COMMON RANDOM FIXED POINT RESULT IN SYMMETRIC SPACE

[1]Manoj kumar Shukla , [2]Arjun Kumar Mehra

[1]Department of Mathematics, Institute for Excellence in Higher Education,Bhopal
manojshukla_10@yahoo.com

[2]Research Scholor, Rani Durgavati Vishwavidyalaya, Jabalpur,(MP),India
marjunmehra1111@gmail.com

Abstract
Under random iteration scheme we study necessary conditions for convergence to a common fixed point of two pair of JSR random operators satisfying generalized contractive condition in the framework of symmetric spaces.

Mathematical Subject Classification(2000): 54H25,47H10

Keywords: Symmetric space, JSR random operator, common fixed point.

1.**Introduction and Preliminaries**

Random fixed point theorems are stochastic generalization of classical fixed point theorems. Random fixed point theory is used to obtained the solution of nonlinear random system. Beg [1,2], Beg and Shahzad [3,5] studied the structure of common random fixed points and random coincidence points of a pair of compatible random operator. Recently, Beg and shahzad [4,5] had used different iteration process to obtain common random fixed points.Recently, Some results are obtained by Mehta e.l.[6].In this paper we study necessary conditions for convergence to a common fixed point of two pair of JSR random operator satisfying generalized contractive conditions in Polish and symmetric spaces.

Throughout this paper (Ω,Σ) denote a measurable space. A symmetric function on a set X is non-negative real valued function d on XxX such that for all x, y $\in$X we have

(i) d(x,y) = 0 if and only if x=y, and

(ii) d(x,y) = d(y,x)

Let d be a symmetric function on X. For $\varepsilon > 0$ and x $\in$X , B(x, ε) denote the spherical open ball centered at x and radius ε. A topology t(d) on X is given by U $\in$ t(d) if and only if for each x $\in$U, B(x, ε) $\subset$ U for some $\varepsilon > 0$.

Here $x_n \to x$ as $n\to\infty$ if and only if $x_n \to x$ in topology t(d) Let F be a subset of X. A mapping $\xi:\Omega\to X$ is *measurable* if $\xi^{-1}(U)\in\Sigma$ for each open subset U of X. The mapping $T:\Omega\times F\to F$ is a random map if and only if for each fixed $x\in F$, the mapping $T(.,x):\Omega\to F$ is measurable. The mapping T is continuous if for each $\omega\in\Omega$, the mapping $t(\omega,.):F\to X$ is continuous. A measurable mapping $\xi:\Omega\to X$ is a random fixed point of random map

$T:\Omega\times F\rightarrow F$ if and only if $T(\omega,\xi(\omega))=\xi(\omega)$ for each $\omega\in\Omega$.We denote the set of random fixed point of a random map T by RF(T) and the set of all measurable mapping from Ω into a symmetric space X by $M(\Omega,X)$. We denote the nth iterate $T(\omega,T(\omega,\ldots\ldots\ldots,T(\omega,x)$ of T by $T_n(\omega,x)$.

Definition 1.1 Let X be a separable complete space i.e.X is Polish space. Random operators $S,T:X\rightarrow X$ is said to be T**-JSR mapping** if

$\alpha d(T(\omega,\xi n(\omega)),S(\omega,\xi n(\omega))) \leq \alpha\, d(T(\omega,T(\xi n(\omega)),T(\omega,\xi n(\omega))) = 0 \ \forall\omega\in\Omega$

where α= lim sup or lim inf and $\{(\xi n(\omega)\}$ is a sequence in X such that lim $T(\omega,\xi n(\omega))$ = lim $S(\omega,\xi n(\omega)) = \xi(\omega)$.

Definition 1.2 Let X be a polish space. Random operators $S,T:\Omega\times X\rightarrow X$ are said to be weakly T-JSR if $T(\omega,\xi(\omega))=S(\omega,\xi(\omega))$ for some ξ in $M(\Omega,X)$, then

$$S(\omega,T(\omega,\xi(\omega)))\leq T(\omega,T(\omega,\xi(\omega))).$$

Definition 1.3 Let $\{x_n\},\{y_n\}$ be two sequences in symmetric space (X,d) and x,y in X. The space satisfy the following conditions

(a) $\lim_{n\to\infty} d(x_n,x) = \lim_{n\to\infty} d(x_n,y) = 0$ then $x = y$

(b) $\lim_{n\to\infty} d(x_n,x) = \lim_{n\to\infty} d(x_n,y_n) = 0$ then $\lim_{n\to\infty} d(y_n,x) = 0$

Definition 1.4 Let $\{x_n\}$, $\{y_n\}$ be two sequences in metric space (X,d) and x in X. The space X is said to satisfy condition (H_E) if

$\lim_{n\to\infty} d(x_n,x) = \lim_{n\to\infty} d(x,y_n) = 0$ then $\lim_{n\to\infty} d(x_n,y_n) = 0$

Definition 1.5 Let d be a symmetric function on X. The two random mappings

$S,T:\Omega xX\rightarrow X$ are said to satisfy property (I) if there exists a sequence $\{\xi_n\}$ in $M(\Omega,X)$ such that for some ξ in $M(\Omega,X)$,

$\lim_{n\to\infty} d(T(\omega,\xi n(\omega),\xi(\omega))) = \lim_{n\to\infty} d(S(\omega,\xi n(\omega),\xi(\omega))) = 0 \ \forall\omega\in\Omega.$

1. MAIN RESULTS

THEOREM 2.1: Let (X,d) be a Symmetric space that satisfy (I) and (H_E). Let S and T are S- JSR random operators from ΩxX to X which satisfy the property (I).Moreover, $\forall$ $x,y\in X$ and $\omega\in\Omega$ we have

$$d(T(\omega,x),T(\omega,y)) \le [\max\{d(S(\omega,x),S(\omega,y)),d(S(\omega,x),T(\omega,x))+d(S(\omega,y),T(\omega,y)),$$
$$d(S(\omega,x),T(\omega,y))+d(S(\omega,y),T(\omega,x))\}] \qquad \text{.........(2.1)}$$

If $T(\omega,X) \subset S(\omega,X)$ and one of $T(\omega,X)$ or $S(\omega,X)$is a complete subspace of X for every $\omega\in\Omega$,then T and S have unique and common random fixed point.

Proof : Since random operators S and T satisfy the property (I) therefore there exists a sequence $\{\xi_n\}$ in $M(\Omega,X)$ such that for some $\xi \in M(\Omega,X)$ and for every $\omega\in\Omega$

$$\lim_{n\to\infty} d(T(\omega,\xi_n(\omega),\xi(\omega))) = \lim_{n\to\infty} d(S(\omega,\xi_n(\omega),\xi(\omega))) = 0 \qquad \text{.........(2.2)}$$

Then by property (H_E), we have $\lim_{n\to\infty} d(T(\omega,\xi_n(\omega)),S(\omega,\xi_n(\omega))) =0$ for every $\omega\in\Omega$.

Now, suppose $S(\omega,X)$ is a complete subspace of X for every $\omega\in\Omega$. Let $\xi_1:\Omega\to X$ be the limit of the sequence of measurable mapping $\{S(\omega,\xi_n(\omega))\}$ and $S(\omega,\xi_n(\omega))$ in $S(\omega,X)$ for every $\omega\in\Omega$ and $n\in N$. Since X is separable, therefore $\xi_1 \in M(\Omega,X)$.

Moreover $\xi_1(\omega)\in S(\omega,X)$ for every $\omega\in\Omega$. Then this allows obtaining the measurable mapping $\bar{\xi}:\Omega\to X$ such that $\xi(\omega) = S(\omega, \bar{\xi}(\omega))$. Now for every $\omega\in\Omega$ we show that that $T(\omega, \bar{\xi}(\omega)) = S(\omega, \bar{\xi}(\omega))$. Consider

$$d(T(\omega, \bar{\xi}(\omega)),T(\omega,\xi_n(\omega))) \le \max[d(S(\omega, \bar{\xi}(\omega)),S(\omega,\xi_n(\omega))),$$
$$\{d(S(\omega, \bar{\xi}(\omega)),T(\omega, \bar{\xi}(\omega)))+ d(S(\omega,\xi_n(\omega)),T(\omega,\xi_n(\omega)))\}/2,$$
$$\{d(S(\omega, \bar{\xi}(\omega)),T(\omega,\xi_n(\omega)))+d(S(\omega,\xi_n(\omega)),T(\omega, \bar{\xi}(\omega)))\}/2]$$

$$d(T(\omega, \bar{\xi}(\omega)),T(\omega,\xi_n(\omega))) \le \max[d(\xi(\omega)),S(\omega,\xi_n(\omega))),$$
$$\{d(\xi(\omega)),T(\omega, \bar{\xi}(\omega)))+d(S(\omega,\xi_n(\omega)),T(\omega,\xi_n(\omega)))\}/2,$$
$$\{d(\xi(\omega)),T(\omega,\xi_n(\omega)))+d(S(\omega,\xi_n(\omega)),T(\omega, \bar{\xi}(\omega)))\}/2]$$

Now by (2.2) and on taking limit we obtain

$$d(T(\omega, \bar{\xi}(\omega)),\xi(\omega))\le d(\xi(\omega),T(\omega, \bar{\xi}(\omega)))$$

which is contradiction therefore $\xi(\omega) = T(\omega, \bar{\xi}(\omega)) \Rightarrow T(\omega, \bar{\xi}(\omega)) = S(\omega, \bar{\xi}(\omega))$.

The weak T-JSR of random mapping T and S implies that for some $\bar{\xi}$ in $M(\Omega,X)$.

Now we show that $T(\omega,T(\omega, \bar{\xi}(\omega))) = T(\omega, \bar{\xi}(\omega))$ for every $\omega\in \Omega$. Consider

$$d(T(\omega, \bar{\xi}(\omega)),T(\omega,\xi(\omega))) \le \max[d(S(\omega, \bar{\xi}(\omega)),S(\omega,\xi(\omega))),$$
$$\{d(S(\omega, \bar{\xi}(\omega)),T(\omega, \bar{\xi}(\omega)))+d(S(\omega,\xi(\omega)),T(\omega,\xi(\omega)))\}/2,$$
$$\{d(S(\omega, \bar{\xi}(\omega)),T(\omega,\xi(\omega))),d(S(\omega,\xi(\omega)),T(\omega, \bar{\xi}(\omega)))\}/2]$$
$$\le \max[d(T(\omega, \bar{\xi}(\omega)),S(\omega,T(\omega, \bar{\xi}(\omega))),$$
$$\{d(T(\omega, \bar{\xi}(\omega)),T(\omega, \bar{\xi}(\omega)))+d(S(\omega,T(\omega, \bar{\xi}(\omega))),T(\omega,T(\omega, \bar{\xi}(\omega)))\}/2,$$
$$\{d(T(\omega, \bar{\xi}(\omega)),T(\omega,T(\omega, \bar{\xi}(\omega)))+d(S(\omega,T(\omega, \bar{\xi}(\omega))),T(\omega, \bar{\xi}(\omega)))\}/2]$$
$$\le \max[d(T(\omega, \bar{\xi}(\omega)),T(\omega,T(\omega, \bar{\xi}(\omega))),$$
$$\{d(T(\omega, \bar{\xi}(\omega)),T(\omega, \bar{\xi}(\omega)))+d(T(\omega,T(\omega, \bar{\xi}(\omega))),T(\omega,T(\omega, \bar{\xi}(\omega)))\}/2,$$
$$\{d(T(\omega, \bar{\xi}(\omega)),T(\omega,T(\omega, \bar{\xi}(\omega)))+d(T(\omega,T(\omega, \bar{\xi}(\omega))),T(\omega, \bar{\xi}(\omega)))\}/2]$$
$$\le d(T(\omega, \bar{\xi}(\omega)),T(\omega,T(\omega, \bar{\xi}(\omega)))] < d(T(\omega, \bar{\xi}(\omega)),T(\omega,\xi(\omega)))$$

Which is contradictions,therefore $T(\omega,T(\omega, \bar{\xi}(\omega))) = T(\omega, \bar{\xi}(\omega))$ i.e. $T(\omega, \bar{\xi}(\omega))$ is a random fixed point of T.Again,

$$d(S(\omega,T(\omega, \bar{\xi}(\omega))),T(\omega, \bar{\xi}(\omega))) \le d(T(\omega,T(\omega, \bar{\xi}(\omega))),T(\omega, \bar{\xi}(\omega)))$$

$$= d(T(\omega, \bar{\xi}(\omega)),T(\omega, \bar{\xi}(\omega))) = 0.$$

It implies that $T(\omega, \bar{\xi}(\omega))$ is also fixed point of S. Thus $T(\omega, \bar{\xi}(\omega))$ is common fixed point of S and T. The proof is similar when $T(\omega,X)$ is supposed to be complete subspace of X for every $\omega\in\Omega$ as $T(\omega,X) \subset S(\omega,X)$.

Uniqueness:- Let ν and $\bar{\nu}$ from Ω to X are two common fixed point of S and T .

Let $\nu(\omega) \neq \bar{\nu}(\omega)$ then by contraction we have

$$d(\nu(\omega), \bar{\nu}(\omega)) = d(T(\omega,\nu(\omega)),T(\omega, \bar{\xi}(\omega)))$$
$$\leq \max[d(S(\omega,\nu(\omega)),S(\omega, \bar{\nu}(\omega))),$$
$$\{d(S(\omega,\nu(\omega)),T(\omega,\nu(\omega)))+d(S(\omega, \bar{\nu}(\omega)),T(\omega, \bar{\nu}(\omega)))\}/2,$$
$$\{d(S(\omega,\nu(\omega)),T(\omega, \bar{\nu}(\omega))),d(S(\omega, \bar{\nu}(\omega)),T(\omega,\nu(\omega)))\}/2]$$
$$\leq d(\nu(\omega), \bar{\nu}(\omega))$$

which contradiction, therefore $\nu(\omega) = \bar{\nu}(\omega)$ for every $\omega\in\Omega$.

Example1:- Let $\Omega = [0,1]$ and Σ be the sigma algebra of Lebesgue's measurable subset of [0,1]. Let X = R with $d(x,y) = a^{|x-y|} - 1$, where $a > 1$ and clearly d is symmetric on R. Define random operators S and T from ΩxX to X as

$S(\omega,x) = (1-\omega^2+2x)/3$ and $T(\omega,x) = (1-\omega^2+3x)/4$.

Also sequence of mapping $\xi_n : \Omega \to X$ is defined by $\xi_n(\omega) = 1+(1/n) -\omega^2$ for every $\omega\in\Omega$ and $n \in N$. Define measurable mapping $\xi : \Omega \to X$ as $\xi(\omega) = 1-\omega^2$ for every $\omega\in\Omega$.

$$\lim_{n\to\infty} d(T(\omega,\xi_n(\omega)),\xi(\omega)) = \lim_{n\to\infty} a^{|T(\omega,\xi n(\omega)) - \xi(\omega)|} - 1 = \lim_{n\to\infty} a^{3/4n} - 1 = 0$$

and

$$\lim_{n\to\infty} d(S(\omega,\xi n(\omega)),\xi(\omega))=\lim_{n\to\infty} a^{|T(\omega,\xi n(\omega))-\xi(\omega)|}-1= \lim_{n\to\infty} a^{2/3n} -1=0$$

Clearly S and T satisfy property I.

Example2:- Let S and T from ΩxX to X as $S(\omega,x) = (1-\omega^2-2x)$ and $T(\omega,x) = (1-\omega^2-3x)/2$ and let $\bar{\xi}(\omega) = 1-\omega^2$ for every $\omega\in\Omega$. Then

$$T(\omega, \bar{\xi}(\omega)) = -1+\omega^2 = S(\omega, \bar{\xi}(\omega)) \Rightarrow S(\omega,T(\omega, \bar{\xi}(\omega))) < T(\omega,T(\omega, \bar{\xi}(\omega))).$$

Thus S and T are weakly T-JSR operators.

Also in **Example 1** S and T are weakly T-JSR operators and $T(\omega, \bar{\xi}(\omega)) = 1-\omega^2$ is a unique random fixed point of S and T.

References

[1] Beg,I.,Random fixed points of random operators satisfying semi **c**ont-ractivity conditions, Mathematica Japonica 46(1997), no. 151-155.

[2] Beg,I., Approximation of random fixed points in normed spaces, Nonlinear Analysis 51(2002), no.8 1363-1372

[3] Beg,I., and N. Shahzad, Random fixed points of random multivalued operators on polish spaces, Nonlinear Analysis 20(1993), no.7, 835-847 .

[4] Beg,I.,and N. Shahzad, Random fixed points for nonexpansive and contractive-type random operators on Banach spaces, Journal of

Applied Mathematics and stochastic Analysis 7(1994),no.4, 569-580.

[5] Beg,I.,and N. Shahzad, Common random fixed points of random multi-valued operators on metric space, Bollettino della Unione Mathematica Italiana, Serie VII.A 9 (1995) no.3 493-503

[6] Smriti Mehta and Vanita Ben Dhagat" Some Fixed Point Theorem in Polish Spaces" Applied Mathematical Sciences, Vol.4, 2010, no.28, 1395-1403.