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DELTA-CONVERGENCE FOR ASYMPTOTICALLY NONEXPANSIVE MAPPING IN CAT(K) SPACE THROUGH MODIFIED S- ITERATION PROCEDURE

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Abstract

In this paper we establish convergence theorem for asymptotically non expansive mapping in CAT (k) space and will approximate fixed point through modified S-iteration procedure.

Keywords: Fixed point ,Asymptotically nonexpansive mapping, Δ-convergence,CAT (k) space, S-iteration
MSC: 47H10, 54H25.

Introduction

The concept of Δ convergence in a general metric space was introduced by Lim [22]. In 2008, Kirk and Panyanak [18] used the notion of convergence introduced by Lim [22] to prove in the CAT(0) space and analogous of some Banach space results which involve weak convergence. Further, Dhompongsa and Panyanak [9] obtained Δ convergence theorems for the Picard, Mann and Ishikawa iterations in a CAT(0) space.

Kirk ([17, 18]) first studied the theory of fixed point in CAT(k) spaces. Later on, many authors generalized the notion of CAT(k) given in17,18 [18, 19], mainly focusing on CAT(0) spaces (see e.g., [1,5,6,8,14,19,21,28,25,29]). The results of a CAT(0)space can be applied to any CAT(k space with $k \leq 0$ since any CAT(k) space is a CAT(k')space for every $k \geq k'$ (see in [3]). Although, CAT (k) spaces for $k > 0$,were studied by some authors (see e.g., [10, 13, 24]).

For a real number k, a CAT(k) space is a geodesic metric space whose geodesic triangle is thinner than the corresponding comparison triangle in a model space with curvature k.The precise definition is given below.The term‘CAT(k)’was coined byGromov [12].The initials are in honor of Cartan, Alexandrov and Toponogov, each of whom considered similar conditions in varying degrees of generality.

The well known Mann and Ishikawa iteration process are given below :

1) The Mann iteration process is defined by the sequence $\{x_n\}$,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n \quad , \quad n \geq 1 \end{cases}$$

where $\{\alpha_n\}$ is a sequence in (0,1).

2) Further ,the Ishikawa iteration process is defined by the sequence $\{x_n\}$,

$$\begin{cases} x_1 \in K \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n \end{cases} \quad n \geq 1 ,$$

where $\{\alpha_n\}$ and$\{ \beta_n \}$ are a sequence in (0,1). This iteration process reduces to the Mann iteration process when $\beta_n = 0$ for all n≥1.

3) In 2007, Agarwal, O’Regan and Sahu [2] introduced the S-iteration process in Banach space,

$$\begin{cases} x_1 \in K \\ x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Ty_n, \\ y_n = (1-\beta_n)x_n + \beta_n Tx_n \end{cases} \quad n \geq 1,$$

where $\{\alpha_n\}$ and$\{\ \beta_n\ \}$ are a sequence in (0,1). Note that (3) is independent of (2) (and hence (1)). They showed that their process independent of those of Mann and Ishikawa and converges faster than both of these (see 2[[1], Proposition 3.1]).

4) Schu [27], in 1991, considered the modified Mann iteration process which is a generalization of the Mann iteration process,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n \end{cases}, \quad n \geq 1$$

where $\{\alpha_n\}$ is a sequence in (0,1).

5) Tan and Xu [30], in 1994, studied the modified Ishikawa iteration process which is a generalization of the Ishikawa iteration process,

$$\begin{cases} x_1 \in K \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n \end{cases} \quad n \geq 1,$$

where $\{\alpha_n\}$ and$\{\ \beta_n\ \}$ are a sequence in (0,1). This iteration process reduces to the Mann iteration process when β_n = 0 for all n≥1.

6) In 2007, Agarwal, O'Regan and Sahu [2] introduced the modified S- iteration process in Banach space,

$$\begin{cases} x_1 \in K \\ x_{n+1} = (1-\alpha_n)T^n x_n + \alpha_n T^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n \end{cases} \quad n \geq 1,$$

where $\{\alpha_n\}$ and$\{\ \beta_n\ \}$ are a sequence in (0,1). Note that (6) is independent of (5) (and hence (4)). Also (6) reduces to (3) when $T^n = T$ for all n≥1.

Recently, Sahin and Basarir [26] modified the iteration process (6) in a CAT(0) space as follows:

Let Kbe a nonempty closed convex subset of a complete CAT(0) space $X$and $T: K \to K$ be an asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is a sequence generated iteratively by

7) $$\begin{cases} x_1 \in K \\ x_{n+1} = (1-\alpha_n)T^n x_n \oplus \alpha_n T^n y_n, \\ y_n = (1-\beta_n)x_n \oplus \beta_n T^n x_n \end{cases} \quad n \geq 1,$$

where $\{\alpha_n\}$ and$\{\ \beta_n\ \}$ are a sequences such that 0≤ $\alpha_n, \beta_n \leq 1$ for all n ≥1 .

They studied modified S-iteration process and established some strong convergence results under some suitable conditions which generalize some results of Khan and Abbas [15].

Very recently, Kumam, Saluja and Nashine [20] studied modified S-iteration process and investigated the existence and convergence theorems in the setting of CAT(0) spaces for a class of mappings which is wider than that of asymptotically nonexpansive mappings as follows:

8) $$\begin{cases} x_1 \in K \\ x_{n+1} = (1-\alpha_n)T^n x_n \oplus \alpha_n S^n(y_n), \\ y_n = (1-\beta_n)x_n \oplus \beta_n T^n(x_n) \end{cases} \quad n \geq 1,$$

where $\{\alpha_n\}$ and{ β_n } are in [0,1]for all n ≥1.
Inspired by 8) and6) Plern Saipara et.al introduced a new iterative scheme which is given as follows

Let K be a nonempty closed convex subset of a complete CAT(k) space X and $T : k \to k$ be unifo-rmly continuous total asymptotically nonexpansive mapping with $F(T) \neq \emptyset$ .Suppose that $\{x_n\}$ is generated iteratively by

9) $$\begin{cases} x_1 \in K \\ x_{n+1} = (1-\alpha_n)T^n x_n \oplus \alpha_n T^n(y_n), \\ y_n = (1-\beta_n)x_n \oplus \beta_n T^n(x_n) \end{cases} \quad n \geq 1,$$

where $\{\alpha_n\}$ and{ β_n } are in [0,1]for all n ≥1.

We will establish Δ convergence theorem for asymptotically non expansive mapping in CAT (k) space with help of this iterative scheme.

Preliminary and Lemmas:

This class of asymptotically nonexpansive mappings was to introduced by Goebel and Kirk[11] . He proved that if K is a non empty bounded closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive self-mapping of K, then T has a fixed point. Chidume et al.[7] further generalized the class of asymptotically nonexpansive mappin- gs introduced by Goebel and Kirk[11], and proposed the concept of nonself asymptotically nonexpansive mapping The iterative approximation problem for asymptotically nonexpansive mapping and asymptotically nonexpansive type mapping were studied by many authors (see, e.g,16, 27) in a Banach space and a CAT(0) space.
Let (X, ρ) be a metric space . A geodesic path joining $x \in X$ to $y \in Y$ (or, more briefly, a geodesic from x to y) is a map from a closed interval [0, l] ⊂ R to X such that $\gamma(0) = x$,and $\gamma(l) = y$ and ρ (γ (t), γ (t')) = |t− t'| for all t , t' ∈ [0,l]. In particular, ρ is an isometry, and ρ (x, y) =l .The image $\gamma(0, l)$ of γ is called a geodesic (or metric) segment joining x and y.
When it is unique this geodesic segment is denoted by $[x, y]$. This means that $z \in [x, y]$ if and only if there exists $\alpha \in$ [0, l] such that ,

ρ (x, z) = $(1-\alpha)\rho$ (x, y) and ρ (y, z) = α ρ (x, y).
In this case, we write $z = \alpha x \oplus (1-\alpha)y$. The space$(X, \rho)$ is said to be a geodesic space (D −geodesic space) if every two points of X (every two points of distance smaller than D) are joined by a geodesic, and X is said to be uniquely geodesic (D −uniquely geodesic) if there is exactly one geodesic joining x and y for each $x, y \in X$ (for $x, y \in X$ $and \rho$ (x, y) $< D$). A subset K of X is said to be convex K if includes every geodesic segment joining any two of its points. The set K is said to be bounded if,

diam(K) :=sup $\{\rho$ $(x, y): x, y \in K\} < \infty$.
Now we introduce the model spaces M_k^n , for more details on these spaces the reader is referred to3[3].Let $n \in N$. We denote by E^n the metric space R^nendowed with the usual Euclidean distance.
We denote by (.|.)the Euclidean scalar product in R^n, that is,

$(x \mid y)$ =$x_1 y_1$ +…………$x_n y_n$, where x= $(x_1, \dots\dots x_n)$, y= $y_1, \dots\dots y_n$.
LetS^n denote the n - dimensional sphere defined by

S^n =$\{x = x_1, \dots . x_{n+1} \in R^{n+1} : (.|.) = 1\}$,

with metric d_{S^n} = arcos $((x \mid y))$, $x, y \in S^n$.
Let $E^{n,1}$ denote the vector space R^{n+1}endowed with the symmetric bilinear form which associates to vectors $u = (u_1, \dots \dots u_{n+1})$ and $v = (v_1, \dots \dots v_{n+1})$ the real number defined by
$\langle u|v\rangle = -u_{n+1}v_{n+1} + \sum_{i=1}^{n} u_i\, v_i$.

Let H^n denote the hyperbolic n- space defined by

$H^n = \{ u = (u_1, \dots . . u_{n+1}), \in E^{n,1} : \langle u|u\rangle = -1, u_{n+1} > 1 \}$
with metric d_{H^n} ,
such that, cos $d_{H^n}(x, y) = \langle u|v\rangle\ x, y \in H^n$.

Definition 2.1.(a) Let (X, d)be a metric space and K be its subset. Then $T: K \to K$ is said to be nonexpansive if $d(T_x, T_y) \leq d(x, y)$ for all $x, y \in K$ and asymptotically nonexpansive if there exists a sequence $\{ u_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} u_n = 0$ such that $d(T^n x, T^n y) \leq (1 + u_n)d(x, y)$ for all, $y, x \in K$ and n ≥1
A point $x \in K$ is called a fixed point if of T if $x = Tx$. We denote $F(T)$ the set of fixed of T.A sequence $\{x_n\}$ in K is called approximate fixed point sequence for T(AFPS) in short if,

$\lim_{n\to\infty} \rho(x_n, Tx_n)$ =0.

Definition 2.1.(b) Given $K \in R$, we denote M_k^n by the following metric spaces:

(1) If $k = 0$ then M_0^n is the Euclidean space E^n.
(2) If $k > 0$ then M_k^n is obtained from the spherical space S^n by multiplying the distance function by the constant $\frac{1}{\sqrt{k}}$.
(3) If $k < 0$ then M_k^n is obtained from the hyperbolic space S^n by multiplying the distance function by the constant $\frac{1}{\sqrt{-k}}$.

A geodesic triangle $\triangle(x, y, z)$ in a geodesic metric space (X, ρ) consists of three points x, y, z in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for geodesic triangle $\triangle\ (x, y, z)$ in (X, ρ) is a triangle $\triangle\ (\bar{x}, \bar{y}, \bar{z})$ in M_k^2 such that

$$\rho\ (x, y) = d_{M_k^2}(\bar{x}, \bar{y}),\ \rho\ (x, z) = d_{M_k^2}(\bar{x}, \bar{z}),\ \rho\ (z, x) = d_{M_k^2}(\bar{x}, \bar{z})$$

If $k \leq 0$ then such a comparison triangle always exists in M_k^2 . If $k > 0$ then such a triangle exists whenever $\rho\ (x, y) + \rho\ (y, z) + \rho\ (z, x) < 2D_k$,where $D_k = \frac{\pi}{\sqrt{k}}$. A point $\bar{p} \in [\bar{x}, \bar{y}]$ is called comparison point for $p \in [\ x, y]$ if $\rho\ (x, p) = d_{M_k^2}(\bar{x}, \bar{p})$.

A geodesic triangle $\triangle(x, y, z)$ in X is said to satisfy the CAT(k) inequality if for any $p, q \in\ (x, y, z)$ and for their comparison points $\bar{p}, \bar{q} \in \triangle\ (\bar{x}, \bar{y}, \bar{z})$, one has

$$\rho(p, q) \leq d_{M_k^2}(\bar{p}, \bar{q})$$

Definition 2.2 If $k \leq 0$ then X is called CAT(k) space if and only if X is a geodesic space such that all of its geodesic triangles satisfies the CAT(k) inequality . If $k > 0$ then X is called CAT(k) space if and only if X isD_k- geodesic and any geodesic triangle $\triangle(x, y, z)$ in X with $\rho\ (x, y) + \rho\ (y, z) + \rho\ (z, x) < 2D_k$ satisfies the CAT(k) inequality.

In CAT(0) space (X, ρ), if $x, y, z \in\ X$, then the CAT(0) inequality implies

$$\rho^2\left(x, \tfrac{1}{2}y \oplus \tfrac{1}{2}z\right) \leq \tfrac{1}{2}\rho^2(x, y) + \tfrac{1}{2}\ \rho^2(x, z) - \tfrac{1}{4}\rho^2(y, z). \qquad \text{(CN)}$$

This is the (CN) inequality of Bruhat and Tits [4]. This inequality is extended by Dhompongsa and Panyanak [9] as

$$\rho^2(x, (1 - \alpha)y \oplus \alpha z) \leq (1 - \alpha)\rho^2(x, y) + \alpha\rho^2(x, z) - (1 - \alpha)\alpha\rho^2(y, z). \qquad \text{(CN}^*\text{)}$$

for all $\alpha \in [0,1]$ and $x, y, z \in\ X$. Infact if X is a geodesic space then the following statements are equivalent:

(1) X is a CAT(0) space;

(2) X satisfies (CN) ;

(3) X satisfies (CN*).

Let $R \in (0,2]$, Recall that a geodesic space (X,ρ) is said to be R -convex see[23] if for any three points $x, y, z \in X$, we have

$$\rho^2(x,(1-\alpha)y\oplus\alpha z) \leq (1-\alpha)\rho^2(x,y) + \alpha\rho^2(x,z) - (1-\alpha)\alpha\rho^2(y,z). \qquad (2.1)$$

It follows from (CN*). that a geodesic space (X,ρ) is a CAT(0) space if and only if $\rho(X,\rho)$ is R –convex for $R = 2$. The following lemma is a consequence of Proposition 3.1 in [23].

Lemma 2.3 Let $k > 0$ and (X,ρ) be a CAT(k) space with diam(X) $\leq \frac{\frac{\pi}{2}-\varepsilon}{\sqrt{k}}$ for some $\varepsilon \in (0,\frac{\pi}{2})$.Then (X,ρ) is R –convex for R=$(\pi - 2\varepsilon)\tan(\varepsilon)$.

Lemma 2.4 Let $k > 0$ and (X,ρ) be a complete CAT(k) space with diam(X) $\leq \frac{\frac{\pi}{2}-\varepsilon}{\sqrt{k}}$ for some $\varepsilon \in (0,\frac{\pi}{2})$,then

$$\rho((1-\alpha)x\oplus\alpha y, z) \leq (1-\alpha)\rho(x,z)+\rho(y,z)$$

for all $\alpha \in [0,1]$ and $x, y, z \in X$

We now collect some elementary facts about CAT(k) space.Most of them are proved in the setting of CAT(1) space. For completeness, we state the results in CAT(k) with $k > 0$

Let $\{x_n\}$ be a bounded sequence in CAT(k) space (X,ρ). For $x \in X$,we set

r$(x,\{x_n\})$ =$limsup_{n\to\infty}\rho\ (x\{,x_n\})$

The asymptotic radius r$(\{x_n\})$ of $\{x_n\})$ is given by

r$(\{x_n\})$ =inf{ r$(x,\{x_n\})$: $x \in X$, }.

The asymptotic center A $(\{x_n\}$ of $\{x_n\}$ is a set,

A$(\{x_n\}$ ={ $x \in X$: r$(x,\{x_n\})$ $=$ r$(\{x_n\})$}.

It is known from [10] that in a CAT(k) space with diam(X)$< \frac{2\pi}{2\sqrt{k}}$, A$(\{x_n\}$ consists of exactly one point. We now give the concept of Δ convergence and collect some of its basic properties.

Definition 2.5 ([19],[22]). A sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$, if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write Δ-$lim_n x_n$= x and call x is the Δ limit of x_n.

Lemma 2.6 ([25]) Let $k > 0$ and (X,ρ) be a complete CAT(k) space with diam(X) $\leq \frac{\frac{\pi}{2}-\varepsilon}{\sqrt{k}}$ for some $\varepsilon \in (0,\frac{\pi}{2})$,then the following statement hold:

(i) every sequence in X has a Δ-convergence subsequence;

(ii) If $\{x_n\} \sqsubseteq X$ and Δ-$lim_n x_n$= x, then $x \in \bigcap_{k=1}^{\infty} \overline{conv}\{x_k, x_{k+1}, ...\}$, where $\overline{conv(A)}$ =∩ $\{B: B \sqsupseteq A \ and\ B\ is\ closed\ and\ convex\}$.

Lemma 2.7 Let $k > 0$ and (X,ρ) be a complete CAT(k) space with diam(X) $\leq \frac{\frac{\pi}{2}-\varepsilon}{\sqrt{k}}$ for some $\varepsilon \in (0,\frac{\pi}{2})$. If a sequence $\{x_n\}$ in X with A$(\{x_n\})$ = x and let $\{u_n\}$ be a subsequence of $\{x_n\}$ with A$(\{u_n\})$ = $\{u\}$ and the sequence $\{\ \rho\ (x_n, u)\}$ converges then $x = u$.

Lemma 2.8 [30] Let $\{S_n\}$ and $\{t_n\}$ be two sequences of non negative real numbers satisfying the inequality $s_{n+1} \leq s_n + t_n$ for all n$\in N$. If $\sum_{n=1}^{\infty} t_n < +\infty$, then $\lim_{n\to\infty} s_n$ exists.

Definition 2.9 Let (X,ρ) be a metric space and K be a non-empty subset. Then $T: K \to K$ is said to be semi-compact if for a sequence x_n in K with $\lim_{n\to\infty}\rho(x_n, Tx_n) = 0$ then their exists subsequence x_{n_k} of x_n such that $x_{n_k} \to p \in K$.

3 Main results

Now, we shall introduce existence theorems

Theorem3.1. Let $k > 0$ and (X, ρ) be a complete CAT(k) space with diam(X) $\leq \frac{\frac{\pi}{2}-\varepsilon}{\sqrt{k}}$ for some $\varepsilon \in (0, \frac{\pi}{2})$.Let K be a non-empty closed convex subset of X and let $T: K \to K$ be a asymptotically non expansive mapping. Then T has a fixed point.

Proof .Fix $x \in K$ .We consider the sequence $\{ T^n x\}_{n=1}^{\infty}$ as a bounded sequence in K .Let φ be a function defined by

$$\varphi: K \to [0,\infty), \quad \varphi(r)=\lim_{n\to\infty} sup\ \rho(T^n x, r) \ \text{ for all } r \in K.$$

Then their exists $z \in K$ such that $\varphi(z) = \inf\{\varphi(r): r \in K\}$. Since T is asymptotically non expansive mapping ,for each n,m $\in N$,we have

$$\rho(T^{n+m}x, T^m z) \leq (1+u_n)(T^m)\big(\rho(T^n x. z)\big)$$

On taking limit as $n \to \infty$, we obtain

$$\varphi(T^m z) \ \leq (1+u_n)(T^m)\varphi(z)$$

For any m$\in N$. This implies that

$$\lim_{m\to\infty} \varphi(T^m z) \leq \ \varphi(z).$$

(A)

In view of inequality (CN*) we obtain that

$$\rho\left(T^n x, \frac{(T^m z \oplus T^h z)}{2}\right)^2 \leq \frac{1}{2}\rho(T^n x, T^m z)^2 + \frac{1}{2}\rho(T^n x, T^h z)^2$$
$$-\frac{R}{8}\ \rho(T^m x, T^h z)^2,$$

On taking limit as $n \to \infty$,

$$\varphi(z)^2 \leq \varphi\left(\frac{T^m z \oplus T^h z}{2}\right)^2$$
$$\leq \frac{1}{2}\varphi(T^m z)^2 + \frac{1}{2}\varphi(T^h z)^2 - \frac{R}{8}\ \rho(T^m x, T^h z)^2$$

This gives

$$\frac{R}{8}\ \rho(T^m x, T^h z)^2 \leq \frac{1}{2}\varphi(T^m z)^2 + \frac{1}{2}\varphi(T^h z)^2 - \varphi(z)^2$$

(B)

From (A) and(B) we have $\lim_{m,h\to\infty} \rho(T^m x, T^h z) \leq 0$.Therefore, $\{ T^n z\}_{n=1}^{\infty}$ is a Cauchy sequence in K and hence converges to some point $v \in K$. Since T is continuous,

$$Tv = T\left(\lim_{n\to\infty} T^n z\right) = \lim_{n\to\infty} T^{n+1} z = v.$$

This shows that T has a fixed point in K .

Theorem3.2 . Let $k > 0$ and (X, ρ) be a complete CAT(k) space with diam(X) $\leq \frac{\frac{\pi}{2}-\varepsilon}{\sqrt{k}}$ for some $\varepsilon \in (0, \frac{\pi}{2})$.Let K be a non-empty closed convex subset of X and let $T: K \to K$ be a asymptotically non expansive mapping .If $\{x_n\}$ is an AFPS for T such that $\Delta \to \lim_{n\to\infty} x_n = w$, then $w \in K$ and $w = T(w)$.

Proof . By lemma 2.6 we get $w \in K$.

By theorem 3.1 we have , $\varphi(r)=\lim_{n\to\infty} sup\ \rho(T^n x, r)$ for all $r \in K$.
Since $\lim_{n\to\infty} \rho(x_n, T) = 0$
for some m $\in N$, this implies that
$\varphi(r)=\lim_{n\to\infty} sup\ \rho(T^m x_n, r)$ for each $r \in K$ and m $\in N$ (c)

Taking $r= T^m w$ in (c) we get

$$\varphi(T^m w) = \lim_{n\to\infty} sup\ \rho(T^m x_n, T^m w)$$

$$\leq \lim_{n\to\infty} sup(1+u_n)(T^m)\big(\rho(x_n, w)\big)$$

Hence

$$\lim_{m\to\infty} sup\varphi(T^m w) \leq \varphi(w) \tag{d}$$

By lemma 2.4 we have,

$$\rho\left(x_n, \frac{w\oplus T^m w}{2}\right)^2 \leq \frac{1}{2}\,\rho(x_n, w)^2 + \frac{1}{2}\,\rho(x_n, T^m w)^2 - \frac{R}{8}\,\rho(w, T^m w)^2$$

where $R=(\pi - 2\varepsilon)\tan(\varepsilon)$
Since $\Delta - \lim_{n\to\infty} x_n = w$,letting $n \to \infty$,we get

$$\varphi(w)^2 \leq \varphi\left(\frac{w\oplus T^m w}{2}\right)^2$$
$$\leq \frac{1}{2}\varphi(w)^2 + \frac{1}{2}\,\varphi(T^m w)^2 - \frac{R}{8}\,\rho(w, T^m w)^2$$

This gives,

$$\rho(w, T^m w)^2 \leq \frac{4}{R}[\varphi(T^m w)^2 - \varphi(w)^2]. \tag{e}$$

By (d) and (e), we have $\lim_{m\to\infty}\rho(w, T^m w)$ =0. Since T is continuous,
$Tw = T\left(\lim_{m\to\infty} T^m w\right) = \lim_{n\to\infty} T^{m+1} w = w.$
This shows that T has a fixed point in K.

Lemma 3.3 . Let $k > 0$ and (X, ρ) be a complete CAT(k) space with diam(X) $\leq \frac{\frac{\pi}{2}-\varepsilon}{\sqrt{k}}$ for some $\varepsilon \in (0, \frac{\pi}{2})$.Let K be a non-empty closed convex subset of X and let $T: K \to K$ be a asymptotically non expansive mapping with$\sum_{n=1}^{\infty} u_n < \infty$. Let $\{x_n\}$ be a sequence in K defined by (9) where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequencesin (0,1) such that liminf$\alpha_n\beta_n(1-\beta_n)$> 0.Then $\{x_n\}$ is an AFPS for T and $\rho(x_n, p)$ exists for all $p \in F(T)$.

Proof .Step-1 : We first prove that $lim_{n\to\infty}\rho(x_n, p)$ exists.
From theorem 3.1 , we have $F(T) \neq \emptyset$. Let $p \in F(T)$ andM= diam(K). Since T is asymptotically non expansive mapping ,by lemma 2.4 we have

$$\rho(y_n, p)=\rho\big((1-\beta_n)x_n \oplus \beta_n T^n x_n, p\big)$$
$$\leq (1-\beta_n)\rho(x_n, p) + \beta_n \rho(T^n x_n, p)$$
$$\leq (1-\beta_n)\rho(x_n, p) + \beta_n\big((1+u_n)\rho(x_n, p)\big)$$
$$\leq (1+u_n)\rho(x_n, p)$$
$$\rho(x_{n+1}, p) = \rho\big((1-\alpha_n)T^n x_n \oplus \alpha_n T^n y_n, p\big)$$
$$\leq (1-\alpha_n)\, d(T^n x_n, p) + \alpha_n \rho(T^n y_n, T^n p)$$
$$\leq (1-\alpha_n)\{(1+u_n)\rho(x_n, p)\} + \alpha_n\{(1+u_n)\rho(y_n, p)\}$$

$$\leq (1-\alpha_n)(1+u_n)\rho(x_n, p) + \alpha_n(1+u_n)\,(1+u_n)\rho(x_n, p)$$

$$\leq (1+u_n)^2\,\rho(x_n, p)$$

$$\leq (1 + A_n)\, \rho(x_n, p) \qquad \text{(f)}$$

Where $A_n = 2u_n + u_n{}^2$. Since by hypothesis of the theorem $\sum_{n=1}^{\infty} u_n < \infty$, it follows from Lemma (2.8) that $lim_{n\to\infty} \rho(x_n, p)$ exists.

Step-2 We will prove that $lim_{n\to\infty} \rho(x_n, T(x_n)) = 0$.
Next,we show that $\{x_n\}$ is an AFPS for T. In view of (2.1), and (f) we have

$$\begin{aligned}
\rho^2(x_{n+1}.p) &= \rho^2\big((1-\alpha_n)T^n x_n \oplus \alpha_n T^n y_n, p\big)\\
&\leq (1-\alpha_n)\rho^2(T^n(x_n), p) + \alpha_n \rho^2(T^n(y_n), p) - \tfrac{R}{2}\alpha_n(1-\alpha_n)\rho^2(T^n(x_n), T^n(y_n))\\
&\leq (1-\alpha_n)\rho^2(T^n(x_n), T^n(p)) + \alpha_n \rho^2(T^n(y_n), T^n(p))\\
&\leq (1-\alpha_n)\big((1+(u_n)\rho(x_n,p)\big)^2 + \alpha_n\big((1+(u_n)\rho(y_n,p)\big)^2\\
&\leq (1-\alpha_n)\rho^2(x_n,p) + (1-\alpha_n)A_n\rho^2(x_n,p) + \alpha_n\rho^2(y_n,p) + \alpha_n A_n \rho^2(y_n,p)\\
&\leq (1-\alpha_n)(1+A_n)\rho^2(x_n,p) + \alpha_n(1+A_n)\rho^2(y_n,p)\\
&\leq (1-\alpha_n)\rho^2(x_n,p) + \alpha_n\rho^2(y_n,p) + B_n \qquad \text{(g)}
\end{aligned}$$

Again by (2.1) , we have

$$\begin{aligned}
\rho^2(y_n,p) &\leq \rho^2\big((1-\beta_n)x_n \oplus \beta_n T^n x_n, p\big)\\
&\leq (1-\beta_n)\rho^2(x_n,p) + \beta_n\rho^2(T^n(x_n), T^n(p)) - \tfrac{R}{2}\beta_n(1-\beta_n)\rho^2((x_n, T^n(x_n))\\
&\leq (1-\beta_n)\rho^2(x_n,p) + \beta_n\big((1+(u_n)\rho(x_n,p)\big)^2 - \tfrac{R}{2}\beta_n(1-\beta_n)\rho^2((x_n, T^n(x_n))\\
&\leq (1-\beta_n)\rho^2(x_n,p) + \beta_n(1+A_n)\rho^2(x_n,p) - \tfrac{R}{2}\beta_n(1-\beta_n)\rho^2((x_n, T^n(x_n))\\
&\leq (1+A_n\beta_n)\rho^2(x_n,p) - \tfrac{R}{2}\beta_n(1-\beta_n)\rho^2((x_n, T^n(x_n))
\end{aligned}$$

Putting in (g) we get

$$\begin{aligned}
\rho^2(x_{n+1}.p) &\leq (1-\alpha_n)\rho^2(x_n,p) + \alpha_n[(1+A_n\beta_n)\rho^2(x_n,p) - \tfrac{R}{2}\,\beta_n(1-\beta_n)\rho^2((x_n, T^n(x_n))]\\
&\quad + B_n\\
&\leq \rho^2(x_n,p) + \alpha_n[A_n\beta_n\rho^2(x_n,p) - \tfrac{R}{2}\,\beta_n(1-\beta_n)\rho^2((x_n, T^n(x_n))] + B_n
\end{aligned}$$

Implies

$$\tfrac{R}{2}\alpha_n\,\beta_n(1-\beta_n)\rho^2((x_n, T^n(x_n)) \leq \rho^2(x_n,p) - \rho^2(x_{n+1}.p) + B_n + CA_n \text{ ,} \exists C, D \geq 0$$

Since $\sum_{n=1}^{\infty} u_n < \infty$ and hence $\sum_{n=1}^{\infty} A_n < \infty$, we have

$$\sum_{n=1}^{\infty} \alpha_n\,\beta_n(1-\beta_n)\rho^2((x_n, T^n(x_n)) < \infty$$

This implies by

$$\liminf_{n\to\infty} \alpha_n\,\beta_n(1-\beta_n) > 0 \qquad \text{(h)}$$

$$\lim_{n\to\infty} \rho((x_n, T^n(x_n)) = 0$$

By the uniform continuity of T ,we have

$$\lim_{n\to\infty} \rho((x_n, T^{n+1}(x_n)) = 0 \qquad \text{(i)}$$

It follows from (h) and definition of x_{n+1} and y_n that

$$\begin{aligned}
\rho(x_n, x_{n+1}) &= \rho\big(x_n, (1-\alpha_n)T^n x_n \oplus \alpha_n T^n(y_n)\big)\\
&\leq (1-\alpha_n)\,\rho(x_n, T^n(x_n)) + \alpha_n\,\rho(x_n, T^n(y_n))\\
&\leq \rho(x_n, T^n(x_n)) + \rho(x_n, T^n(y_n))\\
&\leq \rho(x_n, T^n(x_n)) + \rho(x_n, T^n(x_n)) + \rho(T^n(x_n), T^n(y_n))\\
&\leq 2\,\rho(x_n, T^n(x_n)) + \rho(T^n(x_n), T^n(y_n))\\
&\leq 2\,\rho(x_n, T^n(x_n)) + (1+u_n)\,\rho(x_n, y_n)\\
&\leq 2\,\rho(x_n, T^n(x_n)) + (1+u_n)\,\rho(x_n, (1-\beta_n)x_n \oplus \beta_n T^n x_n, p)\\
&\leq 2\,\rho(x_n, T^n(x_n)) + (1+u_n)[(1-\beta_n)\rho(x_n, x_n) + \beta_n\,\rho(x_n, T^n(x_n))]
\end{aligned}$$

(j)

By (h),(i) and(j) we have,
$\to 0as$ n$\to \infty$

$$\rho(x_n, T(x_n)) \leq \rho(x_n, x_{n+1}) + \rho(x_{n+1}, T^n(x_{n+1}))$$

$$+ \rho(T^{n+1}(x_{n+1}), T^{n+1}(x_n)) + \rho(T^{n+1}(x_n), T(x_n))$$

$$\leq \rho(x_n, x_{n+1}) + \rho(x_{n+1}, T^n(x_{n+1})) + (1 + u_n)\rho(x_{n+1}, x_n)$$

$$+ \rho(T^{n+1}(x_n), T(x_n)) \to 0as \text{ n} \to \infty$$

Theorem-3.4 . Let $k > 0$ and (X, ρ) be a complete CAT(k) space with diam(X) $\leq \frac{\frac{\pi}{2}-\varepsilon}{\sqrt{k}}$ for some $\varepsilon \in (0, \frac{\pi}{2})$.Let K be a non-empty closed convex subset of X and let $T: K \to K$ be a asymptotically non expansive mapping with$\sum_{n=1}^{\infty} u_n < \infty$. Let $\{x_n\}$ be a sequence in K defined by (9) where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequencesin (0,1) such that liminf$\alpha_n \beta_n (1-\beta_n)$> 0.Then $\{x_n\}$ Δ
converges to fixed point of T .

Proof. Let $W_W(\{x_n\})$:=U A($\{u_n\}$) where the union is taken for all subsequences $\{u_n\}$ of $\{x_n\}$.
We first show that $W_W(\{x_n\}) \sqsubseteq F(T)$. Let $u \in W_W(\{x_n\})$ then their exists subsequences $\{u_n\}$ of $\{x_n\}$ such that A($\{u_n\}) = \{u\}$ .By lemma 2.6 their exists subsequences $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_n v_n = v \in K$. By lemma 3.3 and theorem 3.1 we have $v \in F(T)$. Since $\lim_n \rho(x_n, v)$ exists , so $u = v$ by lemma 2.7. This shows that $W_W(\{x_n\}) \sqsubseteq F(T)$.

Next we show that , Δ converges to point in $F(T)$, it is sufficient to show that $W_W(\{x_n\})$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ such that A($\{u_n\}) = \{u\}$ and let A($\{x_n\}$)={ x } . Since $u \in W_W(\{x_n\}) \sqsubseteq F(T)$, by lemma 3.3 $\lim_n \rho(x_n, u)$ exists and by lemma 2.7 we have $x = u$.
This completes the proof.

Theorem-3.5 . Let $k > 0$ and (X, ρ) be a complete CAT(k) space with diam(X) $\leq \frac{\frac{\pi}{2}-\varepsilon}{\sqrt{k}}$ for some $\varepsilon \in (0, \frac{\pi}{2})$.Let K be a non-empty closed convex subset of X and let $T: K \to K$ be a asymptotically non expansive mapping with$\sum_{n=1}^{\infty} u_n < \infty$. Let $\{x_n\}$ be a sequence in K defined by (9) where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequencesin (0,1) such that liminf$\alpha_n \beta_n (1-\beta_n)$> 0. Suppose that T^m is semi-compact for some m $\in N$. Then sequence $\{x_n\}$ converges strongly to a fixed point of T .

Proof. By lemma 3.3,$lim_n \rho\left(x_n, T(x_n)\right)$ =0.Since T is uniformly continuous ,we have

$$\rho\left(x_n, T^m(x_n)\right) \leq \rho\left(x_n, T(x_n)\right) + \rho\left(T(x_n), T^2(x_n)\right) + \ldots\ldots + \rho\left(T^{m-1}(x_n), T^m(x_n)\right) \to 0$$

as n$\to \infty$ .That is $\{x_n\}$ is an AFPS for T^m. By definition 2.9 their exist a subsequence $\left\{x_{n_j}\right\}$ of $\{x_n\}$ and $p \in K$ such that$\lim_{j \to \infty} x_{n_j} = p$ .Again by the uniform continuity of T ,we have

$$\rho(T(p), p) \leq \rho\left(T(p), T\left(x_{n_j}\right)\right) + \rho\left(T\left(x_{n_j}\right), x_{n_j}\right) + \rho\left(x_{n_j}, p\right) \to 0 \text{ as j} \to \infty.$$

That is, $p \in F(T)$ .By lemma 3.3, $lim_n \rho(x_n, p)$ exists, thus p is the strong limit of the sequence $\{x_n\}$ itself .

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