On Jordan Generalized (σ,τ) -Higher Reverse Derivations of Gamma–Rings

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Abstract:

Let M be a Γ-ring and σ^n,τ^n be two higher endomorphisms of a Γ-ring M, for all $n \in N$ in the present paper we show that under certain conditions of M, every Jordan generalized (σ,τ)-higher reverse derivation of a Γ-Ring M is a generalized (σ,τ)-higher reverse derivation

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1- Introduction:

Let M and Γ be two additive a belian groups, suppose that there is a mapping from M×Γ×M $\longrightarrow$ M (the image of (a,α,b) being denoted by $a\alpha b$, $a, b \in$ M and $\alpha \in \Gamma$). Satisfying for all $a, b, c \in$ M and $\alpha, \beta \in \Gamma$:

(i) $(a + b)\alpha c = a\alpha c + b\alpha c$

$a(\alpha + \beta)c = a\alpha c + a\beta c$

$a\alpha(b + c) = a\alpha b + a\alpha c$

(ii) $(a\alpha b)\beta c = a\alpha(b\beta c)$

Then M is called a Γ-ring. This definition is due to Barnes [1].

Let M be Γ-ring then M is called 2-torsion free if $2a = 0$ implies $a = 0$, for every $a \in$ M, this definition is due to [3].

Let M be a Γ-ring and d: M $\longrightarrow$ M be an additive mapping (that is $d(a + b) = d(a) + d(b)$) of a Γ-ring M into itself then d is called a derivation on M if : $d(a\alpha b) = d(a)\alpha b + a\alpha d(b)$, for all $a, b \in$ M and $\alpha \in \Gamma$ and d is called a Jordan derivation on M if $d(a\alpha a) = d(a)\alpha a + a\alpha d(a)$, for all $a \in$ M and $\alpha \in \Gamma$, [2].

Let M be a Γ-ring, an additive mapping F: M $\longrightarrow$ M is called a generalized derivation on M if there exists a derivation d: $M \longrightarrow M$, such that:

$F(a\alpha b) = F(a)\alpha b + a\alpha d(b)$, for all $a, b \in M$ and $\alpha \in \Gamma$.

And F is called Jordan generalized derivation if there exists a Jordan derivation d: $M \longrightarrow M$, such that:

$F(a\alpha a) = F(a)\alpha a + a\alpha d(b)$, for all $a, b \in M$ and $\alpha \in \Gamma$, [2] .

Let M be a Γ-ring and σ,τ be tow endomorphisms of M. such that d: $M \longrightarrow M$ be an additive mapping. Then d is called (σ,τ)-derivation of M if:

$d(a\alpha b) = d(a)\,\alpha\,\tau(b) + \sigma(a)\,\alpha\, d(b)$, for all $a, b \in M$, $\alpha \in \Gamma$.

And d is called a Jordan (σ,τ)-derivation of M if:

$d(a\alpha a) = d(a)\,\alpha\,\tau(a) + \sigma(a)\,\alpha\, d(a)$, for all $a \in M$, $\alpha \in \Gamma$, [5].

Let M be a Γ-ring and σ,τ be tow endomorphisms of M. such that F: $M \longrightarrow M$ be an additive mapping. Then F is called a generalized (σ,τ)-derivation of M if there exists a (σ,τ)-derivation d: $M \longrightarrow M$, such that:

$F(a\alpha b) = F(a)\,\alpha\,\tau(b) + \sigma(a)\,\alpha\, d(b)$, for all $a, b \in M$, $\alpha \in \Gamma$.

Let M be a Γ-ring and σ,τ be tow endomorphisms of M. such that F: $M \longrightarrow M$ be an additive mapping. Then F is called a Jordan generalized (σ,τ)-derivation of M if there exists a Jordan (σ,τ)- derivation d: $M \longrightarrow M$, such that:

$F(a\alpha a) = F(a)\,\alpha\,\tau(a) + \sigma(a)\,\alpha\, d(a)$, for all $a \in M$, $\alpha \in \Gamma$, [5] .

Let M be a Γ-ring and d: $M \longrightarrow M$ be an additive mapping of a Γ-ring M into itself then d is called reverse derivation on M if

$d(a\alpha b) = d(b)\alpha a + b\alpha d(a)$, for all $a, b \in M$ and $\alpha \in \Gamma$.

Let M be a Γ-ring and d: $M \longrightarrow M$ be an additive mapping of a Γ-ring M into itself then d is called a Jordan reverse derivation on M if

$d(a\alpha a) = d(a)\alpha a + a\alpha d(a)$, for all $a \in M$ and $\alpha \in \Gamma$, [4].

Let M be a Γ-ring and F: $M \longrightarrow M$ be an additive mapping of a Γ-ring M into itself then F is called generalized reverse derivation on M if there exists a reverse derivation d: $M \longrightarrow M$, such that , $F(a\alpha b) = F(b)\alpha a + b\alpha d(a)$, for all $a, b \in M$ and $\alpha \in \Gamma$.

Let M be a Γ-ring and F: $M \longrightarrow M$ be an additive mapping of a Γ-ring M into itself then F is called a Jordan generalized reverse derivation on M if there exists a Jordan reverse derivation d: $M \longrightarrow M$, such that

$F(a\alpha a) = F(a)\alpha a + a\alpha d(a)$, for all $a \in M$ and $\alpha \in \Gamma$, [6].

Let M be a *Γ*-ring and F= $(f_i)_{i\in N}$ be a family of additive mappings of M, such that $f_0 = id_M$ then F is called a generalized higher reverse derivation of M if there exists a higher reverse derivation D = $(d_i)_{i\in N}$ of M , such that for every *a* , b∈ M, α ∈ *Γ* and n ∈N

$$f_n(a\alpha b) = \sum_{i+j=n} f_i(b)\alpha d_j(a)$$

And F is called a Jordan generalized higher reverse derivation of M if there exists a Jordan higher reverse derivation D = $(d_i)_{i\in N}$ of M , such that for every *a* , b∈ M, α ∈ *Γ* and n ∈N

$$f_n(a\alpha a) = \sum_{i+j=n} f_i(a)\alpha d_j(a)$$, [6].

Now, the main purpose of this paper is that every Jordan generalized (σ,τ)- higher reverse derivation of a 2-torsion free Γ-ring M into itself , such that $a\alpha b\beta a = a\beta b\alpha a$, for all *a*, *b* ∈ M and α, β ∈ Γ is a Jordan generalized triple (σ,τ)-higher reverse derivation .

2- Jordan generalized (σ,τ)-Higher Reverse Derivations on Γ-Ring :

Definition (2.1):

Let F = $(f_i)_{i\in N}$ be a family of additive mappings of a Γ-ring M into itself, such that $f_0 = id_M$ and σ,τ be two endomorphisms of M. F is called a **generalized (σ,τ)-higher reverse derivation** if there exists a (σ,τ)- higher reverse derivation D = $(d_i)_{i\in N}$ of M , such that

$$f_n(a\alpha b) = \sum_{i+j=n} f_i(\sigma^{n-i}(b))\alpha d_j(\tau^{n-j}(a))$$, for all *a*, *b* ∈ M, α ∈ Γ and n ∈ N.

Example (2.2):

Let R be a ring and f = $(fi)_{i\in N}$ be a generalized (σ,τ)-higher reverse derivation on R . Then there exists a (σ,τ)- higher reverse derivation d = $(d_i)_{i\in N}$ of R , such that
for all *a* , b ∈ R and n ∈ N

$$f_n(ab) = \sum_{i+j=n} f_i(\sigma^{n-i}(b))d_j(\tau^{n-j}(a)) \cdot$$

Let M = $M_{1\times 2}(R)$ and $\Gamma = \left\{ \begin{pmatrix} n \\ 0 \end{pmatrix}, n \in N \right\}$. Then M is a Γ-ring. We define

F= $(F_i)_{i\in N}$ be a family of additive mappings of M, such that $F_n((a \quad b)) = (f_n(a) \quad f_n(b))$.
Then there exists a (σ,τ)- higher reverse derivation D = $(d_i)_{i\in N}$ of M , such that for all *a* , b ∈ M , α ∈ Γ and n ∈ N $D_n((a \quad b)) = ((d_n(a) \quad d_n(b))$.
Let σ_1^n, τ_1^n be two endomorphisms of M, such that $\sigma_1^n((a \quad b)) = ((\sigma(a) \quad \sigma(b))$, $\tau_1^n((a \quad b)) = ((\tau(a) \quad \tau(b))$.
Then F is a generalized (σ,τ)-higher reverse derivation.

Definition (2.3):

Let F = $(f_i)_{i\in N}$ be a family of additive mappings of a Γ-ring M into itself, such that f_0 = id_M and σ,τ be two endomorphisms of M. F is called a **Jordan generalized (σ,τ)-higher reverse derivation** if there exists a Jordan (σ,τ)- higher reverse derivation D = $(d_i)_{i\in N}$ of M , such that

$$f_n(a\alpha a)=\sum_{i+j=n} f_i(\sigma^{n-i}(a))\alpha d_j(\tau^{n-j}(a))$$, for all $a\in$ M, α ∈ Γ and n ∈ N.

Definition (2.4):

Let F = $(f_i)_{i\in N}$ be a family of additive mappings of a Γ-ring M into itself, such that f_0 = id_M and σ,τ be two endomorphisms of M. F is called a **Jordan generalized triple (σ,τ)-higher reverse derivation** if there exists a Jordan triple (σ,τ)- higher reverse derivation D = $(d_i)_{i\in N}$ of M , such that

$$f_n(a\alpha b\beta a)=f_n(a)\beta a\alpha b+\sum_{\substack{i<n\\ i+j+k=n}} f_i(\sigma^{n-i}(a))\beta d_j(\sigma^k\tau^i(b))\alpha d_k(\tau^{n-k}(a))$$

for all a, b ∈ M, α, β ∈ Γ and n ∈ N.

Lemma (2.5):

Let F = $(f_i)_{i\in N}$ be a Jordan generalized triple (σ,τ)-higher reverse derivations on a Γ-ring M into itself. Then for all a, b, c ∈ M, α, β ∈ Γ and n ∈ N

(i) $$f_n(a\alpha b+b\alpha a)=\sum_{i+j=n} f_i(\sigma^{n-i}(b))\alpha d_j(\tau^{n-j}(a))+\sum_{i+j=n} f_i(\sigma^{n-i}(a))\alpha d_j(\tau^{n-j}(b))$$

(ii) $$f_n(a\alpha b\beta a+a\beta b\alpha a)=f_n(a)\beta a\alpha b+\sum_{\substack{i<n\\ i+j+k=n}} f_i(\sigma^{n-i}(a))\beta d_j(\sigma^k\tau^i(b))\alpha d_k(\tau^{n-k}(a))+$$
$$f_n(a)\alpha a\beta b+\sum_{\substack{i<n\\ i+j+k=n}} f_i(\sigma^{n-i}(a))\alpha d_j(\sigma^k\tau^i(b))\beta d_k(\tau^{n-i}(a))$$

(iii) If M is a 2-torsion free Γ-ring.

$$f_n(a\alpha b\alpha a)=f_n(a)\alpha a\alpha b+\sum_{\substack{i<n\\ i+j+k=n}} f_i(\sigma^{n-i}(a))\alpha d_j(\sigma^k\tau^i(b))\alpha d_k(\tau^{n-k}(a))$$

(iv) $$f_n(a\alpha b\beta c+c\alpha b\beta a)=f_n(c)\beta a\alpha b+\sum_{\substack{i<n\\ i+j+k=n}} f_i(\sigma^{n-i}(c))\beta d_j(\sigma^k\tau^i(b))\alpha d_k(\tau^{n-k}(a))+$$
$$f_n(a)\beta c\alpha b+\sum_{\substack{i<n\\ i+j+k=n}} f_i(\sigma^{n-i}(a))\beta d_j(\sigma^k\tau^i(b))\alpha d_k(\tau^{n-k}(c))$$

(v) In particular, if M is a 2-torsion free commutative Γ-ring

$$f_n(a\alpha b\beta c)=f_n(c)\beta a\alpha b+\sum_{\substack{i<n\\ i+j+k=n}} f_i(\sigma^{n-i}(c))\beta d_j(\sigma^k\tau^i(b))\alpha d_k(\tau^{n-k}(a))$$

(vi) $$f_n(a\alpha b\alpha c+c\alpha b\alpha a)=f_n(c)\alpha a\alpha b+\sum_{\substack{i<n\\ i+j+k=n}} f_i(\sigma^{n-i}(c))\alpha d_j(\sigma^k\tau^i(b))\alpha d_k(\tau^{n-k}(a))+$$
$$f_n(a)\alpha c\alpha b+\sum_{\substack{i<n\\ i+j+k=n}} f_i(\sigma^{n-i}(a))\alpha d_j(\sigma^k\tau^i(b))\alpha d_k(\tau^{n-k}(a))$$

Proof:

(i) Replacing $a+b$ for a in the Definition (2.3), we get:

$$f_n((a+b)\alpha(a+b)) = \sum_{i+j=n} f_i(\sigma^{n-i}(a+b))\alpha d_j(\tau^{n-j}(a+b))$$

$$= \sum_{i+j=n} f_i(\sigma^{n-i}(a)+\sigma^{n-i}(b))\alpha d_j(\tau^{n-j}(a)+\tau^{n-j}(b))$$

$$= \sum_{i+j=n} f_i(\sigma^{n-i}(a))\alpha d_j(\tau^{n-j}(a)) + \sum_{i+j=n} f_i(\sigma^{n-i}(a))\alpha d_j(\tau^{n-j}(b)) +$$

$$\sum_{i+j=n} f_i(\sigma^{n-i}(b))\alpha d_j(\tau^{n-j}(a)) + \sum_{i+j=n} f_i(\sigma^{n-i}(b))\alpha d_j(\tau^{n-j}(b)) \quad \ldots(1)$$

On the other hand:

$$f_n((a+b)\alpha(a+b)) = f_n(a\alpha a + a\alpha b + b\alpha a + b\alpha b)$$

$$= \sum_{i+j=n} f_i(\sigma^{n-i}(a))\alpha d_j(\tau^{n-j}(a)) + \sum_{i+j=n} f_i(\sigma^{n-i}(b))\alpha d_j(\tau^{n-j}(b)) + f_n(a\alpha b + b\alpha a) \quad \ldots(2)$$

Comparing (1) and (2), we get:

$$f_n(a\alpha b + b\alpha a) = \sum_{i+j=n} f_i(\sigma^{n-i}(b))\alpha d_j(\tau^{n-j}(a)) + \sum_{i+j=n} f_i(\sigma^{n-i}(a))\alpha d_j(\tau^{n-j}(b))$$

(ii) Replace $a\beta b + b\beta a$ for b in (i), we get:

$$f_n(a\alpha(a\beta b + b\beta a) + (a\beta b + b\beta a)\alpha a)$$

$$= f_n(a\alpha(a\beta b) + a\alpha(b\beta a) + (a\beta b)\alpha a + (b\beta a)\alpha a)$$

$$= f_n((a\alpha a)\beta b + (a\alpha b)\beta a) + (a\beta b)\alpha a + (b\beta a)\alpha a)$$

$$= \sum_{i+j=n} f_i(\sigma^{n-i}(b))\beta d_j(\tau^{n-j}(a\alpha a)) + \sum_{i+j=n} f_i(\sigma^{n-i}(a))\beta d_j(\tau^{n-j}(a\alpha b)) +$$

$$\sum_{i+j=n} f_i(\sigma^{n-i}(a))\alpha d_j(\tau^{n-j}(a\beta a)) + \sum_{i+j=n} f_i(\sigma^{n-i}(a))\alpha d_j(\tau^{n-j}(b\beta a))$$

$$=\sum_{i+j=n} f_i(\sigma^{n-i}(b))\beta(\sum_{r+s=j} d_r(\sigma^{j-r}\tau^{n-j}(a))\alpha d_s(\tau^{j-s}\tau^{n-j}(a)))+$$

$$\sum_{i+j=n} f_i(\sigma^{n-i}(a))\beta(\sum_{e+f=j} d_e(\sigma^{j-e}\tau^{n-j}(b))\alpha d_f(\tau^{j-f}\tau^{n-j}(a)))+$$

$$\sum_{i+j=n} f_i(\sigma^{n-i}(a))\alpha(\sum_{p+q=j} d_p(\sigma^{j-p}\tau^{n-j}(b))\beta d_q(\tau^{j-q}\tau^{n-j}(a)))+$$

$$\sum_{i+j=n} f_i(\sigma^{n-i}(a))\alpha(\sum_{x+y=j} d_x(\sigma^{j-x}\tau^{n-j}(a))\beta d_y(\tau^{j-y}\tau^{n-j}(b)))$$

$$=\sum_{i+r+s=n} f_i(\sigma^{n-i}(b))\beta d_r(\sigma^{s}\tau^{i}(a))\alpha d_s(\tau^{n-s}(a))+$$

$$\sum_{i+e+f=n} f_i(\sigma^{n-i}(a))\beta d_e(\sigma^{f}\tau^{i}(b))\alpha d_f(\tau^{n-f}(a))+$$

$$\sum_{i+p+q=n} f_i(\sigma^{n-i}(a))\alpha d_p(\sigma^{q}\tau^{i}(b))\beta d_q(\tau^{n-q}(a))+$$

$$\sum_{i+x+y=n} f_i(\sigma^{n-i}(a))\alpha d_x(\sigma^{y}\tau^{i}(a))\beta d_y(\tau^{n-y}(b))$$

$$=f_n(b)\beta a\alpha a+\sum_{\substack{i<n\\ i+j+k=n}} f_i(\sigma^{n-i}(b))\beta d_j(\sigma^{k}\tau^{i}(a))\alpha d_k(\tau^{n-k}(a))+$$

$$f_n(a)\beta a\alpha b+\sum_{\substack{i<n\\ i+j+k=n}} f_i(\sigma^{n-i}(a))\beta d_j(\sigma^{k}\tau^{i}(b))\alpha d_k(\tau^{n-k}(a))+$$

$$f_n(a)\alpha a\beta b+\sum_{\substack{i<n\\ i+j+k=n}} f_i(\sigma^{n-i}(a))\alpha d_j(\sigma^{k}\tau^{i}(b))\beta d_k(\tau^{n-k}(a))+$$

$$f_n(a)\alpha b\beta a+\sum_{\substack{i<n\\ i+j+k=n}} f_i(\sigma^{n-i}(a))\alpha d_j(\sigma^{k}\tau^{i}(a))\beta d_k(\tau^{n-k}(b))$$

…(1)

On the other hand:

$$f_n(a\alpha(a\beta b+b\beta a)+(a\beta b+b\beta a)\alpha a)$$
$$=f_n(a\alpha a\beta b+a\alpha b\beta a+a\beta b\alpha a+b\beta a\alpha a)$$
$$=f_n(b)\beta a\alpha a+\sum_{\substack{i<n\\ i+j+k=n}} f_i(\sigma^{n-i}(b))\beta d_j(\sigma^{k}\tau^{i}(a))\alpha d_k(\tau^{n-k}(a))+$$
$$f_n(a)\alpha b\beta a+\sum_{\substack{i<n\\ i+j+k=n}} f_i(\sigma^{n-i}(a))\alpha d_j(\sigma^{k}\tau^{i}(a))\beta d_k(\tau^{n-k}(b))+f_n(a\alpha b\beta a+a\beta b\alpha a)$$

…(2)

Comparing (1) and (2), we get:

$$f_n(a\alpha b\beta a+a\beta b\alpha a)=$$
$$=f_n(a)\beta a\alpha b+\sum_{\substack{i<n\\ i+j+k=n}} f_i(\sigma^{n-i}(a))\beta d_j(\sigma^{k}\tau^{i}(b))\alpha d_k(\tau^{n-k}(a))+$$
$$f_n(a)\alpha a\beta b+\sum_{\substack{i<n\\ i+j+k=n}} f_i(\sigma^{n-i}(a))\alpha d_j(\sigma^{k}\tau^{i}(b))\beta d_k(\tau^{n-k}(a))$$

(iii) Replace α for β in (ii), we get:

$$f_n(a\alpha b\alpha a + a\alpha b\alpha a) = 2f_n(a\alpha b\alpha a)$$

Since M is a 2-torsion free Γ-ring

$$= f_n(a)\alpha a\alpha b + \sum_{\substack{i<n\\ i+j+k=n}} f_i(\sigma^{n-i}(a))\alpha d_j(\sigma^k\tau^i(b))\alpha d_k(\tau^{n-k}(a))$$

(iv) Replace $a+c$ for a in Definition (2.4), we get:

$$f_n((a+c)\alpha b\beta(a+c)) = f_n(a+c)\beta(a+c)\alpha b +$$
$$\sum_{\substack{i<n\\ i+j+k=n}} f_i(\sigma^{n-i}(a+c))\beta d_j(\sigma^k\tau^i(b))\alpha d_k(\tau^{n-j}(a+c))$$
$$= f_n(a)\beta a\alpha b + \sum_{\substack{i<n\\ i+j+k=n}} f_i(\sigma^{n-i}(a))\beta d_j(\sigma^k\tau^i(b))\alpha d_k(\tau^{n-j}(a)) +$$
$$f_n(c)\beta a\alpha b + \sum_{\substack{i<n\\ i+j+k=n}} f_i(\sigma^{n-i}(c))\beta d_j(\sigma^k\tau^i(b))\alpha d_k(\tau^{n-j}(a)) +$$
$$f_n(a)\beta c\alpha b + \sum_{\substack{i<n\\ i+j+k=n}} f_i(\sigma^{n-i}(a))\beta d_j(\sigma^k\tau^i(b))\alpha d_k(\tau^{n-j}(c)) +$$
$$f_n(c)\beta c\alpha b + \sum_{\substack{i<n\\ i+j+k=n}} f_i(\sigma^{n-i}(c))\beta d_j(\sigma^k\tau^i(b))\alpha d_k(\tau^{n-j}(c))$$

…(1)

On the other hand

$$f_n((a+c)\alpha b\beta(a+c)) = f_n(a\alpha b\beta a + a\alpha b\beta c + c\alpha b\beta a + c\alpha b\beta c)$$
$$= f_n(a)\beta a\alpha b + \sum_{\substack{i<n\\ i+j+k=n}} f_i(\sigma^{n-i}(a))\beta d_j(\sigma^k\tau^i(b))\alpha d_k(\tau^{n-k}(a)) +$$
$$f_n(c)\beta c\alpha b + \sum_{\substack{i<n\\ i+j+k=n}} f_i(\sigma^{n-i}(a))\beta d_j(\sigma^k\tau^i(b))\alpha d_k(\tau^{n-k}(c)) + f_n(a\alpha b\beta c + c\alpha b\beta a)$$

…(2)

Compare (1) and (2), we get:

$$f_n(a\alpha b\beta c + c\alpha b\beta a) = f_n(c)\beta a\alpha b + \sum_{\substack{i<n\\ i+j+k=n}} f_i(\sigma^{n-i}(c))\beta d_j(\sigma^k\tau^i(b))\alpha d_k(\tau^{n-k}(a)) +$$
$$f_n(a)\beta c\alpha b + \sum_{\substack{i<n\\ i+j+k=n}} f_i(\sigma^{n-i}(a))\beta d_j(\sigma^k\tau^i(b))\alpha d_k(\tau^{n-k}(c))$$

(v) By (iv) and since M is a 2-torsion free commutative Γ-ring, we get:

$$f_n(a\alpha b\beta c + a\alpha b\beta c) = 2f_n(a\alpha b\beta c)$$
$$= f_n(c)\beta a\alpha b + \sum_{\substack{i<n\\ i+j+k=n}} f_i(\sigma^{n-i}(c))\beta d_j(\sigma^k\tau^i(b))\alpha d_k(\tau^{n-k}(a))$$

(vi) Replace α for β in (iv), we get:

$$f_n(a\alpha b\alpha c + c\alpha b\alpha a) = f_n(c)\alpha a\alpha b + \sum_{\substack{i<n\\ i+j+k=n}} f_i(\sigma^{n-i}(c))\alpha d_j(\sigma^k\tau^i(b))\alpha d_k(\tau^{n-k}(a)) +$$
$$f_n(a)\alpha c\alpha b + \sum_{\substack{i<n\\ i+j+k=n}} f_i(\sigma^{n-i}(a))\alpha d_j(\sigma^k\tau^i(b))\alpha d_k(\tau^{n-k}(c))$$

Definition (2.6):

Let $F = (f_i)_{i\in N}$ be a Jordan generalized (σ,τ)-higher reverse derivation of a Γ-ring M into itself, then for all a, $b \in$ M, $\alpha \in \Gamma$ and $n \in N$, we define

$$\delta_n = f_n(a\alpha b) - \sum_{i+j=n} f_i(\sigma^{n-i}(b))\alpha d_j(\tau^{n-j}(a))$$

Lemma (2.7):

Let $F = (f_i)_{i\in N}$ be a Jordan generalized (σ,τ)-higher reverse derivation of a Γ-ring M into itself, then for all $a, b, c \in$ M, $\alpha, \beta \in \Gamma$ and $n \in N$:

(i) $\delta_n(a,b)_\alpha = -\delta_n(b,a)_\alpha$

(ii) $\delta_n(a+b,c)_\alpha = \delta_n(a,c)_\alpha + \delta_n(b,c)_\alpha$

(iii) $\delta_n(a,b+c)_\alpha = \delta_n(a,b)_\alpha + \delta_n(a,c)_\alpha$

(iv) $\delta_n(a,b)_{\alpha+\beta} = \delta_n(a,b)_\alpha + \delta_n(a,b)_\beta$

Proof:

(i) By Lemma (2.5) (i), we get:

$$f_n(a\alpha b + b\alpha a) = \sum_{i+j=n} f_i(\sigma^{n-i}(b))\alpha d_j(\tau^{n-j}(a)) + \sum_{i+j=n} f_i(\sigma^{n-i}(a))\alpha d_j(\tau^{n-j}(b))$$

$$f_n(a\alpha b) - \sum_{i+j=n} f_i(\sigma^{n-i}(b))\alpha d_j(\tau^{n-j}(a)) = -(f_n(b\alpha a) - \sum_{i+j=n} f_i(\sigma^{n-i}(a))\alpha d_j(\tau^{n-j}(b)))$$

$\delta_n(a,b)_\alpha = -\delta_n(b,a)_\alpha$

(ii)
$$\delta_n(a+b,c)_\alpha = f_n((a+b)\alpha c) - \sum_{i+j=n} f_i(\sigma^{n-i}(c))\alpha d_j(\tau^{n-j}(a+b))$$

$$= f_n(a\alpha c + b\alpha c) - \sum_{i+j=n} f_i(\sigma^{n-i}(c))\alpha d_j(\tau^{n-j}(a)) - \sum_{i+j=n} f_i(\sigma^{n-i}(c))\alpha d_j(\tau^{n-j}(b))$$

$$= f_n(a\alpha c) - \sum_{i+j=n} f_i(\sigma^{n-i}(c))\alpha d_j(\tau^{n-j}(a)) + f_n(b\alpha c) - \sum_{i+j=n} f_i(\sigma^{n-i}(c))\alpha d_j(\tau^{n-j}(b))$$

$= \delta_n(a,c)_\alpha + \delta_n(b,c)_\alpha$

(iii)
$$\delta_n(a,b+c)_\alpha = f_n(a\alpha(b+c)) - \sum_{i+j=n} f_i(\sigma^{n-i}(b+c))\alpha d_j(\tau^{n-j}(a))$$

$$= f_n(a\alpha b + a\alpha c) - \sum_{i+j=n} f_i(\sigma^{n-i}(b))\alpha d_j(\tau^{n-j}(a)) - \sum_{i+j=n} f_i(\sigma^{n-i}(c))\alpha d_j(\tau^{n-j}(a))$$

$$= f_n(a\alpha b) - \sum_{i+j=n} f_i(\sigma^{n-i}(b))\alpha d_j(\tau^{n-j}(a)) + f_n(a\alpha c) - \sum_{i+j=n} f_i(\sigma^{n-i}(c))\alpha d_j(\tau^{n-j}(a))$$

$= \delta_n(a,b)_\alpha + \delta_n(a,c)_\alpha$

(iv)
$$\delta_n(a,b)_{\alpha+\beta} = f_n(a(\alpha+\beta)b) - \sum_{i+j=n} f_i(\sigma^{n-i}(b))(\alpha+\beta)d_j(\tau^{n-j}(a))$$

$$= f_n(a\alpha b) - \sum_{i+j=n} f_i(\sigma^{n-i}(b))\alpha d_j(\tau^{n-j}(a)) + f_n(a\beta b) - \sum_{i+j=n} f_i(\sigma^{n-i}(b)\beta d_j(\tau^{n-j}(a))$$

$= \delta_n(a,b)_\alpha + \delta_n(a,b)_\beta$

Remark (2.8):

Note that $F = (f_i)_{i\in N}$ is a generalized (σ,τ)-higher reverse derivation of a Γ-ring M into itself if and only if $\delta_n = 0$, for all $n \in N$.

3- The Main Result :

Theorem (3.1):

Let $F = (f_i)_{i\in N}$ be a Jordan generalized (σ,τ)-higher reverse derivation of a Γ-ring M into itself, then $\delta_n = 0$, for all $n \in N$.

Proof:

By Lemma (2.5) (i), we get

$$f_n(a\alpha b + b\alpha a) = \sum_{i+j=n} f_i(\sigma^{n-i}(b))\alpha d_j(\tau^{n-j}(a)) + \sum_{i+j=n} f_i(\sigma^{n-i}(a))\alpha d_j(\tau^{n-j}(b)) \quad \ldots(1)$$

On the other hand

$$f_n(a\alpha b + b\alpha a) = f_n(a\alpha b) + f_n(b\alpha a) = f_n(a\alpha b) + \sum_{i+j=n} f_i(\sigma^{n-i}(a))\alpha d_j(\tau^{n-j}(b)) \quad \ldots(2)$$

Compare (1) and (2), we get:

$$f_n(a\alpha b) = \sum_{i+j=n} f_i(\sigma^{n-i}(b))\alpha d_j(\tau^{n-j}(a))$$

$$f_n(a\alpha b) - \sum_{i+j=n} f_i(\sigma^{n-i}(b))\alpha d_j(\tau^{n-j}(a)) = 0$$

By Definition (2.4), we get:

$\phi_n = 0$, for all $n \in N$.

Corollary (3.2):

Every Jordan generalized (σ,τ)-higher reverse derivation of a Γ-ring M is a generalized (σ,τ)-higher reverse derivation of M

Proof:

By Theorem (3.1), we get $\phi_n = 0$, for all $n\in N$ and by Remark (2.8) we get the require result.

Proposition (3.3):

Every Jordan generalized (σ,τ)-higher reverse derivation of a 2-torsion free Γ-ring M into itself, such that $a\alpha b\beta a = a\beta b\alpha a$, for all $a, b \in M$ and $\alpha, \beta \in \Gamma$ is a Jordan generalized triple (σ,τ)-higher reverse derivation .

Proof:

Let $F = (f_i)_{i\in N}$ be a Jordan generalized (σ,τ)-higher reverse derivation of a Γ-ring M into itself.

Replace $a\beta b + b\beta a$ for b in Lemma (2.5) (i), we get:

$$f_n(a\alpha(a\beta b + b\beta a) + (a\beta b + b\beta a)\alpha a)$$
$$= f_n(a\alpha(a\beta b) + a\alpha(b\beta a) + (a\beta b)\alpha a + (b\beta a)\alpha a)$$
$$= f_n((a\alpha a)\beta b + (a\alpha b)\beta a) + (a\beta b)\alpha a + (b\beta a)\alpha a)$$
$$= \sum_{i+j=n} f_i(\sigma^{n-i}(b))\beta d_j(\tau^{n-j}(a\alpha a)) + \sum_{i+j=n} f_i(\sigma^{n-i}(a))\beta d_j(\tau^{n-j}(a\alpha b)) +$$
$$\sum_{i+j=n} f_i(\sigma^{n-i}(a))\alpha d_j(\tau^{n-j}(a\beta a)) + \sum_{i+j=n} f_i(\sigma^{n-i}(a))\alpha d_j(\tau^{n-j}(b\beta a))$$

$$= \sum_{i+j=n} f_i(\sigma^{n-i}(b))\beta(\sum_{r+s=j} d_r(\sigma^{j-r}\tau^{n-j}(a))\alpha d_s(\tau^{j-s}\tau^{n-j}(a))) +$$

$$\sum_{i+j=n} f_i(\sigma^{n-i}(a))\beta(\sum_{e+f=j} d_e(\sigma^{j-e}\tau^{n-j}(b))\alpha d_f(\tau^{j-f}\tau^{n-j}(a))) +$$

$$\sum_{i+j=n} f_i(\sigma^{n-i}(a))\alpha(\sum_{p+q=j} d_p(\sigma^{j-p}\tau^{n-j}(b))\beta d_q(\tau^{j-q}\tau^{n-j}(a))) +$$

$$\sum_{i+j=n} f_i(\sigma^{n-i}(a))\alpha(\sum_{x+y=j} d_x(\sigma^{j-x}\tau^{n-j}(a))\beta d_y(\tau^{j-y}\tau^{n-j}(b)))$$

$$= \sum_{i+r+s=n} f_i(\sigma^{n-i}(b))\beta d_r(\sigma^{s}\tau^{i}(a))\alpha d_s(\tau^{n-s}(a)) +$$

$$\sum_{i+e+f=n} f_i(\sigma^{n-i}(a))\beta d_e(\sigma^{f}\tau^{i}(b))\alpha d_f(\tau^{n-f}(a)) +$$

$$\sum_{i+p+q=n} f_i(\sigma^{n-i}(a))\alpha d_p(\sigma^{q}\tau^{i}(b))\beta d_q(\tau^{n-q}(a)) +$$

$$\sum_{i+x+y=n} f_i(\sigma^{n-i}(a))\alpha d_x(\sigma^{y}\tau^{i}(a))\beta d_y(\tau^{n-y}(b))$$

$$= f_n(b)\beta a\alpha a + \sum_{\substack{i<n \\ i+j+k=n}} f_i(\sigma^{n-i}(b))\beta d_j(\sigma^{k}\tau^{i}(a))\alpha d_k(\tau^{n-k}(a)) +$$

$$f_n(a)\beta a\alpha b + \sum_{\substack{i<n \\ i+j+k=n}} f_i(\sigma^{n-i}(a))\beta d_j(\sigma^{k}\tau^{i}(b))\alpha d_k(\tau^{n-k}(a)) +$$

$$f_n(a)\alpha a\beta b + \sum_{\substack{i<n \\ i+j+k=n}} f_i(\sigma^{n-i}(a))\alpha d_j(\sigma^{k}\tau^{i}(b))\beta d_k(\tau^{n-k}(a)) +$$

$$f_n(a)\alpha b\beta a + \sum_{\substack{i<n \\ i+j+k=n}} f_i(\sigma^{n-i}(a))\alpha d_j(\sigma^{k}\tau^{i}(a))\beta d_k(\tau^{n-k}(b))$$

$$= f_n(b)\beta a\alpha a + \sum_{\substack{i<n \\ i+j+k=n}} f_i(\sigma^{n-i}(b))\beta d_j(\sigma^{k}\tau^{i}(a))\alpha d_k(\tau^{n-k}(a)) +$$

$$2(f_n(a)\beta a\alpha b + \sum_{\substack{i<n \\ i+j+k=n}} f_i(\sigma^{n-i}(a))\beta d_j(\sigma^{k}\tau^{i}(b))\alpha d_k(\tau^{n-k}(a))) +$$

$$f_n(a)\alpha b\beta a + \sum_{\substack{i<n \\ i+j+k=n}} f_i(\sigma^{n-i}(a))\alpha d_j(\sigma^{k}\tau^{i}(a))\beta d_k(\tau^{n-k}(b))$$

Since M is a 2-torsion free Γ-ring , then

$$= f_n(b)\beta a\alpha a + \sum_{\substack{i<n \\ i+j+k=n}} f_i(\sigma^{n-i}(b))\beta d_j(\sigma^{k}\tau^{i}(a))\alpha d_k(\tau^{n-k}(a)) +$$

$$f_n(a)\beta a\alpha b + \sum_{\substack{i<n \\ i+j+k=n}} f_i(\sigma^{n-i}(a))\beta d_j(\sigma^{k}\tau^{i}(b))\alpha d_k(\tau^{n-k}(a)) +$$

$$f_n(a)\alpha b\beta a + \sum_{\substack{i<n \\ i+j+k=n}} f_i(\sigma^{n-i}(a))\alpha d_j(\sigma^{k}\tau^{i}(a))\beta d_k(\tau^{n-k}(b))$$

…(1)

On the other hand :

$$f_n(a\alpha(a\beta b+b\beta a)+(a\beta b+b\beta a)\alpha a)$$
$$=f_n(a\alpha a\beta b+a\alpha b\beta a+a\beta b\alpha a+b\beta a\alpha a)$$

Since $a\alpha b\beta a = a\beta b\alpha a$, for all $a, b \in$ M and $\alpha, \beta \in \Gamma$

$$=f_n(a\alpha a\beta b+a\alpha b\beta a+a\beta b\alpha a+b\beta a\alpha a)$$

$$=f_n(b)\beta a\alpha a+\sum_{i+j+k=n}^{i<n} f_i(\sigma^{n-i}(b))\beta d_j(\sigma^k\tau^i(a))\alpha d_k(\tau^{n-k}(a))+$$

$$f_n(a)\alpha b\beta a+\sum_{i+j+k=n}^{i<n} f_i(\sigma^{n-i}(a))\alpha d_j(\sigma^k\tau^i(a))\beta d_k(\tau^{n-k}(b))+2f_n(a\alpha b\beta a)\ldots(2)$$

Compare (1), (2) and since M is a 2-torsion free Γ-ring , we have :

$$f_n(a\alpha b\beta a)=f_n(a)\beta a\alpha b+\sum_{i+j+k=n}^{i<n} f_i(\sigma^{n-i}(a))\beta\, d_j(\sigma^k\tau^i(b))\alpha d_k(\tau^{n-k}(a)).$$

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