or by adding vertices and edges to deform G into a complete graph B_ℓ by a simple comparing with;

$$\pi^{\varepsilon(B_m)} = (\{[v_1, v_2]\}, \{[v_1, v_3], [v_2, v_3]\}, \dots, \{[v_1, v_m], \dots, [v_{m-1}, v_m]\});$$

that has exponent vector $d^{B_m} = (1,2, \dots, m-1)$, or;

$$\pi^{\varepsilon(B_\ell)} = (\{[v_1, v_2]\}, \{[v_1, v_3], [v_2, v_3]\}, \dots, \{[v_1, v_\ell], \dots, [v_{\ell-1}, v_\ell]\});$$

that has exponent vector $d^{B_\ell} = (1,2, \dots, \ell-1)$.

For case (1): if d_{v_k} is the number of the edges of $\Pi^{\mathcal{E}}$ that contain v_k as a vertex, then we will add $(m-1) - d_{v_k}$ edges, for $1 < k \leq m$ in order to connect v_k with the other $(m-1) - d_{v_k}$ vertices of V to produce a complete graph B_m.

For case (2), we will add $\ell - m$ vertices and $k - d_k$ edges to the block Π_k^{ε}, for $2 < k \leq \ell$ in order to deform G into B_ℓ.

However, we can deform G into a hypersolvable graph G^1 by deleting every non-singular block of $\Pi^{\mathcal{E}}$, i.e. we will delete the s edges that related to the s singular blocks of $\Pi^{\mathcal{E}}$. But by using this procedure, the resulting deformed arrangement G^1 is either supersolvable or hypersolvable which is not supersolvable. So we need to iterate the process until we require our deformed supersolvable graph G^b where b presents the repetition number of the process. Via this deformation method, there is no vertex will be deleted. On the other hand, every k-cycle of G with no chord will be broken, for $k \geq 4$.

In the following we emphasis a special kind of graphs:

2.4. Construction:

Let $G = (V, \varepsilon)$ be a hypersolvable graph with hypersolvable partition $\Pi^G = (\Pi^V, \Pi^{\mathcal{E}})$ and a hypersolvable ordering $\trianglelefteq$. Assume, $\Pi^{\mathcal{E}}$ has just one singular block and it is the last one, i.e. $rk(\Pi_m^{\varepsilon}) = |V| = m$. Since G is not supersolvable, hence it has a k- circuit (cycle) with no chord, $k \geq 4$. In this case, there is just one m-circuit say;

$$C = (\{v_{i_1}, \dots, v_{i_2}\}, \{[v_{i_1}, v_{i_2}], [v_{i_2}, v_{i_3}], \dots, [v_{i_{m-1}}, v_{i_m}], [v_{i_1}, v_{i_m}];$$

with no chord and the edge $[v_{i_1}, v_{i_m}]$ of Π_m^{ε} is the maximal one of the k-circuit C via $\trianglelefteq$. Due the deformation method (2.3.), G can be deformed into a supersolvable graph by deleting the edge $[v_{i_1}, v_{i_m}]$ of the block Π_m^{ε}. Put, $G^1 = (V^1 = V, \varepsilon^1 = \varepsilon - \Pi_m^{\varepsilon})$ to be the deformed supersolvable graph. Definitely, G^1 is a supersolvable graph.

2.5. Corollary:

Suppose we have the conclusions of construction (2.4.). Then $p(M(\mathcal{A}_G)) = m - 2$.

Proof: This is a direct result of theorem (2.2.). ■

2.6. Example:

Every m-generic graph $G = (V, \varepsilon)$ is a graph with just one singular block is the block Π_m^{ε} and $p(M(\mathcal{A}_G)) = m - 2$. For example, the graph in figure (1.) is a 5-generic graph with $p(M(\mathcal{A}_G)) = 3$ and figure (2.) shows its supersolvable deformed graph by deleting the last edge;

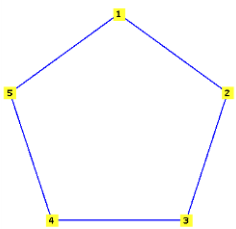

Figure 1. A 5-generic graph **Figure 2.** A defomed graph of 5-generic graph

2.7. Construction:

Let $G=(V,\varepsilon)$ be a hypersolvable graph with hypersolvable partition $\Pi^G=(\Pi^V,\Pi^{\mathcal{E}})$ and a hypersolvable ordering $\trianglelefteq$. Assume, $\Pi^{\mathcal{E}}$ has just one singular block and it is Π^{ε}_{m-1}, i.e. $rk(\Pi^{\varepsilon}_{m-1})=|V|-1=m-1$. Thus, G has a k- circuit (cycle) with no chord, $k\geq 4$. Actually, G has just one k- circuit say;

$$C=(\{v_{i_1},\dots,v_{i_2}\},\{[v_{i_1},v_{i_2}],[v_{i_2},v_{i_3}],\dots,[v_{i_{k-1}},v_{i_k}],[v_{i_1},v_{i_k}];$$

with no chord and the edge $[v_{i_1},v_{i_k}]$ of Π^{ε}_{m-1} is the maximal one of the k-circuit C via $\trianglelefteq$. For this case, we cannot guess that G can be deformed into a supersolvable graph by deleting just one edges and example (2.9.) demonstrate this goal.

2.8. Corollary:

Suppose we have the conclusions of construction (2.7.). Then $p(M(\mathcal{A}_G))=k-2$.

Proof: This is a direct result of theorem (2.2.). ■

2.9. Example:

Let G and G' be the graphs shown in figure (3.) and figure (4.) respectively. then each one of them has $p(M(\mathcal{A}_G))=2$.

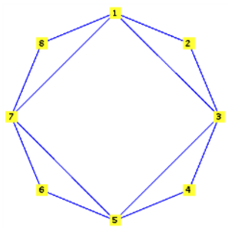

Figure 3. The graph G

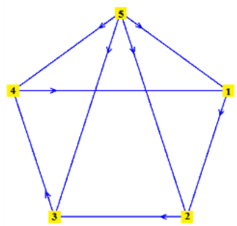

Figure 4. The graph G'

Each one of them can be deformed easly by deleting edges into a supersolvabe graphs as shown in figure (5.) and figure (6.) respectively:

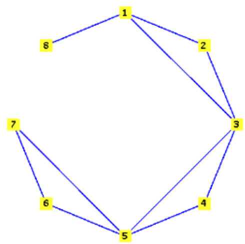

Figure 5. A deformed graph of G

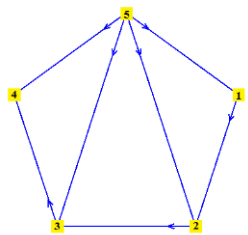

Figure 6. A deformed graph of G'

2.10. Example:

Let G, G' and G'' be the graphs shown in the figures (2.10.1.), (2.10.2.) and (2.10.3.) respectvely. The graph G has $p(M(\mathcal{A}_G))=2$ with $s=15$ singular blocks of $\Pi^{\mathcal{E}}$, the graph G' has $p(M(\mathcal{A}_{G'}))=3$ with $s=31$ singular block of $\Pi^{\mathcal{E}'}$ and the graph G'' has $p(M(\mathcal{A}_{G''}))=3$ with $s=27$ singular blocks of $\Pi^{\mathcal{E}''}$. Deduce that, in spit of, each one of the graphs G, G' and G'' has no triangle, they are not generic.

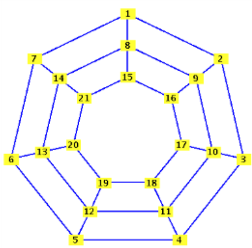

Figure 7. The graph G

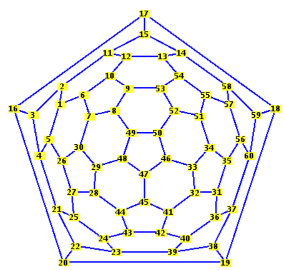

Figure 8. The graph G'

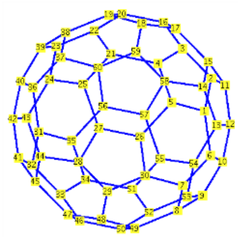

Figure 9. The graph G''

It is clear that, to deform any one of the graphs above by just deleting edges will be more complecated and it cannot be by applying the method for just one step.

3. The cohomological ring of a hypersolvable graphic arrangement

In this section we restricted, a construction of the cohomological ring of the complement of any hypersolvable arrangement discribed in [3], on the complement of any hypersolvable graphic arrangement by using the hypersolvable partition of a graph structure, as follows:

3.1. Theorem:

Let G be a supersolvable graph with a supersolvable partition $\Pi^G = (\Pi^V, \Pi^\mathcal{E})$ and a hypersolvable ordering $\trianglelefteq$ and let Π' be its induced supersolvable partition on $\mathcal{A}_G$. Then $NBC(\mathcal{A}_G) = S(\Pi')$ and for $1 \leq k \leq rk(G) = \ell - 1$;

$$b_k(\boldsymbol{H}^*(M(\mathcal{A}_G))) = \sum_{i_1=1}^{\ell-k-1} \sum_{i_2=i_1+1}^{\ell-k} \dots \sum_{i_k=i_{k-1}+1}^{\ell-1} d_{i_1} d_{i_2} \dots d_{i_k}.$$

and the cohomological ring $\boldsymbol{H}^*(M(\mathcal{A}_G))$ can be determined by the following commutative diagram:

$$\begin{array}{ccccccccccccc}
0 \to & \boldsymbol{H}^{\ell-1}(M(\mathcal{A}_G)) & \xrightarrow{\partial^H_{\ell-1}} & \boldsymbol{H}^{\ell-2}(M(\mathcal{A}_G)) & \xrightarrow{\partial^H_{\ell-2}} \cdots \xrightarrow{\partial^H_2} & \boldsymbol{H}^1(M(\mathcal{A}_G)) & \xrightarrow{\partial^H_1} & \boldsymbol{H}^1(M(\mathcal{A}_G)) & \xrightarrow{\partial^H_0} 0 \\
& \omega^{-1}_{\ell-1} \downarrow & & \omega^{-1}_{\ell-2} \downarrow & & \omega^{-1}_1 \downarrow & & \omega^{-1}_0 \downarrow & \\
0 \to & \boldsymbol{A}_{\ell-1}(\mathcal{A}_G) & \xrightarrow{\partial^A_{\ell-1}} & \boldsymbol{A}_{\ell-2}(\mathcal{A}_G) & \xrightarrow{\partial^A_{\ell-2}} \cdots \xrightarrow{\partial^A_2} & \boldsymbol{A}_1(\mathcal{A}_G) & \xrightarrow{\partial^A_1} & \boldsymbol{A}_0(\mathcal{A}_G) & \xrightarrow{\partial^A_0} 0 \\
& \psi^{-1}_{\ell-1} \downarrow & & \psi^{-1}_{\ell-2} \downarrow & & \psi^{-1}_1 \downarrow & & \psi^{-1}_0 \downarrow & \\
0 \to & \boldsymbol{NBC}_{\ell-1}(\mathcal{A}_G) & \xrightarrow{\partial^N_{\ell-1}} & \boldsymbol{NBC}_{\ell-2}(\mathcal{A}_G) & \xrightarrow{\partial^N_{\ell-2}} \cdots \xrightarrow{\partial^N_2} & \boldsymbol{NBC}_1(\mathcal{A}_G) & \xrightarrow{\partial^N_1} & \boldsymbol{NBC}_0(\mathcal{A}_G) & \xrightarrow{\partial^N_0} 0 \\
& \mathcal{J}_{\ell-1} \downarrow & & \mathcal{J}_{\ell-2} \downarrow & & \mathcal{J}_1 \downarrow & & \mathcal{J}_0 \downarrow & \\
0 \to & (\Pi')_{\ell-1} & \xrightarrow{\partial^\pi_{\ell-1}} & (\Pi')_{\ell-2} & \xrightarrow{\partial^\pi_{\ell-2}} \cdots \xrightarrow{\partial^\pi_2} & (\Pi')_1 & \xrightarrow{\partial^\pi_1} & (\Pi')_0 & \xrightarrow{\partial^\pi_0} 0
\end{array}$$

Proof: This is a direct result of theorem ((2.4.), [3]) and theorem (1.15.), where the K-chain map $\mathcal{J}_*: \boldsymbol{NBC}_*(\mathcal{A}_G) \to (\Pi')_*$ is the unique K-isomorphism that extends the one to one correspondence between the bases of $\boldsymbol{NBC}_*(\mathcal{A}_G)$ and $(\Pi')_*$;

$$\mathcal{J}_*: \{e_B | B \in NBC(\mathcal{A}_G) = S(\Pi')\} \to \{q_B | B \in S(\Pi')\},$$

that defined as, $\mathcal{J}_*(e_B) = q_B,\ B \in S(\Pi')$. ■

3.2. Theorem:

Let G be a hypersolvable graph with a hypersolvable partition $\Pi^G = (\Pi^V, \Pi^\mathcal{E})$ and a hypersolvable ordering $\trianglelefteq$ such that $rk(\mathcal{A}_G) = m - 1 < \ell$, i.e. G is not supersolvable. Then, due theorem (2.2.);

$$2 \leq p(\mathcal{A}_G) = c - 2 \leq m - 2;$$

and for $1 \leq k \leq c - 2$;

$$NBC_k(\mathcal{A}_G) = S_k(\Pi'),\ NBC_{c-1}(\mathcal{A}_G) = S_{c-1}(\Pi') \backslash S_{c-1}(\Pi') \cap BC_{c-1}(\mathcal{A}_G);$$

and for $c \leq k \leq m - 1$, $NBC_k(\mathcal{A}_G) \subset S_k(\Pi')$. The cohomological group $\boldsymbol{H}^*(M(\mathcal{A}_G))$ can be determined by the following commutative diagrams:

$$
\begin{array}{ccccccccc}
\boldsymbol{H}^{c-1}(M(\mathcal{A}_G)) & \xrightarrow{\partial^H_{c-1}} & \boldsymbol{H}^{c-2}(M(\mathcal{A}_G)) & \xrightarrow{\partial^H_{c-2}} \cdots \xrightarrow{\partial^H_{2}} & \boldsymbol{H}^{1}(M(\mathcal{A}_G)) & \xrightarrow{\partial^H_{1}} & \boldsymbol{H}^{1}(M(\mathcal{A}_G)) & \xrightarrow{\partial^H_{0}} & 0 \\
\omega^{-1}_{c-1}\downarrow & & \omega^{-1}_{c-2}\downarrow & & \omega^{-1}_{1}\downarrow & & \omega^{-1}_{0}\downarrow & & \\
\boldsymbol{A}_{c-1}(\mathcal{A}_G) & \xrightarrow{\partial^A_{c-1}} & \boldsymbol{A}_{c-2}(\mathcal{A}_G) & \xrightarrow{\partial^A_{c-2}} \cdots \xrightarrow{\partial^A_{2}} & \boldsymbol{A}_{1}(\mathcal{A}_G) & \xrightarrow{\partial^A_{1}} & \boldsymbol{A}_{0}(\mathcal{A}_G) & \xrightarrow{\partial^A_{0}} & 0 \\
\psi^{-1}_{c-1}\downarrow & & \psi^{-1}_{c-2}\downarrow & & \psi^{-1}_{1}\downarrow & & \psi^{-1}_{0}\downarrow & & \\
\boldsymbol{NBC}_{c-1}(\mathcal{A}_G) & \xrightarrow{\partial^N_{c-1}} & \boldsymbol{NBC}_{c-2}(\mathcal{A}_G) & \xrightarrow{\partial^N_{c-2}} \cdots \xrightarrow{\partial^N_{2}} & \boldsymbol{NBC}_{1}(\mathcal{A}_G) & \xrightarrow{\partial^N_{1}} & \boldsymbol{NBC}_{0}(\mathcal{A}_G) & \xrightarrow{\partial^N_{0}} & 0 \\
\not{f}_{c-1}\downarrow & & \mathcal{J}_{m-2}\downarrow & & \mathcal{J}_{1}\downarrow & & \mathcal{J}_{0}\downarrow & & \\
(\Pi')_{c-1} & \xrightarrow{\partial^\pi_{c-1}} & (\Pi')_{c-2} & \xrightarrow{\partial^\pi_{c-2}} \cdots \xrightarrow{\partial^\pi_{2}} & (\Pi')_{1} & \xrightarrow{\partial^\pi_{1}} & (\Pi')_{0} & \xrightarrow{\partial^\pi_{0}} & 0
\end{array}
$$

and;

$$
\begin{array}{ccccc}
0 \to \boldsymbol{H}^{m-1}(M(\mathcal{A}_G)) & \xrightarrow{\partial^H_{m-1}} \cdots \xrightarrow{\partial^H_{c+1}} & \boldsymbol{H}^{c}(M(\mathcal{A}_G)) & \xrightarrow{\partial^H_{c}} \\
\omega^{-1}_{m-1}\downarrow & & \omega^{-1}_{c}\downarrow & \\
0 \to \boldsymbol{A}_{m-1}(\mathcal{A}_G) & \xrightarrow{\partial^A_{m-1}} \cdots \xrightarrow{\partial^A_{c+1}} & \boldsymbol{A}_{c}(\mathcal{A}_G) & \xrightarrow{\partial^A_{c}} \\
\psi^{-1}_{m-1}\downarrow & & \psi^{-1}_{c}\downarrow & \\
0 \to \boldsymbol{NBC}_{m-1}(\mathcal{A}_G) & \xrightarrow{\partial^N_{m-1}} \cdots \xrightarrow{\partial^N_{c+1}} & \boldsymbol{NBC}_{c}(\mathcal{A}_G) & \xrightarrow{\partial^N_{c}} \\
\not{f}_{m-1}\downarrow & & \not{f}_{c}\downarrow & \\
0 \to (\Pi')_\ell \xrightarrow{\partial^\pi_{\ell}} \cdots \xrightarrow{\partial^\pi_{m+1}} (\Pi')_m \xrightarrow{\partial^\pi_{m}} (\Pi')_{m-1} & \xrightarrow{\partial^\pi_{m-1}} \cdots \xrightarrow{\partial^\pi_{c+1}} & (\Pi')_{c} & \xrightarrow{\partial^\pi_{c}}
\end{array}
$$

Proof: According theorem ((2.5), [3]) and theorem (2.2.), our claim is proved, where the K-chain map $\not{f}_*: \boldsymbol{NBC}_*(\mathcal{A}_G) \to (\Pi')_*$ is the unique K-injective chain map that extends the one to one mapping that embedding the NBC basis of $\boldsymbol{NBC}_*(\mathcal{A}_G)$ of the basis of $(\Pi')_*$;

$$\not{f}_*: \{e_B | B \in NBC(\mathcal{A}_G) \subseteq S(\Pi')\} \to \{q_B | B \in S(\Pi')\},$$

that defined as, $\not{f}_*(e_B) = q_B,\ B \in NBC(\mathcal{A}_G)$. Recall the definition of $p(\mathcal{A}_G) = c - 2$ of theorem (2.2.). In fact, for $1 \le k \le c-2$, since $NBC_k(\mathcal{A}_G) = S_k(\Pi')$, hence, $\not{f}_k = \mathcal{J}_k: \boldsymbol{NBC}_k(\mathcal{A}_G) \to (\Pi')_k$ is an isomorphism. Moreover, for $c-1 \le k \le m-1$, the homomorphism $\not{f}_k = \mathcal{J}_k: \boldsymbol{NBC}_k(\mathcal{A}_G) \to (\Pi')_k$ is a monomorphism since $NBC_k(\mathcal{A}_G) \subset S_k(\Pi')$. ■

3.3. Corollary:

Let G be a hypersolvable graph with hypersolvable partition, $\Pi^G = (\Pi^V, \Pi^{\mathcal{E}})$ such that $m \ge 4$ and it has an exponent vector $d = (1, \dots, 1)$, i.e. G has no triangle. Then we have the following:

1. If $|\mathcal{E}| = \ell = m - 1$, then G is supersolvable and the cohomological ring has a structure as shown in theorem (3.1.) and for $1 \le j \le m-1$, $b_j(\boldsymbol{H}^*(M(\mathcal{A}_G))) = \binom{m-1}{j}$.

2. If $c(G) = \ell = m$, then G is generic have just one m-cycle and for $1 \leq j < m-1$, $b_j(\boldsymbol{H}^*(M(\mathcal{A}_G))) = |NBC_j(\mathcal{A}_G)| = \binom{m}{j}$ and $b_{m-1}(\boldsymbol{H}^*(M(\mathcal{A}_G))) = m-1$. Due theorem (3.2.), the cohomological ring $\boldsymbol{H}^*(M(\mathcal{A}_G))$ can be determined as the following commutative diagram;

$$\begin{array}{ccccccccccc}
0 \to & \boldsymbol{H}^{m-1}(M(\mathcal{A}_G)) & \xrightarrow{\partial^H_{m-1}} & \boldsymbol{H}^{m-2}(M(\mathcal{A}_G)) & \xrightarrow{\partial^H_{m-2}} \cdots \xrightarrow{\partial^H_2} & \boldsymbol{H}^1(M(\mathcal{A}_G)) & \xrightarrow{\partial^H_1} & \boldsymbol{H}^1(M(\mathcal{A}_G)) & \xrightarrow{\partial^H_0} 0 \\
& \omega^{-1}_{m-1} \downarrow & & \omega^{-1}_{m-2} \downarrow & & \omega^{-1}_1 \downarrow & & \omega^{-1}_0 \downarrow & \\
0 \to & \boldsymbol{A}_{m-1}(\mathcal{A}_G) & \xrightarrow{\partial^A_{m-1}} & \boldsymbol{A}_{m-2}(\mathcal{A}_G) & \xrightarrow{\partial^A_{m-2}} \cdots \xrightarrow{\partial^A_2} & \boldsymbol{A}_1(\mathcal{A}_G) & \xrightarrow{\partial^A_1} & \boldsymbol{A}_0(\mathcal{A}_G) & \xrightarrow{\partial^A_0} 0 \\
& \psi^{-1}_{m-1} \downarrow & & \psi^{-1}_{m-2} \downarrow & & \psi^{-1}_1 \downarrow & & \psi^{-1}_0 \downarrow & \\
0 \to & \boldsymbol{NBC}_{m-1}(\mathcal{A}_G) & \xrightarrow{\partial^N_{m-1}} & \boldsymbol{NBC}_{m-2}(\mathcal{A}_G) & \xrightarrow{\partial^N_{m-2}} \cdots \xrightarrow{\partial^N_2} & \boldsymbol{NBC}_1(\mathcal{A}_G) & \xrightarrow{\partial^N_1} & \boldsymbol{NBC}_0(\mathcal{A}_G) & \xrightarrow{\partial^N_0} 0 \\
& \mathcal{J}_{m-1} \downarrow & & \mathcal{J}_{m-2} \downarrow & & \mathcal{J}_1 \downarrow & & \mathcal{J}_0 \downarrow & \\
0 \to (\Pi')_m \xrightarrow{\partial^\pi_m} & (\Pi')_{m-1} & \xrightarrow{\partial^\pi_{m-1}} & (\Pi')_{m-2} & \xrightarrow{\partial^\pi_{m-2}} \cdots \xrightarrow{\partial^\pi_2} & (\Pi')_1 & \xrightarrow{\partial^\pi_1} & (\Pi')_0 & \xrightarrow{\partial^\pi_0} 0
\end{array}$$

3. If $c(G) \leq m-1 < \ell$, then G is neither supersolvable nor generic and for $1 \leq j \leq c(G)-2$, $b_j(\boldsymbol{H}^*(M(\mathcal{A}_G))) = |NBC_j(\mathcal{A}_G)| = \binom{m}{j}$ and $b_{c-1}\left(\boldsymbol{H}^*(M(\mathcal{A}_G))\right) = \binom{m}{c-1} - |BC_{c-1}(\mathcal{A}_G)| = \binom{m}{c-1} - |O|$, where $O = \{C \subseteq G | C \text{ is a } c-\text{circuit with no chord}\}$ and the cohomological ring $\boldsymbol{H}^*(M(\mathcal{A}_G))$ can be determined as shown in theorem (3.2.).

Proof: Due theorem (2.2.4) in [5] and theorem (2.2.), corollary claim is down. ■

4. The second skeleton of the minimal CW complex for a hypersolvable graphic arrangements

This section contains an algorithm to comput the second skeleton of the complement of a hypersolvable graphic arrangement by using a fashoin of its fundamental group as iterated semi direct product that presented in [4] by Cohen and Suciu. This algorithm technique has previously been introduced by Switzer in [16]. So we will agree this algorithm without proof and see [4, 12, 16] as evidences. In [3], Al-Taai and the author was firstly used this technique in order to give a topological interpretation for vanishing of higher homotopy groups of the complement of a hypersolvable arrangement when we deformed it by Jambu's and Papadima's deformation method, so for general case we refer the reader to [3].

We start by reviewing the definitions and basic facts that we needed for the algorithm:

4.1. Definition: [12]

A topological space X with the following properties:

1. X is homotopy equivalent to a connected, finite type CW complex;

2. The homology groups $H_*(X)$ are torsion free, and;

3. The cup product $\cup : \wedge H^1(X) \to H^*(X)$ is surjective;

is said to be p-minimal, for some non-negative integer p, if it has the homotopy type of a CW complex $\boldsymbol{K}$ such that the number of k-cells in $\boldsymbol{K}$ is $b_k(X) = rk(H^*(X))$, for all $k \leq p$. We called X minimal if it is p-minimal, for all p.

4.2. Definition: [4]

Assume each of $G_1, \dots, G_\ell$ be a group, and for $1 \le i < j < \ell$, the action $\alpha_j^i : G_i \to Aut(G_j)$ satisfying the compatibility conditions, $\alpha_k^j\left(g_j^{\alpha_k^i(g_i)}\right) = (\alpha_k^j(g_i))^{-1}\alpha_k^i(g_j)\alpha_k^j(g_i)$, for $i < j < k$. Then, we define the iterated semi direct product of $G_1, \dots, G_\ell$ with respect to the actions α_j^i to be the group;

$$G = G_\ell \propto_{\alpha_\ell} G_{\ell-1} \propto_{\alpha_{\ell-1}} \dots \propto_{\alpha_2} G_1,$$

where for each $1 \le k \le \ell$, the partial iteration $G^k = G_k \propto_{\alpha_k} G^{k-1}$ is defined by the homomorphism $\alpha_k : G^{k-1} \to Aut(G_k)$ with a restriction to G_k; $\alpha_{k/G_p} : G_p \to Aut(G_k)$, $1 \le p < k \le \ell$.

4.3. Definition: [11]

Let $\mathcal{A}$ be a complex central essential r-arrangement with complement $M(\mathcal{A}) \subseteq \mathbb{C}^r$. Define a stratification $\mathfrak{I}$ of $\mathbb{C}^r$ as follows:

1. For each $X \in L(\mathcal{A})$, determine the arrangement $\mathcal{A}^X = \{H \cap X \mid H \in \mathcal{A} \backslash \mathcal{A}_X \text{ and } H \cap X \neq \emptyset\}$ of X, where $\mathcal{A}_X = \{H \in \mathcal{A} | X \subseteq H\} \subseteq \mathcal{A}$, and;
2. Dfine M^X to be the complement of $\mathcal{A}^X$ of X.

Notice that the family $\{M^X\}_{X \in L(\mathcal{A})}$ forms a stratification of $\mathbb{C}^r$ with top dimensional stratum $M(\mathcal{A})$ and each strata is a convex relatively open sets of X.

We emphasize that, Switzer in [16] showed that, for any topological space X, one can construct a CW complex Y (as showed in the following construction), and a weak homotopy equivalence $f : X \to Y$ and this construction is unique up to homotopy.

4.4. Construction:

Let G be a supersolvable graph with a supersolvable graphic $\ell = (m-1)$-arrangement $\mathcal{A}_G$. Then $\mathcal{A}_G$ has a maximal chain of modular elements say;

$$\mathbb{C}^\ell = X_0 < \dots < X_\ell = \{(0, \dots, 0);$$

which induces a supersolvable composition series;

$$\{H\} = \mathcal{A}_{X_1} \subset \dots \subset \mathcal{A}_{X_\ell} = \mathcal{A}_G \dots \text{(4.4.1.)}$$

$\mathcal{A}_G$ is a fiber type arrangement and the composition series (4.4.1.) creates a tower of fibrations;

$$M(\mathcal{A}_G) = M(\mathcal{A}_{X_\ell}) \xrightarrow{p_{\ell-1}} M(\mathcal{A}_{X_{\ell-1}}) \xrightarrow{p_{\ell-1}} \dots \xrightarrow{p_1} M(\mathcal{A}_{X_1}) = M(H) = \mathbb{C}\backslash\{0\};$$

with fiber F^k of p_k homeomorphic to $\mathbb{C}$ with d_k points removed and the fundamental group of the complement $\pi = \pi_1(M(\mathcal{A}_G))$ asserts a fashion of iterated semi direct product of finitely generated groups $\pi = F_{d_\ell} \propto_{\alpha_\ell} F_{d_{\ell-1}} \propto_{\alpha_{\ell-1}} \dots \propto_{\alpha_2} F_{d_1}$, where $F_{d_k} = \langle g_{1,k}, \dots, g_{d_k,k} \rangle$ is free on d_k generators. This creates a nice partition $\Pi = (\Pi_1, \dots, \Pi_\ell)$ as follows;

1. Put $\Pi_1 = \mathcal{A}_{X_1}$ and we will choose $H \in \mathcal{A}_{X_1}$ to be the minimal hyperplane via the fundamental group order that generats F_{d_1}, and;
2. For $2 \le k \le \ell$, put $\Pi_k = \mathcal{A}_{X_k} \backslash \mathcal{A}_{X_{k-1}}$ and oder the hyperplanes of Π_k via the topological ordering that induced from the structure of F_{d_k} as free group with $g_{1,k}, \dots, g_{d_k,k}$ generators and preserve the fundamental group structure as;

$$\pi = \left\langle g_{i,k};\ \begin{matrix} 1 \le i \le d_k \\ 1 \le k \le \ell \end{matrix} \middle| \alpha_k^{j,p}(g_{i,k}) = g_{j,p}^{-1} g_{i,k} g_{j,p};\ \begin{matrix} 1 \le j \le d_p \\ 1 \le p < k \end{matrix} \right\rangle \qquad \text{(4.4.2.)}$$

where each $\alpha_k^{j,p} = \alpha_k(g_{j,p}) \in Aut(F_{d_k})$.

We will construct the second skeleton of a (finite type) minimal CW-complex structure of $M(\mathcal{A}_G)$ as a $K(\pi, 1)$ space that given in ([21], section 6.44, p. 95) induced from the presentation (4.4.2.) above as follows:

S1. Partitioned $\mathbb{C}^{\ell}$ by the stratification defined in definition (4.3.).

S2. Choose any point in $M(\mathcal{A}_G)$, say e^0 and put $M(\mathcal{A}_G)^{0} = \{e^0\}$ to be the 0th-skeleton of $M(\mathcal{A}_G)$.

S3. For each $H \in \mathcal{A}_G$, fixed a 1-cell e_H^1 and an attaching mapping $\varphi_H^1 : \partial e_H^1 \to \{e^0\}$ attached the boundaries of e_H^1 with e^0. Take, $M(\mathcal{A}_G)^{1} = \vee_{H\in\mathcal{A}} S^1 = \vee_{k=1}^{\ell}(\vee_{i=1}^{d_k} S^1_{g_{i,k}})$ to be the 1st- skeleton of $M(\mathcal{A}_G)$. Geometricly, for each $H \in \mathcal{A}_G$ we go around the stratum M^H and return into e^0 by e_H^1. Clearly; $\pi_0(M(\mathcal{A}_G), e^0) \cong \pi_0(M(\mathcal{A}_G)^1, e^0) \cong 0$, since $M(\mathcal{A}_G)^1$ is path connected.

S4. The following short exact sequence represents the presentation (4.4.2);

$$0 \to \langle r_\gamma;\quad 1 \le \gamma \le b_2(\mathcal{A}_G)\rangle \xrightarrow{\beta} \langle g_{i,k};\quad \begin{matrix} 1 \le i \le d_k \\ 1 \le k \le \ell \end{matrix}\rangle \xrightarrow{\alpha} \pi_1(M(\mathcal{A}_G), e^0) \to 0.$$

For each relation r_γ, choose a map $\varphi_\gamma^2 : (S^1, e^0) \to (M(\mathcal{A}_G)^1, e^0)$ by meaning of $\beta(r_\gamma)$. Attach 2-cells e_γ^2 of $M(\mathcal{A}_G)^1$ by the maps φ_γ^2 to create the second skeleton of $M(\mathcal{A}_G)$ as the following disjoint union;

$$M(\mathcal{A}_G)^{2} = M(\mathcal{A}_G)^{1} \coprod_{\substack{\gamma=1 \\ \varphi_\gamma^2}}^{b_2(M(\mathcal{A}_G))} e_\gamma^2 = M(\mathcal{A}_G)^{1} \coprod_{\substack{\gamma=1 \\ \varphi_\gamma^2}}^{b_2(M(\mathcal{A}_G))} S_\gamma^2$$

Put $f_\gamma^2 : (D^2, S^1, s_0) \to (M(\mathcal{A}_G)^1, \varphi_\gamma^2(S^1), e^0)$ be the characteristic map of e_γ^2, for $1 \le \gamma \le b_2(\mathcal{A}_G)$. Thus, $\pi_1(M(\mathcal{A}_G), e^0) \cong \pi_1(M(\mathcal{A}_G)^2, e^0)$ and;

$$M(\mathcal{A}_G)^2 / M(\mathcal{A}_G)^1 = \vee_{\gamma=1}^{b_2(M(\mathcal{A}_G))} S_\gamma^2.$$

4.5. Construction:

Let G be a hypersolvable graph with a hypersolvable graphic $(m-1)$-arrangement $\mathcal{A}_G$ that not supersolvable, recall Jambu's and Papadima's 1-parameter family $\{\tilde{\mathcal{A}}_t\}_{t\in\mathbb{C}}$ of deformed supersolvable arrangements that introduced firstly in [12]. We follow a computation algorithm given in [9] of $\tilde{\mathcal{A}} = \tilde{\mathcal{A}}_1$ for $\mathcal{A}_G$. The arrangement $\tilde{\mathcal{A}}$ is a supersolvable arrangements and it has with $\mathcal{A}_G$ the same Lattice intersection pattern to codimension two $\ell_2(\mathcal{A}_G) = \{B \subseteq \mathcal{A}_G \,||B| \le 3\}$ and isomorphic fundematal groups, i.e. $\pi = \pi_1(M(\mathcal{A}_G)) \cong \pi_1(M(\tilde{\mathcal{A}})) \cong \pi_1(M(\tilde{\mathcal{A}})^2, \tilde{e}^0)$, where $M(\tilde{\mathcal{A}})^2$ is the 2nd skeleton due [3]. Thus;

$$\pi \cong \pi_1(M(\tilde{\mathcal{A}})) = F_{d_\ell} \propto_{\alpha_\ell} F_{d_{\ell-1}} \propto_{\alpha_{\ell-1}} \dots \propto_{\alpha_2} F_{d_1};$$

derived a hypersolvable partition $\Pi = (\Pi_1, \dots, \Pi_\ell)$ by using the one to one coorespondance between $\mathcal{A}_G$ and $\tilde{\mathcal{A}}$. Due this one to one coorespondance reordered the hyperplanes of $\mathcal{A}_G$ via the ordering we defined on the hyperplanes of $\tilde{\mathcal{A}}$ as in construction (4.4.) that induced from the structure of the fundamental group. We will construct the second skeleton of $M(\mathcal{A}_G)$ exactly as designed in construction (4.4.), the items (S1-S4).

4.6. Remark:

The advantage of studying the second skeleton of a hypersolvable graphic arrangement $\mathcal{A}_G$ lies in the fact that, if $X \in L(\mathcal{A}_G)$ and $rk(X) = 2$, then either $|\mathcal{A}_X| = 2$ or 3. In fact, for $1 \le p < k \le \ell$, the colinear relations r_γ, for $1 \le \gamma \le b_2(\mathcal{A}_G)$, among the hyperplanes of $\mathcal{A}$ are associated to the triangles of ε. So, there are just two kinds of relations as follows:

1. If $X = \{H_{j,p}, H_{i,k}\}$, then, the action $\alpha_k^{j,p}(g_{i,k}) = g_{i,k}$ is trivial and the relation will be a usual commutator relation, i.e. $g_{i,k}^{-1} g_{j,p}^{-1} g_{i,k} g_{j,p} = 0$, i.e. we have a torus relation as the following figure:

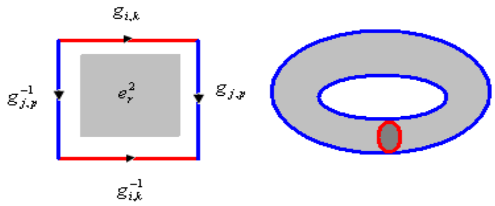

Figure 10. A trivial action . $g_{i,k}^{-1} g_{j,p}^{-1} g_{i,k} g_{j,p} = 0$

For example, if $Q(\mathcal{A}_G) = (x_1 - x_2)(x_2 - x_3)(x_3 - x_4)$, be the defining polynomial of an arrangement $\mathcal{A}_G$, then $\mathcal{A}_G$ is supersolvable graphic arrangement with fundamental group of its complement is;

$$\pi_1(M(\mathcal{A}), e^0) = \langle g_1, g_2, g_3 | \begin{array}{l} g_2 = g_1^{-1} g_2 g_1 \\ g_3 = g_1^{-1} g_3 g_1 \\ g_3 = g_2^{-1} g_3 g_2 \end{array} \rangle.$$

Then, it has second skeleton as;

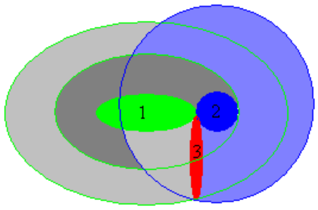

Figure 11. A second skeleton of $\mathcal{A}_G$, $Q(\mathcal{A}_G) = (x_1 - x_2)(x_2 - x_3)(x_3 - x_4)$

2. If $X = \{H_{1,p}, H_{2,k}, H_{3,k}\}$, we have the following relations and attaching mapping via those relations;

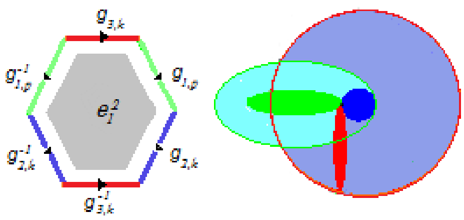

Figure 12. a. Part one of the action, $g_{1,p} g_{2,k} g_{3,k} = g_{3,k} g_{1,p} g_{2,k}$

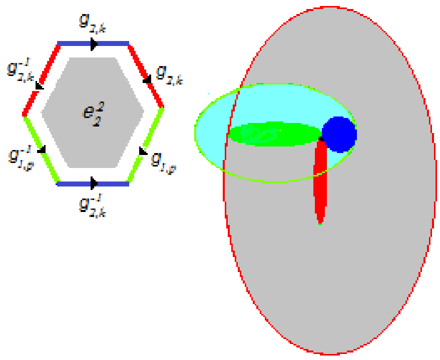

Figure 12. b. Part two of the action, $g_{2,k} g_{3,k} g_{1,p} = g_{3,k} g_{1,p} g_{2,k}$

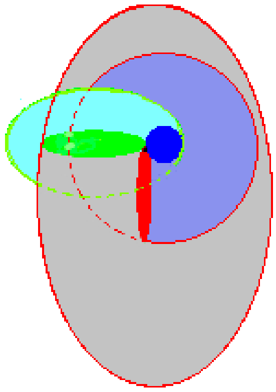

Figure 12. c. The second skeleton represented the action $g_{1,p}g_{2,k}g_{3,k} = g_{3,k}g_{1,p}g_{2,k} = g_{2,k}g_{3,k}g_{1,p}$

The second skeleton given in figure (4.6.5.), is the Minimal CW complex for the supersolvable graphic arrangement $\mathcal{A}_G$ that related to a graph given in figure (4.6.6.) has defining polynomial, $Q(\mathcal{A}_G) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$, and its complement is homotopic to $(S^1 \vee S^1) \times S^1$.

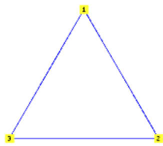

Figure 13.

We mentioned here that, due [13], the relation in figure (4.4.7) is selfed contained in figure (4.5.6.), so there are no attaching cell related to this relation that correspondence to a broken circuit of $\mathcal{A}_G$ via fandamental group order.

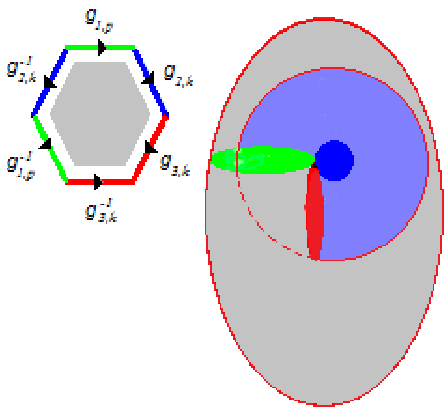

Figure 14. Part three of the action, $g_{1,p}g_{2,k}g_{3,k} = g_{2,k}g_{1,p}g_{3,k}$

By following the fundamental group structure of the complement of any graphic arrangement and the type of the actions among the different blocks of Π that we discussed above, the second skeleton has a regular construction. We leave it to the reader to construct the second skeleton for the section (2) examples as we shown in the following example:

5.2. Example:

Let $\mathcal{A}_G$ be a generic graphic ℓ-arrangement. Then;

$$Q(\mathcal{A}_G) = (x_1 - x_2)(x_2 - x_3)(x_3 - x_4) \ldots (x_{\ell-1} - x_\ell)(x_\ell - x_1),$$

Be its defining polynomial. The fundamental group of its complement $M(\mathcal{A}_G)$ has a structure as;

$$\pi_1(M(\mathcal{A}_G), e^0) = \langle g_1, g_2, \ldots, g_\ell \mid \quad g_k = g_p^{-1} g_k g_p, 1 \le p < k \le \ell \rangle.$$

Due to [9], the deformed arrangement $\tilde{\mathcal{A}}$ of $\mathbb{C}^{\ell+1}$ has a defining polynomial;

$$Q(\tilde{\mathcal{A}}) = (x_1 - x_2)(x_2 - x_3)(x_3 - x_4) \dots (x_{\ell-2} - x_{\ell-1})(x_{\ell-1} - x_1 + x_{\ell+1}).$$

And by applying construction (4.5.), we will construct the second skeleton of $M(\mathcal{A}_G)$ as follows:

S1. Partitioned $\mathbb{C}^\ell$ by the stratification defined in definition (4.3.).

S2. Choose any point in $M(\mathcal{A}_G)$, say e^0 and put $M(\mathcal{A}_G)^{\ 0} = \{e^0\}$ to be the 0th-skeleton of $M(\mathcal{A}_G)$.

S3. For each $H \in \mathcal{A}_G$, fixed a 1-cell e_H^1 and an attaching mapping $\varphi_H^1 : \partial e_H^1 \to \{e^0\}$ attached the boundaries of e_H^1 with e^0. Take, $M(\mathcal{A}_G)^{\ 1} = \vee_{H\in\mathcal{A}} S^1 = \vee_{k=1}^{\ell}(\vee_{i=1}^{d_k} S^1_{g_{i,k}})$ to be the 1st- skeleton of $M(\mathcal{A}_G)$.

S4. The following short exact sequence represents the presentation (4.4.2);

$$0 \to \langle g_k = g_p^{-1} g_k g_p,\ 1 \le p < k \le \ell \rangle \xrightarrow{\beta} \langle g_1, g_2, \dots, g_\ell \rangle \xrightarrow{\alpha} \pi_1(M(\mathcal{A}_G), e^0) \to 0.$$

For each relation $g_k = g_p^{-1} g_k g_p$, $1 \le p < k \le \ell$, choose a map $\varphi^2_{p,k} : (S^1, e^0) \to (M(\mathcal{A}_G)^1, e^0)$ as shown in figure (4.6.1.). the number of the 2-cells $e^2_{p,k}$ of $M(\mathcal{A}_G)^1$ that attached by maps $\varphi^2_{p,k}$ to create the second skeleton of $M(\mathcal{A}_G)$, is $\binom{\ell}{2}$ and the second skeleton will be;

$$M(\mathcal{A}_G)^{\ 2} = M(\mathcal{A}_G)^{\ 1} \coprod_{p=1}^{\ell-1} \underset{\varphi^2_{p,k}}{\coprod_{k=p+1}^{\ell}} e^2_{p,k} = M(\mathcal{A}_G)^{\ 1} \coprod_{p=1}^{\ell-1} \underset{\varphi^2_{p,k}}{\coprod_{k=p+1}^{\ell}} S^2_{p,k}$$

Thus, $\pi_1(M(\mathcal{A}_G), e^0) \cong \pi_1(M(\mathcal{A}_G)^2, e^0) \cong \mathbb{Z}^\ell$. Actually, $M(\mathcal{A}_G)^2 / M(\mathcal{A}_G)^1 = \vee_{p=1}^{\ell-1} \vee_{k=p+1}^{\ell} S^2_{p,k}$.

5. The p^{th} skeleton of the minimal CW complex for a hypersolvable graphic arrangements

This section is devoted to introduce an algorithm to compute the higher skeletons of theminimal CW complex of the complement of a hypersolvable graphic arrangement by using a computation of a presentation of first non-vanishing higher homotopy group introduced in [12] by Papadima and Suciu.

5.1. Construction:

For a supersolvable graphic $\ell = (m-1)$ -arrangement $\mathcal{A}_G$, recall construction (4.4.) for the second skeleton of $M(\mathcal{A}_G)$. We will complete the Minimal CW complex for $M(\mathcal{A}_G)$ by using induction to attach higher cells due Switzer prosedure [16], as follows;

For $2 \le k \le \ell$, if;

$$0 \to \langle r^k_\gamma ; 1 \le \gamma \le b_{k+1}(\mathcal{A}_G) \rangle \xrightarrow{\beta} \langle g^k_\vartheta ; 1 \le \vartheta \le m_k) \rangle \xrightarrow{\alpha} \pi_k(M(\mathcal{A}_G), e^0) \cong 0 \to 0.$$

be the presentation short exact sequence of the k^{th}-higher homotopy group $\pi_k(M(\mathcal{A}_G), e^0)$ such that the set of generatores $\{g^k_\vartheta\}_{\vartheta=1}^{m_k}$ generats $\pi_k(M(\mathcal{A}_G)^k, e^0) \ncong 0$, where m_k represent the number of higher k-holes of $M(\mathcal{A}_G)^k$ and for $1 \le \gamma \le b_{k+1}(\mathcal{A}_G)$, let $\varphi^{k+1}_\gamma : (S^k, s_0) \to (M(\mathcal{A}_G)^k, e^0)$ be the attachin mapping representing the relation $\beta(r^k_\gamma)$ and attach $(k+1)$-cell e^{k+1}_γ by means of φ^{k+1}_γ. But, $\pi_k(M(\mathcal{A}_G), e^0) \cong 0$, so $\beta : \langle r^k_\gamma ; 1 \le \gamma \le b_{k+1}(\mathcal{A}_G) \rangle \to \langle g^k_\vartheta ; 1 \le \vartheta \le m_k) \rangle$ is an isomorphism. Thus, $m_k = b_{k+1}(\mathcal{A}_G) = \sum_{i_1=1}^{\ell-k+1} \sum_{i_2 = i_1+1}^{\ell-k} \dots \sum_{i_{k+1} = i_k+1}^{\ell} d_{i_1} d_{i_2} \dots d_{i_k}$. Put;

$$M(\mathcal{A}_G)^{\ k+1} = M(\mathcal{A}_G)^{\ k} \underset{\varphi^{k+1}_\gamma}{\coprod_{\gamma=1}^{b_{k+1}(M(\mathcal{A}_G))}} e^{k+1}_\gamma = M(\mathcal{A})^{\ k} \underset{\varphi^{k+1}_\gamma}{\coprod_{\gamma=1}^{b_k+1(M(\mathcal{A}_G))}} S^{k+1}_\gamma$$

In the long exact homotopy sequence;

$$\dots \to \pi_{k+1}(M(\mathcal{A}_G)^{k+1}, M(\mathcal{A}_G)^k, e^0) \xrightarrow{d_{k+1}} \pi_k(M(\mathcal{A}_G)^k, e^0) \to \ \pi_k(M(\mathcal{A}_G)^{k+1}, e^0) \to \cdots;$$

we have d_{k+1} is an epimorphism. Therefore, $\pi_k(M(\mathcal{A}_G)^{k+1}, e^0)$ is trivial and $\pi_i(M(\mathcal{A}_G)^{k+1}, e^0) \cong \pi_i(M(\mathcal{A}_G)^k, e^0)$ for $0 \le i < k$.

$$\pi_k(M(\mathcal{A}_G), e^0) = \pi_k(M(\mathcal{A}_G)^{k+1}, e^0) = \begin{cases} \pi_1(M(\mathcal{A}_G)^2, e^0); & if\ k = 1 \\ 0; & if\ k \neq 1 \end{cases}$$

Finally, take the minimal CW complex for $M(\mathcal{A}_G)$, $\coprod_{k=0}^{\ell} M(\mathcal{A}_G)^k$ with the weak toplogty.

5.2. Construction:

For the second skeleton of the minimal CW complex of a hypersolvable graphic $(m-1)$-arrangement that is not supersolvable, recall construction (4.5.). It is known that, $M(\mathcal{A}_G)$ is a p-minimal CW complex, where, $p = p(\mathcal{A}_G) = max\{k \big| b_k(\mathcal{A}_G) = \sum_{i_1=1}^{\ell-k} \sum_{i_2=i_1+1}^{\ell-k+1} \dots \sum_{i_k=i_{k-1}+1}^{\ell} d_{i_1} d_{i_2} \dots d_{i_k}\}$.

From theorem (2.2.), $p = c - 2$. Accordingly, our aim will achived by three parts.

First, embed $\mathcal{A}_G$ of $\mathbb{C}^\ell$ by the arrangement, $\mathcal{A}_G \oplus \mathbb{C}^{\ell-r} = \{H \oplus \mathbb{C}^{\ell-r} \big| H \in \mathcal{A}_G\}$, which its complement $M(\mathcal{A}_G \oplus \mathbb{C}^{\ell-r})$ is a subspace of the complement $M(\tilde{\mathcal{A}})$ of Jambue's-Papadima's deformed arrangement of $\mathcal{A}_G$. Deduce that, $M(\mathcal{A}_G) \cong M(\mathcal{A}_G) \times \{(0, \dots, 0)\}$ is a strong deformation retract of $M(\mathcal{A} \oplus \mathbb{C}^{\ell-r})$. It is to be expected that $(M(\tilde{\mathcal{A}}), M(\mathcal{A}_G))$ is a topological pair, since, $M(\mathcal{A}_G) \simeq M(\mathcal{A}_G \oplus \mathbb{C}^{\ell-r}) \subseteq M(\tilde{\mathcal{A}})$.

Secondly, recall the exact homotopy sequence of higher homotopy groups of the topological pair $(M(\tilde{\mathcal{A}}), M(\mathcal{A}_G))$ from ([16], p. 38);

$$\cdots \to \pi_k(M(\mathcal{A}_G), e^0) \xrightarrow{i_k} \pi_k(M(\tilde{\mathcal{A}}), e^0) \xrightarrow{q_k} \pi_k(M(\tilde{\mathcal{A}}), M(\mathcal{A}_G), e^0)$$
$$\xrightarrow{d_k} \pi_{k-1}(M(\mathcal{A}_G), e^0) \xrightarrow{i_{k-1}} \dots \xrightarrow{d_1} \pi_0(M(\mathcal{A}_G), e^0) \xrightarrow{i_0} \pi_0(M(\tilde{\mathcal{A}}), e^0) \xrightarrow{q_0} \pi_0(M(\tilde{\mathcal{A}}), M(\mathcal{A}_G), e^0) \xrightarrow{d_0} 0$$

where e^0 can be chosen to be any point of $M(\mathcal{A}_G) \times \{(0, \dots, 0)\}$. Papadima and Suciu in [16], proved that $M(\mathcal{A}_G)$ and $M(\tilde{\mathcal{A}})$ have the same $(c-2)^{th}$-skeletons, (i.e. $\pi_k(M(\tilde{\mathcal{A}}), M(\mathcal{A}_G), e^0) = 0$, for $0 \le k \le c-2$) and they have isomorphic k^{th}-higher homotopy groups, $\pi_k(M(\mathcal{A}_G), e^0)$ and $\pi_k(M(\tilde{\mathcal{A}}), e^0)$, for $0 \le k \le c - 3 < r$. Recall construction (5.1.) as a minimal CW complex of $M(\tilde{\mathcal{A}})$ and recall construction (4.5.) as a minimal 2nd skeleton of $M(\mathcal{A}_G)$. For $0 \le k \le c-3$, the isomorphisms, $i_k: \pi_k(M(\mathcal{A}_G), e^0) \to \pi_k(M(\tilde{\mathcal{A}}), e^0)$ and $q_k: \pi_k(M(\tilde{\mathcal{A}}), e^0) \to \pi_k(M(\tilde{\mathcal{A}}), M(\mathcal{A}_G), e^0)$ induced cellular homotopy equivalences between $(c-2)^{th}$-skeletons of $M(\mathcal{A}_G)$ and $M(\tilde{\mathcal{A}})$, $i_k: M(\mathcal{A}_G)^k \to M(\tilde{\mathcal{A}})^k$ and $q_k: M(\tilde{\mathcal{A}})^k \to M(\mathcal{A}_G)^k$.

Thirdly, complete the $c-2$-minimal CW complex for $M(\mathcal{A}_G)$ by using induction to attach higher cells due Switzer prosedure [16], as follows:

For $2 < k \le c-3$,

The homotopy equivalence $q_k: M(\tilde{\mathcal{A}})^k \to M(\mathcal{A}_G)^k$, iduced an isomorphism;

$$q_k: \pi_k(M(\tilde{\mathcal{A}})^k, e^0) \to \pi_k(M(\mathcal{A}_G)^k, e^0).$$

Due construction (5.1.), we have;

$$\begin{array}{c} 0 \to \langle r_\gamma^k; 1 \le \gamma \le b_{k+1}(\mathcal{A}_G) \rangle \xrightarrow{\beta} \langle g_\vartheta^k; 1 \le \vartheta \le m_k) \rangle \xrightarrow{\alpha} \pi_k(M(\tilde{\mathcal{A}}), e^0) \cong 0 \to 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad q_k \downarrow\uparrow\ i_k \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \pi_k(M(\mathcal{A}_G), e^0) \end{array}$$

Thus, the set $\{q_k(g_\vartheta^k)\}_{\vartheta=1}^{m_k(\tilde{\mathcal{A}})}$ generates the homotopy group $\pi_k(M(\mathcal{A}_G)^k, e^0)$ and for $1 \le \gamma \le b_{k+1}(\mathcal{A}_G)$, if $\varphi_\gamma^{k+1}: (S^k, s_0) \to (M(\tilde{\mathcal{A}})^k, e^0)$ be the attaching mapping that representing $\beta(r_\gamma^k)$ of $\pi_k(M(\tilde{\mathcal{A}})^k, e^0)$, put $q_k\varphi_\gamma^k: (S^k, s_0) \to (M(\mathcal{A}_G)^k, e^0)$ to be the attaching mapping that represents the relation $q_k\beta(r_\gamma^k)$ of $\pi_k(M(\mathcal{A}_G)^k, e^0)$. Attach $(k+1)$-cells e_γ^{k+1} by means of $q_k\varphi_\gamma^{k+1}$, for $1 \le \gamma \le b_{k+1}(\mathcal{A})$. Put;

$$M(\mathcal{A}_G)^{\ k+1} = M(\mathcal{A}_G)^{\ k} \coprod_{\substack{\gamma=1 \\ p_k\varphi_\gamma^{k+1}}}^{b_{k+1}(M(\mathcal{A}))} e_\gamma^{k+1} = M(\mathcal{A}_G)^{\ k} \coprod_{p_k\varphi_\gamma^{k+1}}^{b_k+1(M(\mathcal{A}))} S_\gamma^{k+1}.$$

For $k = c-2$:

It is known that, $\pi_{c-2}(M(\mathcal{A}_G)^{c-1}, e^0) = \pi_{c-2}(M(\mathcal{A}_G), e^0) \ncong 0$, i.e. it is not trivial and [12] includes a presentation of it as a $\mathbb{Z}\pi$-module, say;

$$0 \to \langle r_\gamma^{c-2}; 1 \leq \gamma \leq b_{c-1}(\mathcal{A}_G)\rangle \xrightarrow{\beta} \langle g_\vartheta^{c-2}; 1 \leq \vartheta \leq m_{c-2})\rangle \xrightarrow{\alpha} \pi_{c-2}(M(\tilde{\mathcal{A}}), e^0) \to 0.$$

From the following portion;

$$\pi_{c-1}(M(\mathcal{A}_G)^{c-1}, M(\mathcal{A}_G)^{c-2}, e^0) \xrightarrow{d_{c-1}} \pi_{c-2}(M(\mathcal{A}_G)^{c-2}, e^0) \xrightarrow{i'_{c-2}} \pi_{c-2}(M(\mathcal{A}_G), e^0) \ncong 0$$
$$i_{c-2} \downarrow\uparrow q_{c-2}$$
$$\pi_{c-2}(M(\tilde{\mathcal{A}})^{c-2}, e^0)$$

the induced homomorphism, $q_{c-2}\colon \pi_{c-2}(M(\tilde{\mathcal{A}})^{c-2}, e^0) \to \pi_{c-2}(M(\mathcal{A}_G)^{c-2}, e^0)$ is an isomorphism and $i'_{c-2}\colon \pi_{c-2}(M(\mathcal{A}_G)^{c-2}, e^0) \to \pi_{c-2}(M(\mathcal{A}_G), e^0)$ is an epimorphism, since they have the same set of generatores $\{g_\vartheta^{c-2}\}_{\vartheta=1}^{m_{c-2}}$, where $m_{c-2} = \sum_{i_1=1}^{\ell-c+1}\sum_{i_2=i_1+1}^{\ell-c+2} \dots \sum_{i_k=i_{k-1}+1}^{\ell} d_{i_1} d_{i_2} \dots d_{i_k} = b_{c-1}(\tilde{\mathcal{A}})$ represent the number of higher $(c-2)$-holes of $M(\tilde{\mathcal{A}})^{c-2}$ and for $1 \leq \gamma \leq b_{c-1}(\mathcal{A}_G)$, let $\varphi_\gamma^{c-1}\colon (S^{c-1}, s_0) \to (M(\mathcal{A}_G)^{c-2}, e^0)$ be the attachin mapping representing the relation $\beta(r_\gamma^{c-2})$ and attach $(c-1)$-cell e_γ^{c-1} by means of φ_γ^{c-1}. It is worth pointing out that the the number of attaching $(c-1)$-cells is not enough to kill of all the higher $(c-2)$-holes. Put;

$$M(\mathcal{A}_G)^{\,c-1} = M(\mathcal{A}_G)^{\,c-2} \coprod_{\substack{\gamma=1 \\ \varphi_\gamma^{c-1}}}^{b_{c-1}(M(\mathcal{A}_G))} e_\gamma^{c-1} = M(\mathcal{A}_G)^{\,c-2} \coprod_{\substack{\gamma=1 \\ \varphi_\gamma^{c-1}}}^{b_{c-1}(M(\mathcal{A}_G))} S_\gamma^{c-1}$$

with the weak toplogty. Therefore, for $0 \leq k \leq c-2$.

$$\pi_k(M(\mathcal{A}_G), e^0) = \pi_k(M(\mathcal{A}_G)^{k+1}, e^0) = \begin{cases} \pi_1(M(\mathcal{A}_G)^2, e^0); & if\ k = 1 \\ 0; & if\ k = 0\ or\ 1 \leq k \leq c-3 \\ \pi_{c-2}(M(\mathcal{A}_G)^{c-1}, e^0); & if\ k = c-2 \end{cases}$$

5.3. Example:

Recall example (4.7.) of a generic graphic ℓ-arrangement $\mathcal{A}_G$ and its deformed arrangement $\tilde{\mathcal{A}}$ of $\mathbb{C}^{\ell+1}$. One can deduce that, $\tilde{\mathcal{A}}$ has the same lattice with the Boolen arrangement with $\ell+1$ hyperplanes. By applying construction (5.2.), $p\big(M(\mathcal{A}_G)\big) = \ell$-skeleton of $M(\mathcal{A}_G)$ can be considered and suppose $M(\mathcal{A}_G)^{\ell-1}$ be its $\ell-1$-skeleton. Recall the portion;

$$\pi_\ell\big(M(\mathcal{A}_G)^\ell, M(\mathcal{A}_G)^{\ell-1}, e^0\big) \xrightarrow{d_\ell} \pi_{\ell-1}\big(M(\mathcal{A}_G)^{\ell-1}, e^0\big) \xrightarrow{i'_{\ell-1}} \pi_{\ell-1}(M(\mathcal{A}_G), e^0) \ncong 0$$
$$i_{\ell-1} \downarrow\uparrow q_{\ell-1}$$
$$\pi_{\ell-1}\big(M(\tilde{\mathcal{A}})^{\ell-1}, e^0\big)$$

The induced homomorphism, $q_{\ell-1}\colon \pi_{\ell-1}\big(M(\tilde{\mathcal{A}})^{\ell-1}, e^0\big) \to \pi_{\ell-1}\big(M(\mathcal{A}_G)^{\ell-1}, e^0\big)$ is an isomorphism and $i'_{\ell-1}\colon \pi_{\ell-1}\big(M(\mathcal{A}_G)^{\ell-1}, e^0\big) \to \pi_{\ell-1}(M(\mathcal{A}_G), e^0)$ is an epimorphism. Thus, $\pi_{\ell-1}\big(M(\tilde{\mathcal{A}})^{\ell-1}, e^0\big)$ and $\pi_{\ell-1}\big(M(\mathcal{A}_G)^{\ell-1}, e^0\big)$ have the same set of generatores say $\{g_\vartheta^{\ell-1}\}_{\vartheta=1}^{m_{\ell-1}}$, where $m_{\ell-1} = \binom{\ell+1}{\ell} = \ell + 1 = b_\ell(\tilde{\mathcal{A}})$ represent the number of higher $(\ell-1)$-holes of $M(\tilde{\mathcal{A}})^{\ell-1}$ and $b_\ell(\mathcal{A}_G) = \ell - 1$, represents the number of attaching (ℓ)-cells which is not enough to kill of all the higher $(\ell-1)$-holes.

6. Conclusions:

In this paper;

1. The author compute the dimension of the first non vanishing higher homotopy group of the complement for any hypersolvable graphic arrangement that not supersolvable, and related to a hypersolvable graph. It is equal to the dimension of the smallest cycle of the graph with no chord.
2. An algorithm to deform a hypersolvable graph that not supersolvable into a supersolvable graph was stated.
3. A construction of the cohomological ring of the complement for any hypersolvable graphic arrangemen was considered.
4. A construction to compute the minimal CW complex of of the complement for any hypersolvable graphic arrangement was described.

References:

[1] Ali, H. M., (2007), *A topological interpretation for vanishing of higher homotopy groups of a hypersolvable arrangement*, A Ph.D. thesis submitted to College of Science/University of Basrah.

[2] Al-Ta'ai, A. H., Ali, H. M., Majeed, M. A., (2010), *On Orlik-Solomon algebra of a hypersolvable arrangement*, Basra Reserchers J. 36, 19-43.

[3] Al-Ta'ai, A. H. and Ali, H. M., (2011), *A Topological Interpretation for Vanishing of Higher Homotopy Group of a Hypersolvable Arrangement*, J. Basrah Researcges (Sci.), V. 37-C, No. 4, 414-433.

[4] Cohen, D. C. and Suciu, A. I., (7-mar-1995), *Homology of Iterated Semidriect Products of Free Groups*, Available at Arxivaly-geom/9503002-v1.

[5] Fadhil, A.G., (2012), *On The Hypersolvable Graphic Arrangements*, A M.Sc. thesis submitted to College of Science/University of Basrah.

[6] Falk, M. and Randell, R., (1985), *The Lower Central Series of a Fiber Type Arrangement*, Inventions Mathematicae, 82, 77-88.

[7] Hattori, A., (1975), *Topology of C^n Minus a Finite Number of Affine Hyperplanes in General Position*. Journal of Faculty of Science of the University of Tokyo, 22, 205 – 219.

[8] Jambu, M. and Papadima, S., (1998), A *Generalization of Fiber type Arrangements and a new Deformation Method*, Topology, 3, 1135-1164.

[9] Jambu, M. and Papadima, S., (2002), *Deformation of Hypersolvable Arrangements*, Topology and it is Applications, 103-111.

[10] Orlik, P. and Solomon, L., (1980), *Combinatorics and Topology of complements of hyperplanes*, Invent. Math. 56, 167-189.

[11] Orlik, P. and Terao, H., (1992), *Arrangements of Hyperplanes*, Grundlhern Math.Wiss., vol. 300, Springer-Verlag Berlin.

[12] Papadima, S. and Suciu, A. I., (2002), *Higher Homotopy Groups of Arrangements*, Advances in Math. 165, 71-100, preprint arxiv:math. AT/0002251.

[13] Randell, R., (1982), *The Fundamental Group of the Complement of a Union of Complex Hyperplanes*, Inventions Mathematical 96, 103-108.

[14] Randell, R., (2002), *Morse Theory, Milnor Fibers and Minimality of Hyperplane Arrangements*, Proc. Amer. Math. Soc. 130, 2737-2743.

[15] Stanley, R. p., (1972), *Supersolvable Lattices*. Algebra Universalis 2, 197-217.

[16] Switzer, M. R., (1975), *Algebric Topology- Homotopy and Homology*, Springer- Verlag.