Different methods of estimation for generalized inverse Lindley distribution

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Abstract

In this paper, we have considered different methods of estimation of the unknown parameters of GILD. First we briefly describe different methods of estimations, namely maximum likelihood estimators, moments estimators, least squares estimators, weighted least squares, maximum product spacing estimates, methods of minimum distances, method of Cramer-von-Misses and methods of Anderson-Darling and compare them using extensive numerical simulations.

Keyword: Generalized inverse Lindley distribution; Lindley distribution; Bias; Mean squared errors; Method of moment estimators; Least squares estimators; Weighted least squares estimators; Percentiles estimators; Simulations.

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1 Introduction

Lifetime distribution represents an attempt to describe, mathematically, the length of the life of a system or a device. Lifetime distributions are most frequently used in the fields like medicine, engineering etc. Many parametric models such as exponential, gamma, Weibull have been frequently used in statistical literature to analyze lifetime data. But there is no clear motivation for the gamma and Weibull distributions. They only have more general mathematical closed form than the exponential distribution with one additional parameter.

Recently, one parameter Lindley distribution has attracted the researchers for its use in modelling lifetime data, and it has been observed in several papers that this distribution has performed excellently. The Lindley distribution was originally proposed by Lindley in the context of Bayesian statistics, as a counter example of fudicial statistics which can be seen that as a mixture of $\exp(\theta)$ and gamma$(2, \theta)$.

Some of the advances in the literature of Lindley distribution are given by Ghitany et al. (2011) who has introduced a two-parameter weighted Lindley distribution and has pointed that Lindley distribution is particularly useful in modeling biological data from mortality studies. Mahmoudi et al. (2010) have proposed generalized Poisson Lindley distribution. Bakouch et al. (2012) have come up with extended Lindley (EL) distribution, Adamidis and Loukas (1998) have introduced exponential geometric (EG) distribution. Shanker et al. (2013) have introduced a two-parameter Lindley distribution. Zakerzadeh et al. (2012) have proposed a new two parameter lifetime distribution: model and properties. M.K. Hassan (2008) has introduced convolution of Lindley distribution. Ghitany et al. (2013) worked on the estimation of the reliability of a stress-strength system from power Lindley distribution. Elbatal et al. (2013) has proposed a new generalized Lindley distribution.

Definition 1.1. *A random variable X is said to have Lindley distribution with parameter θ if its probability density function is defined as:*

$$g_X(x;\theta) = \frac{\theta^2}{(\theta+1)}(1+x)e^{-\theta x}; x>0, \theta>0 \tag{1}$$

with cumulative distribution function

$$G(x) = 1 - \frac{e^{-\theta x}(1+\theta+\theta x)}{1+\theta} \tag{2}$$

Because of the wide applicability of inverse distributions, Sharma et al. introduced generalized inverse Lindley distribution (GILD)with probability density function

$$f(x,\alpha,\theta) = \frac{\alpha\theta^2}{(1+\theta)}\left[\frac{1+x^\alpha}{x^{2\alpha+1}}\right]e^{-\frac{\theta}{x^\alpha}}, x>0, \alpha>0, \theta>0 \tag{3}$$

and cumulative distribution function

$$F(x,\alpha,\theta) = \left[1+\frac{\theta}{1+\theta}\frac{1}{x^\alpha}\right]e^{-\frac{\theta}{x^\alpha}}, x>0, \alpha>0, \theta>0 \tag{4}$$

where, θ is the scale parameter while the parameter α controls its shape.

2 Maximum likelihood estimation

In this section, we briefly review the MLEs of the parameters of GILD distribution. Let $x_1, x_2, \ldots, x_n$ be a independent and identically distributed (i.i.d) observed random sample of size n from GIL distribution (3). Then, the likelihood function is defined as

$$\ell\left(\underset{\sim}{x},\alpha,\theta\right) = \frac{\alpha^n\theta^{2n}}{(1+\theta)^n}\prod_{i=1}^{n}(1+x_i^\alpha)\prod_{i=1}^{n}x_i^{(-2\alpha-1)}e^{\theta\sum_{i=1}^{n}x_i^{-\alpha}} \tag{5}$$

The log-likelihood function corresponding to (5), is given by

$$\log\ell = n\ln(\alpha) + 2n\ln(\theta) - n\ln(1+\theta) + \sum_{i=1}^{n}\ln(1+x_i^\alpha) - (2\alpha+1)\sum_{i=1}^{n}\ln(x_i) - \theta\sum_{i=1}^{n}x_i^{-\alpha} \tag{6}$$

The maximum likelihood estimates $\hat{\alpha}_{ML}$ and $\hat{\theta}_{ML}$ of σ and λ, respectively can be obtained as the simultaneous solution of the following non-linear equations:

$$\frac{\partial\log\ell}{\partial\alpha} = 0 = \frac{n}{\alpha} + \sum_{i=1}^{n}\frac{\ln(x_i)\,x_i^\alpha}{1+x_i^\alpha} - 2\sum_{i=1}^{n}\ln(x_i) + \theta\sum_{i=1}^{n}\ln(x_i)\,x_i^{-\alpha} \tag{7}$$

$$\frac{\partial\log\ell}{\partial\theta} = 0 = \frac{2n}{1+\theta} - \frac{n}{1+\theta} - \sum_{i=1}^{n}x_i^{-\alpha} \tag{8}$$

2.1 Maximum product spacing estimates

The maximum product spacing (MPS) method has been proposed by [4]. This method is based on an idea that the differences of the consecutive points should be identically distributed. The geometric mean of the differences is given as

$$GM = \sqrt[n+1]{\prod_{i=1}^{n+1}D_i} \tag{9}$$

where, the difference D_i is defined as

$$D_i = \int_{x_{(i-1)}}^{x_{(i)}} f\left(x, \alpha, \theta\right) dx; \quad i = 1, 2, \ldots, n+1. \tag{10}$$

where, $F(x_{(0)}, \alpha, \theta) = 0$ and $F(x_{(n+1)}, \alpha, \theta) = 1$. The MPS estimators $\hat{\alpha}_{PS}$ and $\hat{\theta}_{PS}$ of α and θ are obtained by maximizing the geometric mean (GM) of the differences. Substituting (3) in (17) and taking logarithm of the above expression, we will have

$$LogGM = \frac{1}{n+1} \sum_{i=1}^{n+1} \log\left[F(x_{(i)}, \alpha, \theta) - F(x_{(i-1)}, \alpha, \theta)\right] \tag{11}$$

The MPS estimators $\hat{\alpha}_{PS}$ and $\hat{\theta}_{PS}$ of α and θ can be obtained as the simultaneous solution of the following non-linear equations:

$$\begin{aligned} \frac{\partial LogGM}{\partial \alpha} &= \frac{1}{n+1} \sum_{i=1}^{n+1} \left[\frac{F_{\alpha}^{'}(x_{(i)}, \alpha, \theta) - F_{\alpha}^{'}(x_{(i-1)}, \alpha, \theta)}{F(x_{(i)}, \alpha, \theta) - F(x_{(i-1)}, \alpha, \theta)} \right] \\ &= \frac{1}{n+1} \sum_{i=1}^{n+1} \left[\frac{\frac{\theta}{1+\theta} \frac{\log\left(x_{(i)}\right)}{x_{(i)}^{\alpha}} e^{-\frac{\theta}{x_{(i)}^{\alpha}}} - \frac{\theta}{1+\theta} \frac{\log\left(x_{(i-1)}\right)}{x_{(i-1)}^{\alpha}} e^{-\frac{\theta}{x_{(i-1)}^{\alpha}}}}{\left[1 + \frac{\theta}{1+\theta} \frac{1}{x_{(i)}^{\alpha}}\right] e^{-\frac{\theta}{x_{(i)}^{\alpha}}} - \left[1 + \frac{\theta}{1+\theta} \frac{1}{x_{(i-1)}^{\alpha}}\right] e^{-\frac{\theta}{x_{(i-1)}^{\alpha}}}} \right] = 0 \end{aligned} \tag{12}$$

$$\begin{aligned} \frac{\partial LogGM}{\partial \theta} &= \frac{1}{n+1} \sum_{i=1}^{n+1} \left[\frac{F_{\theta}^{'}(x_{(i)}, \alpha, \theta) - F_{\theta}^{'}(x_{(i-1)}, \alpha, \theta)}{F(x_{(i)}, \alpha, \theta) - F(x_{(i-1)}, \alpha, \theta)} \right] \\ & \frac{1}{n+1} \sum_{i=1}^{n+1} \left[\frac{-\frac{\theta}{(1+\theta)^2} \frac{1+\theta\left(1+x_{(i)}^{\alpha}\right)}{x_{(i)}^{2\alpha}} e^{-\frac{\theta}{x_{(i)}^{\alpha}}} + \frac{\theta}{(1+\theta)^2} \frac{1+\theta\left(1+x_{(i-1)}^{\alpha}\right)}{x_{(i-1)}^{2\alpha}} e^{-\frac{\theta}{x_{(i-1)}^{\alpha}}}}{\left[1 + \frac{\theta}{1+\theta} \frac{1}{x_{(i)}^{\alpha}}\right] e^{-\frac{\theta}{x_{(i)}^{\alpha}}} - \left[1 + \frac{\theta}{1+\theta} \frac{1}{x_{(i-1)}^{\alpha}}\right] e^{-\frac{\theta}{x_{(i-1)}^{\alpha}}}} \right] = 0 \end{aligned}$$

where,
$F_{\alpha}^{'}(x, \alpha, \theta) = \frac{\theta}{1+\theta} \frac{\log(x)}{x^{\alpha}} e^{-\frac{\theta}{x^{\alpha}}}$ and $F_{\theta}^{'}(x, \alpha, \theta) = -\frac{\theta}{(1+\theta)^2} \frac{1+\theta(1+x^{\alpha})}{x^{2\alpha}} e^{-\frac{\theta}{x^{\alpha}}}$.

3 Moments Estimators

It is observed by Sharma et al. that if X follows GILD distribution, then

$$E(X) = \frac{\theta^{\frac{1}{\alpha}}}{\alpha\left(1+\theta\right)} \Gamma\left(\frac{\alpha - 1}{\alpha}\right)\left(\alpha\left(1+\theta\right) - 1\right), \alpha > 1 \tag{13}$$

$$E(X^2) = \frac{\theta^{\frac{2}{\alpha}}}{\alpha\left(1+\theta\right)} \Gamma\left(\frac{\alpha - 2}{\alpha}\right)\left(\alpha\left(1+\theta\right) - 2\right), \alpha > 2$$

The MMEs of the two-parameter GIL distribution can be obtained by equating the first two theoretical moments with the sample moments $\frac{1}{n}\sum_{i=1}^{n} x_i$ and $\frac{1}{n}\sum_{i=1}^{n} x_i^2$ respectively,

$$\frac{1}{n}\sum_{i=1}^{n} x_i = \frac{\theta^{\frac{1}{\alpha}}}{\alpha\left(1+\theta\right)} \Gamma\left(\frac{\alpha - 1}{\alpha}\right)\left(\alpha\left(1+\theta\right) - 1\right) \tag{14}$$

and

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 = \frac{\theta^{\frac{2}{\alpha}}}{\alpha\,(1+\theta)}\Gamma\left(\frac{\alpha-2}{\alpha}\right)(\alpha\,(1+\theta)-2) \tag{15}$$

The method of moments estimators are the roots of the two equations. Similar to the MLEs, such non-linear equations do not have closed form solutions. We can apply numerical method such as Newton-Raphson method to determine the roots.

4 Maximum product spacing estimates

The maximum product spacing (MPS) method has been proposed by Cheng and Amin [Cheng, R. C. H., and N. A. K. Amin. "Estimating parameters in continuous univariate distributions with a shifted origin." Journal of the Royal Statistical Society. Series B (Methodological) (1983): 394-403.] This method is based on an idea that the differences (Spacings) of the consecutive points should be identically distributed.

The geometric mean of the differences is given as

$$GM = \sqrt[n+1]{\prod_{i=1}^{n+1} D_i} \tag{16}$$

where, the difference D_i is defined as

$$D_i = \int_{x_{(i-1)}}^{x_{(i)}} f(x,\beta)\,dx; \quad i = 1, 2, \ldots, n+1. \tag{17}$$

where, $F(x_{(0)}, \beta) = 0$ and $F(x_{(n+1)}, \beta) = 1$. The MPS estimator $\hat{\beta}_{PS}$ of β is obtained by maximizing the geometric mean (GM) of the differences. Substituting (3) in (17) and taking logarithm of the above expression, we will have

$$\begin{aligned} LogGM &= \frac{1}{n+1}\sum_{i=1}^{n+1}\log\left[F(x_{(i)},\beta) - F(x_{(i-1)},\beta)\right] \\ &= \frac{1}{n+1}\sum_{i=1}^{n+1}\log\left[\left[1+\frac{\theta}{1+\theta}\frac{1}{x_{(i)}^{\alpha}}\right]e^{-\frac{\theta}{x_{(i)}^{\alpha}}} - \left[1+\frac{\theta}{1+\theta}\frac{1}{x_{(i-1)}^{\alpha}}\right]e^{-\frac{\theta}{x_{(i-1)}^{\alpha}}}\right] \end{aligned} \tag{18}$$

The MPS estimator $\hat{\beta}_{PS}$ of β can be obtained as the simultaneous solution of the following non-linear equation:

$$\frac{\partial LogGM}{\partial \beta} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F_{\beta}'(x_{(i)},\beta) - F_{\beta}'(x_{(i-1)},\beta)}{F(x_{(i)},\beta) - F(x_{(i-1)},\beta)}\right] = 0 \tag{19}$$

4.1 Methods of Minimum Distances

Most theoretical studies of minimum distance estimation, and most applications, make use of "distance" measures which underlie already-established goodness of fit tests: the test statistic used in one of these tests is used as the distance measure to be minimized.In this subsection we present three estimation methods for $\bar{\alpha}$ and $\bar{\theta}$ based on the minimization, with respect to α and θ, of the goodness-of-fit statistics. This class of statistics is based on the difference between the estimate of the cumulative distribution function and the empirical distribution function.

4.1.1 Method of Cramér-von-Mises

To motivate our choice of Cramer-von Mises type minimum distance estimators, MacDonald (1971) provided empirical evidence that the bias of the estimator is smaller than the other minimum distance estimators. Thus, The Cramé-von Mises estimates $\widehat{\alpha}_{CME}$ and $\widehat{\lambda}_{CME}$ of the parameters α and λ are obtained by minimizing, with respect to α and λ, the function:

$$C(\alpha,\lambda) = \frac{1}{12n} + \sum_{i=1}^{n}\left(F\left(x_{i:n} \mid \alpha,\lambda\right) - \frac{2i-1}{2n}\right)^2$$

$$\frac{1}{12n} + \sum_{i=1}^{n}\left(\left[1+\frac{\theta}{1+\theta}\frac{1}{x_{(i)}^{\alpha}}\right]e^{-\frac{\theta}{x_{(i)}^{\alpha}}} - \frac{2i-1}{2n}\right)^2. \tag{20}$$

These estimates can also be obtained by solving the non-linear equations:

$$\sum_{i=1}^{n}\left(\left[1+\frac{\theta}{1+\theta}\frac{1}{x_{(i)}^{\alpha}}\right]e^{-\frac{\theta}{x_{(i)}^{\alpha}}} - \frac{2i-1}{2n}\right)\frac{\theta}{1+\theta}\frac{\log\left(x_{(i)}\right)}{x_{(i)}^{\alpha}}e^{-\frac{\theta}{x_{(i)}^{\alpha}}} = 0,$$

$$-\sum_{i=1}^{n}\left(\left[1+\frac{\theta}{1+\theta}\frac{1}{x_{(i)}^{\alpha}}\right]e^{-\frac{\theta}{x_{(i)}^{\alpha}}} - \frac{2i-1}{2n}\right)\frac{\theta}{(1+\theta)^2}\frac{1+\theta\left(1+x_{(i)}^{\alpha}\right)}{x_{(i)}^{2\alpha}}e^{-\frac{\theta}{x_{(i)}^{\alpha}}} = 0.$$

4.1.2 Methods of Anderson-Darling and Right-tail Anderson-Darling

The Anderson-Darling test is similar to the Cramrvon Mises criterion except that the integral is of a weighted version of the squared difference, where the weighting relates the variance of the empirical distribution function The Anderson-Darling test was developed by T.W. Anderson and D.A. Darling as an alternative to other statistical tests for detecting sample distributions departure from normality. The Anderson-Darling estimates $\widehat{\alpha}_{ADE}$ and $\widehat{\lambda}_{ADE}$ of the parameters α and λ are obtained by minimizing, with respect to α and λ, the function:

$$A(\alpha,\lambda) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\log F\left(x_{i:n} \mid \alpha,\lambda\right) + \log \overline{F}\left(x_{n+1-i:n} \mid \alpha,\lambda\right)\right\}. \tag{21}$$

These estimates can also be obtained by solving the non-linear equations:

$$\sum_{i=1}^{n}(2i-1)\left[\frac{F_{\alpha}'\left(x_{i:n} \mid \alpha,\lambda\right)}{F\left(x_{i:n} \mid \alpha,\lambda\right)} - \frac{\overline{F}_{\alpha}'\left(x_{n+1-i:n} \mid \alpha,\lambda\right)}{\overline{F}\left(x_{n+1-i:n} \mid \alpha,\lambda\right)}\right] = 0,$$

$$\sum_{i=1}^{n}(2i-1)\left[\frac{F_{\theta}'\left(x_{i:n} \mid \alpha,\lambda\right)}{F\left(x_{i:n} \mid \alpha,\lambda\right)} - \frac{\overline{F}_{\theta}'\left(x_{n+1-i:n} \mid \alpha,\lambda\right)}{\overline{F}\left(x_{n+1-i:n} \mid \alpha,\lambda\right)}\right] = 0,$$

The Right-tail Anderson-Darling estimates $\widehat{\alpha}_{RTADE}$ and $\widehat{\lambda}_{RTADE}$ of the parameters α and λ are obtained by minimizing, with respect to α and λ, the function:

$$R(\alpha,\lambda) = \frac{n}{2} - 2\sum_{i=1}^{n}F\left(x_{i:n} \mid \alpha,\lambda\right) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log \overline{F}\left(x_{n+1-i:n} \mid \alpha,\lambda\right). \tag{22}$$

These estimates can also be obtained by solving the non-linear equations:

$$-2\sum_{i=1}^{n}\frac{F_{\theta}'\left(x_{i:n} \mid \alpha,\lambda\right)}{F\left(x_{i:n} \mid \alpha,\lambda\right)} + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\overline{F}_{\theta}'\left(x_{n+1-i:n} \mid \alpha,\lambda\right)}{\overline{F}\left(x_{n+1-i:n} \mid \alpha,\lambda\right)} = 0,$$

$$-2\sum_{i=1}^{n}\frac{F_{\alpha}'\left(x_{i:n} \mid \alpha,\lambda\right)}{F\left(x_{i:n} \mid \lambda,\sigma\right)} + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\overline{F}_{\alpha}'\left(x_{n+1-i:n} \mid \alpha,\lambda\right)}{\overline{F}\left(x_{n+1-i:n} \mid \alpha,\lambda\right)} = 0.$$

5 Least square estimates

The least square estimators and weighted least square estimators were proposed by Swain, Venkataraman and Wilson (1988) to estimate the parameters of Beta distributions. The least square estimators of the unknown parameters α and λ of LE distribution can be obtained by minimizing

$$\sum_{j=1}^{n}\left[F(X_{(i)})-\frac{j}{n+1}\right]^2$$

with respect to unknown parameters λ and θ. Suppose $F(X_{(i)})$ denotes the distribution function of the ordered random variables $X_{(1)}<X_{(2)}<\cdots<X_{(n)}$ where $\{X_1,X_2,\cdots,X_n\}$ is a random sample of size n from a distribution function $F(\cdot)$. Therefore, in this case, the least square estimators of λ and θ, say $\hat{\lambda}_{LSE}$ and $\hat{\theta}_{LSE}$ respectively, can be obtained by minimizing

$$\sum_{i=1}^{n}\left[\left[1+\frac{\theta}{1+\theta}\frac{1}{x^{\alpha}}\right]e^{-\frac{\theta}{x^{\alpha}}}-\frac{i}{n+1}\right]^2$$

with respect to α and λ.

The least square estimates (LSEs) $\hat{\lambda}_{LS}$ and $\hat{\theta}_{LS}$ of λ and θ are obtained by minimizing

$$LS(\lambda,\theta)=\sum_{j=1}^{n}\left[\left[1+\frac{\theta}{1+\theta}\frac{1}{x^{\alpha}}\right]e^{-\frac{\theta}{x^{\alpha}}}-\frac{j}{n+1}\right]^2 \tag{23}$$

Therefore, $\hat{\lambda}_{LS}$ and $\hat{\theta}_{LS}$ of λ and θ can be obtained as the solution of the following system of equations:

$$\begin{aligned}\frac{\partial LS(\lambda,\theta)}{\partial\alpha}&=\sum_{i=1}^{n}F_{\alpha}^{'}(x_{(i)},\lambda,\theta)\left(F\left(x_{(i)},\lambda,\theta\right)-\frac{i}{n+1}\right)\\&=\sum_{i=1}^{n}\frac{\theta}{1+\theta}\frac{\log(x)}{x^{\alpha}}e^{-\frac{\theta}{x^{\alpha}}}\\&\quad\times\left[\left[1+\frac{\theta}{1+\theta}\frac{1}{x^{\alpha}}\right]e^{-\frac{\theta}{x^{\alpha}}}-\frac{j}{n+1}\right]^2\end{aligned} \tag{24}$$

$$\frac{\partial LS(\lambda,\theta)}{\partial\theta}=\sum_{i=1}^{n}F_{\theta}^{'}(x_{(i)},\lambda,\theta)\left(F\left(x_{(i)},\lambda,\theta\right)-\frac{i}{n+1}\right)$$

$$=-\sum_{i=1}^{n}\frac{\theta}{(1+\theta)^2}\frac{1+\theta(1+x^{\alpha})}{x^{2\alpha}}e^{-\frac{\theta}{x^{\alpha}}} \tag{25}$$

$$\times\left[\left[1+\frac{\theta}{1+\theta}\frac{1}{x^{\alpha}}\right]e^{-\frac{\theta}{x^{\alpha}}}-\frac{j}{n+1}\right]^2 \tag{26}$$

These non-linear can be routinely solved using Newton's method or fixed point iteration techniques. The subroutines to solve non-linear optimization problem are available in R software namely.

5.1 The weighted least square estimators

The weighted least square estimators of the unknown parameters can be obtained by minimizing

$$\sum_{j=1}^{n}w_j\left[F(X_{(j)})-\frac{j}{n+1}\right]^2$$

with respect to α and λ. The weights w_j are equal to $\frac{1}{V(X_{(j)})} = \frac{(n+1)^2(n+2)}{j(n-j+1)}$. Therefore, in this case, the weighted least square estimators of α and λ, say $\hat{\alpha}_{WLSE}$ and $\hat{\lambda}_{WLSE}$ respectively, can be obtained by minimizing

$$\sum_{j=1}^{n} \frac{(n+1)^2(n+2)}{n-j+1} \left[\frac{\left(1-e^{-\lambda x}\right)^{\theta} \left(1+\theta-\theta \log\left(1-e^{-\lambda x}\right)\right)}{1+\theta} - \frac{j}{n+1} \right]^2$$

with respect to α and λ.

Therefore, $\hat{\lambda}_{WLSW}$ and $\hat{\theta}_{WLSW}$ of λ and θ can be obtained as the solution of the following system of equations:

$$\frac{\partial WLS\,(\lambda,\theta)}{\partial \alpha} = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)\theta}{(n-j+1)(1+\theta)} \frac{\log(x)}{x^{\alpha}} e^{-\frac{\theta}{x^{\alpha}}} \times \left[\left[1 + \frac{\theta}{1+\theta}\frac{1}{x^{\alpha}}\right] e^{-\frac{\theta}{x^{\alpha}}} - \frac{j}{n+1} \right]^2 \quad (27)$$

$$\frac{\partial WLS\,(\lambda,\theta)}{\partial \theta} = -\sum_{i=1}^{n} \frac{(n+1)^2(n+2)\theta}{(n-j+1)\,(1+\theta)^2} \frac{1+\theta\,(1+x^{\alpha})}{x^{2\alpha}} e^{-\frac{\theta}{x^{\alpha}}} \quad (28)$$

$$\times \left[\left[1 + \frac{\theta}{1+\theta}\frac{1}{x^{\alpha}}\right] e^{-\frac{\theta}{x^{\alpha}}} - \frac{j}{n+1} \right]^2 \quad (29)$$

These non-linear can be routinely solved using Newton's method or fixed point iteration techniques. The subroutines to solve non-linear optimization problem are available in R software namely *optim()*, *nlm()* and *bbmle()* etc. We used *nlm()* package for optimizing (5), (18) and (23).

6 Simulation algorithms and study

To generate a random sample of size n from GILD, we follow the following steps:

1. Set n, $\Theta = (\beta)$ and initial value x^0.
2. Generate $U \sim Unifrom\,(0,1)$.
3. Update x^0 by using the Newton's formula
   $x^{\star} = x^0 - R(x^0, \Theta)$
   where, $R(x^0, \Theta) = \frac{F_X(x^0,\Theta)-U}{f_X(x^0,\Theta)}$, $F_X(.)$ and $f_X(.)$ are cdf and pdf of GILD, respectively.
4. If $\mid x^0 - x^{\star} \mid \leq \epsilon$, (very small, $\epsilon > 0$ tolerance limit), then store $x = x^{\star}$ as a sample from GILD.
5. If $\mid x^0 - x^{\star} \mid > \epsilon$, then, set $x^0 = x^{\star}$ and go to step 3.
6. Repeat steps 3-5, n times for $x_1, x_2, \ldots, x_n$ respectively.

7 Comparison study of the proposed estimators

This subsection deals with the comparisons study of the proposed estimators in terms of their mean square error on the basis of simulates sample from pdf of GILD with varying sample sizes. For this purpose, we take $\alpha = 2$ and $\theta = 1.5$, arbitrarily and $n = 10, 20, \ldots, 50$. All the algorithms are coded in R, a statistical computing environment and we used algorithm given above for simulations purpose.

We calculate Maximum likelihood estimation(MLE),methods of moments(MM), modified method of moments(MME), Least square estimatior(LSE), Percentile estimation(PE),Maximum product spacing(MPS),Cramer-von Mises estimation(CME) and Anderson Darling estimation(ADE) of β based on each generated sample. This proses is repeated 1000 of times, and average estimates and corresponding mean square errors are computed and also reported in Table 1 and 2.

From Table 1 and Table 2 it can be observed that as sample size increases the mean square error decreases, it proves the consistency of the estimators. Maximum product spacing(MPS) of parameter α and θ is superior than the others methods of estimation.

Table 1: Estimates and mean square errors (in IInd row of each cell) of the proposed estimators with varying sample size for α

	MLE	LSE	PSE	WLS	CVM	AD	RADE	MM
10	2.313455	2.009112	1.830988	2.001844	1.891834	2.21844	1.85142	1.87431
	0.5658535	0.6818355	0.3271637	0.7147259	0.8096587	0.8291452	0.874785	0.8325914
20	2.163287	1.996208	1.880599	2.019074	2.059119	1.933184	2.11358	1.87431
	0.1931416	0.2429763	0.1410891	0.2813676	0.2704694	0.3115214	0.294857	0.2791485
30	2.111635	1.995606	1.903263	2.011532	2.037083	1.93134 8	2.14418	1.88153
	0.1117104	0.1408858	0.09011919	0.161499	0.1508393	0.161415	0.168574	0.1748591
40	2.086004	1.999337	1.918055	2.011878	2.029909	1.974878	2.10243	1.92147
	0.07687683	0.1033162	0.06559104	0.1172614	0.1088453	0.109845	0.108745	0.1045988
50	2.068373	1.997535	1.926667	2.008684	2.021861	1.984878	2.05198	1.974781
	0.06050827	0.08165715	0.05416229	0.09471132	0.08508901	0.095845	0.098785	0.094652
60	2.056900	1.998592	1.933338	2.010485	2.018830	1.984878	2.05198	1.974781
	0.04814758	0.06681029	0.04422282	0.07685404	0.06913374	0.067888	0.068142	0.064101
70	2.050378	1.999597	1.940829	2.011620	2.016889	1.989654	2.04546	1.978812
	0.0411482	0.05691567	0.03829946	0.06535114	0.05864063	0.060432	0.061156	0.059101
80	2.039288	1.996869	1.940636	2.009087	2.012009	1.991023	2.03899	1.980043
	0.03328447	0.04881408	0.03240416	0.0563805	0.05247164	0.051243	0.051156	0.050123
90	2.036175	1.997161	1.946137	2.009747	2.010623	1.992048	2.03147	1.980987
	0.02975332	0.04297147	0.02900838	0.04937122	0.04392276	0.044256	0.045178	0.043582
100	2.034124	1.997009	1.951037	2.008582	2.009096	1.994002	2.02949	1.982486
	0.02716096	0.03795049	0.02648156	0.04372492	0.03868394	0.039147	0.040128	0.038146

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Table 2: Estimates and mean square errors (in IInd row of each cell) of the proposed estimators with varying sample size for θ

	MLE	LSE	PSE	WLS	CVM	AD	RAD	MM
10	1.616402	1.567633	1.518994	1.573889	1.713010	1.753413	1.743125	1.72450
	0.3975645	0.3997802	0.1843967	0.5893906	0.616127	0.631267	0.634276	0.626785
20	1.544120	1.520683	1.503077	1.528378	1.585997	1.653413	1.62125	1.59478
	0.1093108	0.1086653	0.07540591	0.1516769	0.1394856	0.14540465	0.1616234	0.1594192
30	1.520469	1.502694	1.494357	1.504416	1.544735	1.603234	1.59125	1.57452
	0.06314575	0.05972626	0.04892365	0.06866327	0.07001237	0.07292432	0.07066156	0.06901458
40	1.514480	1.500942	1.494799	1.501555	1.531643	1.564799	1.541555	1.521643
	0.04493958	0.0430905	0.03671667	0.04685823	0.04761913	0.04900267	0.04976523	0.04861345
50	1.510075	1.498675	1.494284	1.498944	1.523087	1.533245	1.544256	1.520087
	0.03598292	0.03525313	0.03043308	0.03796338	0.03809246	0.03928608	0.03824638	0.03609245
60	1.507407	1.499047	1.494169	1.499871	1.519356	1.534345	1.549159	1.529456
	0.02818338	0.02879254	0.02439059	0.03126575	0.03073359	0.03237852	0.03324126	0.03172456
70	1.507217	1.500457	1.495731	1.501864	1.517851	1.535285	1.541834	1.529847
	0.02307885	0.02423896	0.02032653	0.02659458	0.02568924	0.02732435	0.02699958	0.02600024
80	1.506089	1.501167	1.495898	1.502699	1.516390	1.527469	1.531258	1.516456
	0.01928134	0.02061999	0.01721479	0.02281223	0.02173624	0.02312479	0.02275469	0.02272587
90	1.507598	1.503045	1.498382	1.504802	1.516601	1.524678	1.530002	1.512891
	0.01724938	0.01825467	0.01553347	0.02030977	0.01919224	0.0199334	0.02130456	0.02015434
100	1.505661	1.501122	1.497361	1.502669	1.513273	1.527529	1.536587	1.501245
	0.01595279	0.01677394	0.01449712	0.01853412	0.0174965	0.01849896	0.01951245	0.0178541

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