# Some Result on fixed point Theorem in Hilbert Space

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**Abstract**

This paper puts up a result regarding the generalization of the Banach contraction principle in the Hilbert space, It consist of four rational square terms in the inequality. Further the corollary of Koparde and Wag mode was obtained by considering vanishing values to some constant towards the end of this result.

**Keywords:** Hilbert space, closed subset, Cauchy sequence, Completeness.

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## 1. Introduction

Ever the beginning of the research in the discipline of fixed point theory and approximation theory was initiated by Banach in 1922 and since then many mathematicians has been working towards the development of this area of knowledge which has been extensively reported in the treatises of Nadler [7], Sehgal [9], kabbab [4],Wrong [13],etc. Also some generalize-tions of Banach fixed point theorem were given by D.S.jaggi [3], Fisher [1], Khare [5]. Gangly and Bandyopadhyay [2], Koparde and Waghmode [6], Pandhare [8], Veerapandi and Anil Kumar [12] nvestigated the properties of fixed points of family of mappings on complete metric spaces and in Hilbert spaces.

Motivated by the above results, the result which is found here is the refinement and sharpens some of the gene-alizations of the result Singh. Th. Manihar [10] and Smart [11] results. The theorem follows with the statement:

## 2. Theorem

### Theorem 2.1

Let X be a closed subset of a Hilbert space and $T: X \rightarrow X$ be a self mapping satisfying the following condition.

$$\| T_x - T_y \|^2 \leq \eta_1 \| y - T_y \|^2 + \eta_2 \frac{\|x-T_x\|^2 [1+\|y-T_y\|^2]}{1+\|x-y\|^2} + \eta_3 \| T_x - T_y \|^2 + \eta_4 \| y - T_x \|^2$$

$$+\eta_5 \| x - y \|^2$$

For each, $y \in X\ and\ x \neq y$ , where $\eta_{1,}\eta_2, \eta_3, \eta_4, \eta_5$ are non-negative reals with $0 \leq \eta_1 + \eta_3 + 2\eta_4 + +\eta_5 < 1$.Then T has a unique fixed point in X.

Proof: For some $x_0 \in X$, we define a sequence $\{ x_n\}$ of iterates of T as follows

$x_1 = Tx_0 , x_2 = Tx_1 , x_3 = Tx_{2,\ldots\ldots,} i.e\ \ x_{n+1} = Tx_n$ , for n = 0,1,2,3……..

Next, we show that $\{ x_n\}$ is a Cauchy sequence in X. for this consider

$$\| x_{n+1} - x_n \|^2 = \| Tx_n - T_{x_{n-1}} \|^2$$

Then by using hypothesis, we have

$$\| x_{n+1} - x_n \|^2 \leq \eta_1 \| x_{n-1} - x_n \|^2 + \eta_2 \frac{\|x_n - x_{n+1}\|^2[1 + \|x_{n-1} - x_n\|^2]}{1 + \|x_n - x_{n-1}\|^2} + \eta_3 \| x_{n+1} - x_n \|^2$$

$$+\eta_4 \| x_{n-1} - x_{n+1} \|^2 + \eta_5 \| x_n - x_{n-1} \|^2$$

$$\leq \eta_1 \| x_{n-1} - x_n \|^2 + \eta_2 \| x_n - x_{n+1} \|^2 + \eta_3 \| x_{n+1} - x_n \|^2 + \eta_4 \| x_{n-1} - x_{n+1} \|^2 + \eta_5 \| x_n - x_{n-1} \|^2$$

$$\leq \| x_{n-1} - x_n \|^2 (\eta_1 + \eta_5) + \eta_2 \| x_n - x_{n+1} \|^2 + \eta_3 \| x_{n+1} - x_n \|^2 + \eta_4 \| x_{n-1} + x_n - x_n - x_{n+1} \|^2$$

$$\leq \| x_{n-1} - x_n \|^2 (\eta_1 + \eta_4 + \eta_5) + \| x_n - x_{n+1} \|^2 (\eta_2 + \eta_3 + \eta_4)$$

$$\| x_{n+1} - x_n \|^2 (1 - \eta_2 - \eta_3 - \eta_4) \leq \| x_{n-1} - x_n \|^2 (\eta_1 + \eta_4 + \eta_5)$$

$$\| x_{n+1} - x_n \|^2 \leq \frac{(\eta_1 + \eta_4 + \eta_5)}{(1 - \eta_2 - \eta_3 - \eta_4)} \| x_{n-1} - x_n \|^2$$

$$\| x_{n+1} - x_n \|^2 \leq \rho(n) \| x_{n-1} - x_n \|^2$$

Where

$$\rho(n) = \frac{\eta_1 + \eta_4 + \eta_5}{(1 - \eta_2 - \eta_3 - \eta_4)}$$

$$\frac{\eta_1 + \eta_4 + \eta_5}{1 - \eta_2 - \eta_3 - \eta_4} \leq 1$$

$$\eta_1 + \eta_4 + \eta_5 \leq 1 - \eta_2 - \eta_3 - \eta_4$$

Clearly $\rho$(n) $<$ 1 . For all n as $0 \leq \eta_1 + \eta_2 + \eta_3 + 2\eta_4 + +\eta_5 < 1$ .repeating the same argument we find some $s < 1\ \ such\ that$:

$$\| x_{n+1} - x_n \|^2 \leq \lambda^n \| x_1 - x_0 \|, where\ \lambda = s^2$$

 

Letting $n \to \infty$, we obtain $\| x_{n+1} - x_n \| \to 0$. It follows that $\{x_n\}$ is a Cauchy sequence in X. there exists a point $\mu \epsilon X$ such that $x_n \to \mu \; as \; n \to \infty$.Also $\{x_{n+1}\} = \{Tx_n\}$ is subsequence of $\{x_n\}$ *converges to the same limit* $\mu$. Since T is continuous we obtain

$$T(\mu) = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = \mu$$

Hence $\mu$ is a fixed point of T in X. Next we show that uniqueness of $\mu$ : if T has another fixed point $v$. $\mu \neq v$, Then

$$\| \mu - v \|^2 = \| T_\mu - T_v \|^2$$

$$\leq \eta_1 \| v - T_v \|^2 + \eta_2 \frac{\|\mu - T_\mu\|^2[1 + \|v - T_v\|^2]}{1 + \|\mu - v\|^2} + \eta_3 \| T\mu - T_v \|^2 + \eta_4 \| v - T\mu \|^2 + \eta_5 \| \mu - v \|^2$$

Which inturn, implies that

$$\| \mu - v \|^2 \leq \eta_3 \| \mu - v \|^2 + \eta_4 \| v - \mu \|^2 + \eta_5 \| \mu - v \|^2$$

$$\leq (\eta_3 + \eta_4 + \eta_5) \; \| \mu - v \|^2$$

This gives a contradiction for $\eta_3 + \eta_4 + \eta_5 < 1$.

Thus $\mu$ is a unique fixed point T in X.

## Theorem 2.2

Let X be a closed subset of a Hilbert space and $T: X \to X$ be a self mapping satisfying the following condition.

$$\| T_x - T_y \|^2 \leq \eta_1 \frac{\|y - T_y\|^2[1 + \|x - T_x\|^2]}{1 + \|x - y\|^2} + \eta_2 \frac{\|y - T_y\|^2[1 + \|y - T_y\|^2]}{1 + \|x - y\|^2} + \eta_3 \| y - T_x \|^2 + \eta_4 \| x - y \|^2$$

For each, $y \in X \; and \; x \neq y$ , where $\eta_1, \eta_2, \eta_3, \eta_4$ are non-negative reals with $0 \leq \eta_1 + \eta_2 + 2\eta_3 + \eta_4 < 1$. Then T has a unique fixed point in X.

Proof: For some $x_0 \in X$, we define a sequence $\{ x_n\}$ of iterates of T as follows

$x_1 = Tx_0 , x_2 = Tx_1 , x_3 = Tx_{2,\ldots\ldots,} i.e \;\; x_{n+1} = Tx_n$ , for n = 0,1,2,3……..

Next, we show that $\{ x_n\}$ is a Cauchy sequence in X. for this consider

$$\| x_{n+1} - x_n \|^2 = \| Tx_n - T_{x_{n-1}} \|^2$$

Then by using hypothesis, we have

$$\| x_{n+1} - x_n \|^2 \leq \eta_1 \frac{\|x_{n-1} - x_n\|^2[1 + \|x_n - x_{n+1}\|^2]}{1 + \|x_n - x_{n-1}\|^2} + \eta_2 \frac{\|x_{n-1} - x_n\|^2[1 + \|x_{n-1} - x_n\|^2]}{1 + \|x_n - x_{n-1}\|^2} + \eta_3 \| x_{n-1} - x_{n+1} \|^2 + \eta_4 \| x_n - x_{n-1} \|^2$$

$$\leq \eta_1 \| x_{n-1} - x_n \|^2 + \eta_2 \| x_{n-1} - x_n \|^2 + \eta_3 \| x_{n-1} - x_{n+1} \|^2 + \eta_4 \| x_n - x_{n-1} \|^2$$

$$\leq \| x_{n-1} - x_n \|^2 \left(\eta_1 + \eta_2 + \eta_4\right) + \eta_3 \| x_{n-1} + x_n - x_n - x_{n+1} \|^2$$

$$\leq \| x_{n-1} - x_n \|^2 (\eta_1 + \eta_2 + \eta_3 + \eta_4) + \eta_3 \| x_{n+1} - x_n \|^2$$

$$\| x_{n+1} - x_n \|^2 (1 - \eta_3) \leq \| x_{n-1} - x_n \|^2 (\eta_1 + \eta_2 + \eta_3 + \eta_4)$$

$$\| x_{n+1} - x_n \|^2 \leq \frac{(\eta_1 + \eta_2 + \eta_3 + \eta_4)}{(1 - \eta_3)} \| x_{n-1} - x_n \|^2$$

$$\| x_{n+1} - x_n \|^2 \leq \rho(n) \| x_{n-1} - x_n \|^2$$

Where

$$\rho(n) = \frac{(\eta_1 + \eta_2 + \eta_3 + \eta_4)}{(1 - \eta_3)}$$

$$\eta_1 + \eta_2 + \eta_3 + \eta_4 \leq 1 - \eta_3$$

$$\eta_1 + \eta_2 + 2\eta_3 + \eta_4 \leq 1$$

Clearly $\rho$(n) < 1 . For all n as $0 \leq \eta_1 + \eta_2 + 2\eta_3 + \eta_4 \leq 1$ .repeating the same argument we find some $s < 1$ $such\ that$:

$$\| x_{n+1} - x_n \|^2 \leq \lambda^n \| x_1 - x_0 \|, where\ \lambda = s^2$$

Letting $n \to \infty$, we obtain $\| x_{n+1} - x_n \| \to 0$. It follows that $\{x_n\}$ is a Cauchy sequence in X. there exists a point $\mu \epsilon X$ such that $x_n \to \mu\ as\ n \to \infty$.Also $\{x_{n+1}\} = \{Tx_n\}$ is subsequence of $\{x_n\}$ $converges\ to\ the\ same\ limit\ \mu$. Since T is continuous we obtain

$$T(\mu) = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = \mu$$

Hence $\mu$ is a fixed point of T in X. Next we show that uniqueness of $\mu$ : if T has another fixed point $v$. $\mu \neq v$, Then

$$\| \mu - v \|^2 = \| T_\mu - T_v \|^2$$

$$\leq \eta_1 \frac{\|v - T_v\|^2 [1 + \|\mu - T_\mu\|^2]}{1 + \|\mu - v\|^2} + \eta_2 \frac{\|v - T_v\|^2 [1 + \|v - T_v\|^2]}{1 + \|\mu - v\|^2} + \eta_3 \| v - T_\mu \|^2 + \eta_4 \| \mu - v \|^2$$

Which intern, implies that

$$\| \mu - v \|^2 \leq \eta_3 \| v - \mu \|^2 + \eta_4 \| \mu - v \|^2$$

$$\| \mu - v \|^2 \leq (\eta_3 + \eta_4) \| \mu - v \|^2$$

This gives a contradiction for $\eta_3 + \eta_4 < 1$.

Thus $\mu$ is a unique fixed point T in X.

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