F_1- Delta -Lifting modules

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Abstract:

Let R be an associative ring with identity and let M be an unitary left R-module. An R-module M is called F_1- lifting, if every fully invariant sub module A of M contains a direct summand B of M such that $B \leq_{ce} A$. In this paper we introduce F_1-δ-lifting as a generalization of F_1-lifting module. We prove similar results of F_1-lifting.

Keywords: F_1-δ-lifting modules ,strongly F_1-δ-lifting modules.

Introduction

Let R be an associative ring with identity, and let M be a unitary left R-module. A sub module N of an R-module M is called small in M denoted by (N<< M), if whenever M = N + K, for K ≤ M implies K = M. [6]. We say that a sub module K is a coessential sub module of N in M (denoted by $K \leq_{ce} N$), if N / K << M / K,[6]. A module M is called lifting or satisfies (D_1) , if every sub module N of M contains a direct summand K of M such that $K \leq_{ce} N$.[4] . A non- zero module M is called hollow, if every proper sub module of M is small in M, [3]. Recall that Z(M) = { m∊ M; ann(m) $\leq_e$ R}, is the singular sub module of M, if Z(M) = M then M is singular , and if Z(M) = 0, then M is non- singular. A sub module N of M is called δ-small in M, if whenever M = N + K, K ≤ M with M / K singular implies M = K [6]. A non- zero module M is called δ- hollow, if every proper sub module of M is δ- small. It is clear that every hollow module is δ-hollow , but the converse in general is not hold. Consider R is a semi simple ring and M is a non- zero R-module. Then M is non-singular and semi simple.For any non- zero sub module N ≤ M, N is a direct summand of M , and hence is not small in M, but every sub module of M (even M itself) Is δ-small. A sub module N of M is called fully invariant, if f(N) ≤ N for every f ∊ $End_R(M)$. In [5] it was introduced F_1-lifting module. An R-module M is called F_1-lifting, if every fully invariant sub module N of M contains a direct summand K of M such that $K \leq_{ce} N$ .According to this definition we introduce another generalization for lifting modules. We introduce F_1- δ-lifting module. A module M is called F_1-δ-lifting, if every invariant sub module N of M contains a direct summand K of M such that $N / K \ll_{\delta} M / K$ and we prove similar results of F_1-lifting module.[5].

§1 F_1-δ- lifting module

In this section we introduce F_1-δ- lifting module and give some properties of this type of module, but first we recall some known results which will be needed in our work.

Lemma(1.1) :-[1.lemma 1.1] Let M be an R-module, then

1. Any sum of fully invariant sub module of M is again a fully invariant sub module of M.
2. Any intersection of fully invariant sub module is again a fully invariant sub module of M.
3. If $X \leq Y \leq M$, such that Y is a fully invariant sub module of M and X is a fully invariant sub module of Y, then X is a fully invariant sub module of M .
4. If $M = \bigoplus_{i\in I} X_i$ and S is a fully invariant sub module of M, then $S = \bigoplus_{i\in I} (X_i \cap S)$ and $(X_i \cap S)$ is a fully invariant sub module of X_i , $\forall\, i \in I$.

Definition (1.2) : Let N , K be sub module of an R - module M such that $K \leq N$, we say that K is a generalized coessential sub module of N (denoted by $K \leq_{\delta ce} N$), if $N / K \ll_{\delta} M / K$.

It is clear that if $K \leq_{ce} N$, then $K \leq_{\delta ce} N$, but the convers in general is not true , if R is semi simple ring and M is non-zero R- module, then M is semi simple and non-singular hence $\forall\, N \leq M$, $0 \leq N$ is not small in $M / 0$, but $N / 0 \ll_{\delta} M / 0$.

Definition(1.3): Let M be an R-module, we say that M is F_1-δ-lifting, if every fully invariant sub module N of M contains a direct summand K such that $K \leq_{\delta ce} N$.

It is clear that every δ- hollow module is F_1-δ-lifting.

By [5]. M is F_1-δ-lifting, if and only if every fully invariant A of M can be written as $A = B \oplus S$ where B is a direct summand of M and $S \ll M$.

In the following we prove a similar result for F_1-δ-lifting module.

Proposition(1.4):- The following are equivalent for an R-module M.

1. M is F_1-δ-lifting
2. Every fully invariant sub module A of M can be written as $A = B \oplus S$, where B is a direct summand of M and $S \ll_\delta M$.
3. Every fully invariant sub module A of M can be written as $A = B \oplus S$, where B is direct summand of M and $S \ll_\delta M$.

Proof: $1 \to 2$) Let A be a fully invariant sub module of M, then by (1) , $\exists\, B \leq M$ such that $A / B \ll_\delta M / B$, hence $\exists\, K \leq M$ such that $M= B \oplus K$. Then $A = A\cap B\oplus A\cap K = B\oplus K \cap A$, take $S = K \cap A$.

Now let $M = K \cap A + K'$ with M / K' singular , Hence $M / B = (K \cap A + B) / B + K' / B= A / B + K' / B$.

$M / B / K' / B \cong M/ K'$ singular, and $A / B \ll_\delta M / B$ then $M = K'$.

$2\to 3$) clear

$3\to 1$) Let A be a fully invariant sub module of M then by (3) $A = B + S$, where $B \leq_\oplus M$ and $S \ll_\delta M$. Let $M = B\oplus C$ for $C \leq M$ and let $M / B = A / B + K / B$, where $(M / B) / (K / B) \cong M / K$ singular then $M= A + K = A + B +S = K + S$. Since M / K singular and $S\ll_\delta M$, then $M = K$. hence $A / B \ll_\delta M / B$.

Theorem(1.5): Let $M = M_1 \oplus M_2$ be a direct summand of F_1- δ- lifting modules, then M is F_1-δ- lifting.

Proof: Let A be a fully invariant sub module of M, then $A = A \cap M_1\oplus A \cap M_2$, and $A \cap M_i$ is fully invariant $\forall\, i = 1,2$, since M_i is F_1-δ- lifting

,i=1,2, then by pro.(1.4) , $A \cap M_i = B_i \oplus S_i$, where B_i is a direct summand of M_i, $S_i \ll_\delta M_i$, $\forall\, i = 1,2$…

Let $B = B_1 \oplus B_2$, and $S = S_1\oplus S_2$, then $A = B\oplus S$ where B is a direct summand of M and $S \ll_\delta M$.

Corollary(1.6): Let $M = M_1\oplus M_2\oplus\ldots\oplus M_n$ be a direct sum of F_1 -δ -lifting module, then M is F_1-δ- lifting.

Corollary (1.7): If M is a finite direct sum of δ- hollow modules, then M is F_1-δ -lifting.

Remark(1.8): It is clear that every lifting module isF_1-δ -lifting , but the convers in general is not true, for example, consider Z - module M = Z / pZ ⊕ Z / p^3Z each Z/ pZ , Z / p^3Z is hollow, hence F_1- δ - lifting therefore by theorem(1.5), M is F_1-δ-lifting, but not lifting,[2].

Recall that a pair(f,P) is called a projective δ - cover for an R- module M, if P is projective and f: P → M is an epimorphisim with Kerf $<<_\delta$ P, [7].

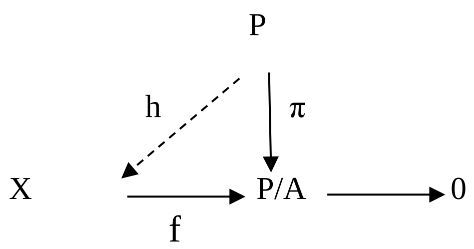
Proposition (1.9): Let P be a projective R-module ,if P is F_1- δ -lifting, then P / A has a projective δ- cover. For every fully sub module A of B.

The convers holds, if for every fully invariant sub module A of P, P / A has a projective δ- cover . f: X → P /A , such that X is uniform.

Proof: Let P be a projective F_1-δ- lifting module and let A be a fully invariant sub module of P, then ∃ B ≤ A, B $\leq_\oplus$ P such that A = B ⊕ S , S$<<_\delta$ P, hence P = B ⊕C, for C ≤ P , then(B + S) / B $<<_\delta$ P / B, thus

π: P / B→ P / (B + S) = P / A , P / B is projective , and Ker π = { w ∈ P/B ; π(w) = A }= { x ∈ B ; x + A = A } = A / B then Ker π = A / B $<<_\delta$ P/ B.

←)) Let A be a fully invariant sub module of P, let f : X → P / A be a projective δ -cover for P / A such that X is uniform. Consider the following diagram.



Since P is projective, then ∃ h : P → X , such that f ∘ h = π .Let x ∈ X, then f(x) ∈ P / A. Since π is an epimorphisim, then ∃ y ∈ P, such that π(y) = f(x),thus f ∘h(y) = f(x), hence x – h(y) ∈ Kerf, therefore X = Kerf + h(p). Since Kerf $<<_\delta$ X and X is uniform, then by [7], Ker f<< X. Hence h(P) = X, then h: P→ X is an epimorphisim thus h splits , therefore P = Ker h ⊕ K for K ≤ P, and A = Ker h ⊕ K∩ A. where Ker h $\leq_\oplus$ P and K ∩ A $<<_\delta$ P.

§2 Strongly F_1- δ – lifting modules.

In this section we introduce a strongly F_1-δ- lifting .According to the definition that appeared in [5]. And we prove some results on this type of modules similar to results of strongly F_1-δ- lifting module [5].

Definition(2.1): Let M be an R-module. We say that M is strongly F_1-δ-lifting , if every fully invariant sub module A of M contains a fully direct summand B of M such that $B \leq_{Gce} A$.

Proposition(2.2): For any R- module M , then following are equivalent:

1. M is a strongly F_1-δ-lifting module .
2. Every fully invariant sub module A of M can be written as $A = B \oplus S$, where B is fully invariant direct summand of M and $S \ll_{\delta} M$.
3. Every fully invariant sub module A of M can be written as $A = B + S$, where B is a fully invariant direct of M and $S \ll_{\delta} M$.

Proposition(2.3): Let M be an F_1- δ-lifting module , if 0 is the only δ- small sub module of M, then every fully invariant sub module of M is strongly F_1-δ – lifting module.

Proof: Let N be a fully invariant sub module of M and let A a fully invariant sub module of N, then A is fully invariant in M [1,Lemma 1.1].Since M is F_1-δ-lifting , then $A = B \oplus S$, where $B \leq_{\oplus} M$ and $S \ll_{\delta} M$. hence $S = 0$, thus $A \leq_{\oplus} M$ therefore N is strongly F_1- δ- lifting module.

Theorem(2.4): A direct summand of a strongly F_1-δ- lifting is again strongly F_1-δ – strongly F_1-δ- liftig.

Proof: Let $M = W \oplus V$, be a strongly F_1- δ- lifting , suppose that A is a fully invariant sub module of W

then ∃ A_2 a fully invariant sub module of V. Such that $A_1 \oplus A_2$ is fully invariant in M see[6]. since M is strongly F_1- δ- lifting then $A_1 \oplus A_2 = B \oplus S$, where $B \leq_{\oplus} M$, B isa fully invariant sub module of M and $S \ll_{\delta} M$, then $B = (B \cap W) \oplus (B \cap V)$ and $B \cap W$ is fully invariant in W also $B \cap W$ is a direct summand of M .

Let $\pi_1: M \to W$ and $\pi_2: M \to V$ then $A_1 = \pi_1(B) + \pi_1(S) = W \cap B + \pi_1(S)$ since $S \ll_{\delta} M$, $\pi_1(S) \ll_{\delta} W$ by [1], therefore W is F_1-δ - lifting.

Proposition(2.5): Let $M = \bigoplus_{i=1} M_i$, and let M_i be a fully invariant sub module of M ,∀ i= 1,…n, then M is strongly F_1- δ -lifting if and only if M_i is strongly F_1-δ - lifting,∀ i= 1,…n.

Proof: →) By prop(3.4)

←) Suppose that $M = \bigoplus_{i=1} M_i$ and M_i is fully invariant ∀ i= 1,…n.

Let N be a fully invariant sub module of M then $N= \bigoplus_{i=1}(N \cap M_i)$ and $N \cap M_i$ is fully invariant,∀ i= 1,…n., since M_i is strongly F_1-δ-lifting , then $N \cap M_i = B_i \oplus S_i$, where B_i is fully invariant direct summand of M_i and $S_i \ll_\delta M_i$,∀ i= 1,…n. Let $B = \bigoplus_{i=1} B_i$ and $S=\bigoplus_{i=1} S_i$ then B = B + S , where $B \leq_\oplus M$ and $S \ll_\delta M$.

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