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On Jordan Triple Homomorphism and Generalized Jordan Triple Homomorphism of Gamma Rings

Fawaz Raad Jarulla[1] Kalyan Kumar Dey [2]
Department of Mathematics,college of Education, Al-Mustansirya University , Iraq[1]
Department of Mathematics , Rajshahi University , Rajshahi, Bangladesh[2]

Abstract:

Let M and M' be two Γ-rings , in the present paper we introduced the concepts of Jordan triple homomorphism , generalized Jordan triple homomorphism on Γ-rings and some Lemmas .

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Key Words: Γ– ring , Jordan homomorphism , Jordan triple homomorphism .

1- Introduction:

Let M and Γ be two additive abelian groups, suppose that there is a mapping from M×Γ×M $\longrightarrow$ M (the image of (a,α,b) being denoted by $a\alpha b$, $a, b \in$ M and $\alpha \in \Gamma$). Satisfying for all $a, b, c \in$ M and $\alpha, \beta \in \Gamma$:

(i) $(a + b)\alpha c = a\alpha c + b\alpha c$

$a(\alpha + \beta) c = a\alpha c + a\beta c$

$a\alpha (b + c) = a\alpha b + a\alpha c$

(ii) $(a\alpha b)\beta c = a\alpha(b\beta c)$

Then M is called a Γ-ring. This definition is due to Barnes [1].

Let M be a Γ-ring, then M is called 2-torsion free if $2a = 0$ implies that $a = 0$, for all $a \in$ M . This definition is due to [2].

An additive mapping θ of a Γ-ring M into a Γ-ring M' is called homomorphism if $\theta(a\alpha b) = \theta(a)\alpha\theta(b)$, for all $a, b \in$ M and $\alpha \in \Gamma$. This definition is due to [1].

An additive mapping θ of Γ-ring M into a Γ-ring M' is called Jordan homomorphism if $\theta(a\alpha b + b\alpha a) = \theta(a)\alpha\theta(b) + \theta(b)\alpha\theta(a)$, for all $a, b \in$ M and $\alpha \in \Gamma$. This definition is due to [3].

Let F be an additive mapping of a Γ-ring M into a Γ-ring M'. F is called a generalized homomorphism if there exists a homomorphism θ from a Γ-ring M into a Γ-ring M', such that $F(a\alpha b) = F(a)\alpha\theta(b)$, for all $a, b \in$ M and $\alpha \in \Gamma$, where θ is called the relating homomorphism . This definition is due to [3].

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And F is called a generalized Jordan homomorphism if there exists a Jordan homomorphism θ from a Γ-ring M into a Γ-ring M', such that

F($a\alpha b + b\alpha a$) = F(a)αθ(b) + F(b)α θ(a), for all $a, b \in$ M and $\alpha \in \Gamma$, where θ is called the relating Jordan homomorphism. This definition is due to [3].

2- Jordan Triple Homomorphism of Γ-Rings :

Definition (2.1):

An additive mapping θ of Γ-ring M into a Γ-ring M' is called Jordan triple homomorphism if $\theta(a\alpha b\beta a) = \theta(a)\alpha\theta(b)\beta\theta(a)$, for all $a, b \in$ M and $\alpha,\beta \in \Gamma$.

Example (2.2):

Let R be a ring , let M = $M_{1\times 2}(R)$, M' = $M'_{1\times 2}(R)$ and $\Gamma = \left\{\begin{pmatrix} n \\ 0 \end{pmatrix}, n \in Z\right\}$ then M and M' be Γ-rings. Let θ be an additive mapping of a Γ-ring M into a Γ-ring M', such that $\theta((a \quad b)) = (a \quad 0)$, for all $(a \quad b) \in$ M we obtain θ is a Jordan triple homomorphism

Lemma (2.3):

Let θ be a Jordan triple homomorphism of a Γ-ring M into a Γ- ring M ', then for all $a, b, c \in$ M, $\alpha, \beta \in \Gamma$ and $n \in N$

(i) $\theta(a\alpha b\beta a + a\beta b\alpha a) = \theta(a)\alpha\theta(b)\beta\theta(a) + \theta(a)\beta\theta(b)\alpha\theta(a)$

(ii) $\theta(a\alpha b\beta c + c\alpha b\beta a) = \theta(a)\alpha\theta(b)\beta\theta(c) + \theta(c)\alpha\theta(b)\beta\theta(a)$

(iii) In particular, if M, M' be two commutative Γ-rings and M' is a 2-torsion free Γ-ring , then

$\theta(a\alpha b\beta c) = \theta(a)\alpha\theta(b)\beta\theta(c)$

(iv) $\theta(a\alpha b\alpha c + c\alpha b\alpha a) = \theta(a)\alpha\theta(b)\alpha\theta(c) + \theta(c)\alpha\theta(b)\alpha\theta(a)$

Proof:

(i) Replace $a\beta b + b\beta a$ for b in Definition Jordan homomorphism , we get :

$\theta(a\alpha(a\beta b + b\beta a) + (a\beta b + b\beta a)\alpha a) = \theta(a)\alpha\theta(a\beta b + b\beta a) + \theta(a\beta b + b\beta a)\alpha\theta(a)$

$= \theta(a)\alpha\theta(a)\beta\theta(b) + \theta(a)\alpha\theta(b)\beta\theta(a) + \theta(a)\beta\theta(b)\alpha\theta(a) + \theta(b)\beta\theta(a)\alpha\theta(a)$ …(1)

On the other hand

$\theta(a\alpha(a\beta b + b\beta a) + (a\beta b + b\beta a)\alpha a) = \theta(a\alpha a\beta b + a\alpha b\beta a + a\beta b\alpha a + b\beta a\alpha a)$

$= \theta(a)\alpha\theta(a)\beta\theta(b) + \theta(b)\beta\theta(a)\alpha\theta(a) + \theta(a\alpha b\beta a + a\beta b\alpha a)$ …(2)

Compare (1) and (2), we get:

$\theta(a\alpha b\beta a + a\beta b\alpha a) = \theta(a)\alpha\theta(b)\beta\theta(a) + \theta(a)\beta\theta(b)\alpha\theta(a)$

(ii) Replace $a + c$ for a in Definition (2.1), we get:

$\theta((a + c)\alpha b\beta(a + c)) = \theta(a + c)\alpha\theta(b)\beta\theta(a + c)$

$= \theta(a)\alpha\theta(b)\beta\theta(a) + \theta(a)\alpha\theta(b)\beta\theta(c) + \theta(c)\alpha\theta(b)\beta\theta(a) + \theta(c)\alpha\theta(b)\beta\theta(c) \ldots$ (1)

On the other hand

$\theta((a + c)\alpha b\beta(a + c)) = \theta(a\alpha b\beta a + a\alpha b\beta c + c\alpha b\beta a + c\alpha b\beta c)$

$= \theta(a)\alpha\theta(b)\beta\theta(a) + \theta(c)\alpha\theta(b)\beta\theta(c) + \theta(a\alpha b\beta c + c\alpha b\beta a)$ …(2)

Compare (1) and (2), we get:

$\theta(a\alpha b\beta c + c\alpha b\beta a) = \theta(a)\alpha\theta(b)\beta\theta(c) + \theta(c)\alpha\theta(b)\beta\theta(a)$

(iii) By (ii) and since M, M' be two commutative Γ-rings and M' is a 2-torsion free Γ-ring , then

$\theta(a\alpha b\beta c + a\alpha b\beta c) = 2\theta(a\alpha b\beta c) = \theta(a)\alpha\theta(b)\beta\theta(c)$

(iv) Replace α for β in (ii), we get:

$\theta(a\alpha b\alpha c + c\alpha b\alpha a) = \theta(a)\alpha\theta(b)\alpha\theta(c) + \theta(c)\alpha\theta(b)\alpha\theta(a)$

Definition (2.4):

Let θ be a Jordan homomorphism of a Γ-ring M into a Γ-ring M',then for all $a, b \in$ M and $\alpha \in \Gamma$, we define

$G(a,b,a)_{\alpha,\beta} = \theta(a\alpha b\beta a) - \theta(a)\alpha\theta(b)\beta\theta(a)$.

Lemma (2.5):

If θ be a Jordan triple homomorphism of a Γ-ring M into a Γ-ring M',then for all $a, b, c, d \in$ M and $\alpha, \beta \in \Gamma$

(i) $G((a + b),c,d)_{\alpha,\beta} = G(a,c,d)_{\alpha,\beta} + G(b,c,d)_{\alpha,\beta}$

(ii) $G(a,(b + c),d)_{\alpha,\beta} = G(a,b,d)_{\alpha,\beta} + G(a,c,d)_{\alpha,\beta}$

(iii) $G(a,b,(c + d))_{\alpha,\beta} = G(a,b,c)_{\alpha,\beta} + G(a,b,d)_{\alpha,\beta}$

Proof:

(i) $G((a + b),c,d)_{\alpha,\beta} = \theta((a + b)\alpha c\beta d) - \theta(a + b)\alpha\theta(c)\beta\theta(d)$

$= \theta(a\alpha c\beta d) - \theta(a)\alpha\theta(c)\beta\theta(d) + \theta(b\alpha c\beta d) - \theta(b)\alpha\theta(c)\beta\theta(d)$

$= G(a,c,d)_{\alpha,\beta} + G(b,c,d)_{\alpha,\beta}$

(ii) $G(a,(b + c),d)_{\alpha,\beta} = \theta(a\alpha(b + c)\beta d) - \theta(a)\alpha\theta(b + c)\beta\theta(d)$

$= \theta(a\alpha b\beta d) - \theta(a)\alpha\theta(b)\beta\theta(d) + \theta(a\alpha c\beta d) - \theta(a)\alpha\theta(c)\beta\theta(d)$

$= G(a,b,d)_{\alpha,\beta} + G(a,c,d)_{\alpha,\beta}$

(iii) $G(a,b,(c + d))_{\alpha,\beta} = \theta(a\alpha b\beta(c + d)) - \theta(a)\alpha\theta(b)\beta\theta(c + d)$

$= \theta(a\alpha b\beta c) - \theta(a)\alpha\theta(b)\beta\theta(c) + \theta(a\alpha b\beta d) - \theta(a)\alpha\theta(b)\beta\theta(d)$

$= G(a,b,c)_{\alpha,\beta} + G(a,b,d)_{\alpha,\beta}$

Proposition (2.6):

Let θ be a Jordan homomorphism from a Γ-ring M into a 2-torsion free Γ-ring M', such that $a\alpha b\beta a = a\beta b\alpha a$, for all $a, b \in$ M and $\alpha, \beta \in \Gamma$, $a'\alpha b'\beta a' = a'\beta b'\alpha a'$, for all $a', b' \in$ M' and $\alpha, \beta \in \Gamma$.Then θ is Jordan triple homomorphism.

Proof:

Replace b by $a\beta b + b\beta a$ in Definition of Jordan homomorphism , we get :

$\theta(a\alpha(a\beta b + b\beta a) + (a\beta b + b\beta a)\alpha a) = \theta(a)\alpha\theta(a\beta b + b\beta a) + \theta(a\beta b + b\beta a)\alpha\theta(a)$

$= \theta(a)\alpha\theta(a)\beta\theta(b) + \theta(a)\alpha\theta(b)\beta\theta(a) + \theta(a)\beta\theta(b)\alpha\theta(a) + \theta(b)\beta\theta(a)\alpha\theta(a)$

Since $a'\alpha b'\beta a' = a'\beta b'\alpha a'$, for all $a', b' \in$ M' and $\alpha, \beta \in \Gamma$, we get:

$= \theta(a)\alpha\theta(a)\beta\theta(b)+2\theta(a)\alpha\theta(b)\beta\theta(a)+\theta(b)\beta\theta(a)\alpha\theta(a)$ …(1)

On the other hand:

$\theta(a\alpha(a\beta b + b\beta a) + (a\beta b + b\beta a)\alpha a) = \theta(a\alpha a\beta b + a\alpha b\beta a + a\beta b\alpha a + b\beta a\alpha a)$

Since $a\alpha b\beta a = a\beta b\alpha a$, for all $a, b \in$ M and $\alpha, \beta \in \Gamma$, we get:

$= \theta(a\alpha a\beta b + b\beta a\alpha a) + 2\theta(a\alpha b\beta a)$

$= \theta(a)\alpha\theta(a)\beta\theta(b) + \theta(b)\beta\theta(a)\alpha\theta(a) + 2\theta(a\alpha b\beta a)$ …(2)

Compare (1) and (2), we get:

$2\theta(a\alpha b\beta a) = 2\theta(a)\alpha\theta(b)\beta\theta(a)$.

Since M' is 2-torsion free Γ-ring , we obtain that θ is Jordan triple homomorphism.

3- Generalized Jordan Triple Homomorphism of Γ-Rings :

Definition (3.1):

An additive mapping F of a Γ-ring M into a Γ-ring M' is called a generalized Jordan triple homomorphism if there exists a Jordan triple homomorphism θ from a Γ-ring M into a Γ-ring M' such that

$F(a\alpha b\beta a) = F(a)\alpha\theta(b)\beta\theta(a)$, for all $a, b \in$ M and $\alpha , \beta \in \Gamma$.

Where θ is called the relating Jordan triple homomorphism.

Example (3.2):

Let R be a ring, let M = $M_{1\times2}(R)$, M' = $M_{1\times2}(R)$ and $\Gamma = \left\{ \begin{pmatrix} n \\ 0 \end{pmatrix}, n \in Z \right\}$.Then M and M' be Γ-rings.Let F be an additive mapping of a Γ-ring M into a Γ-ring M', such that F((*a* *b*)) = (– *a* 0), for all (*a* *b*) ∈ M ,then there exists a homomorphism θ from a Γ-ring M into a Γ-ring M', such that θ((*a* *b*)) = (*a* 0), for all (*a* *b*) ∈ M. Then F is a generalized Jordan triple homomorphism

Lemma (3.3):

Let θ be a generalized Jordan triple homomorphism of a Γ-ring M into a Γ- ring M ', then for all *a*, *b*, *c* ∈ M, α, β ∈ Γ and n ∈ N

(i) F(*a*α*b*β*a* + *a*β*b*α*a*)=F(*a*)αθ(*b*)βθ(*a*) + F(*a*)βθ(*b*)αθ(*a*)

(ii) F(*a*α*b*β*c*+*c*α*b*β*a*)= F(*a*)αθ(*b*)βθ(*c*) + F(*c*)αθ(*b*)βθ(*a*)

(iii) In particular, if M, M' be two commutative Γ-rings and M' is a 2-torsion free Γ-ring , then

F(*a*αbβc) = F(*a*)αθ(*b*)βθ(*c*)

(iv) F(*a*α*b*α*c* +*c*α*b*α*a*)=F(*a*)αθ(*b*)αθ(*c*)+ F(*c*)αθ(*b*)αθ(*a*)

Proof:

(i) Replace *a*β*b* + *b*β*a* for *b* in Definition generalized Jordan homomorphism , we get:

F(*a*α(*a*β*b* + *b*β*a*) + (*a*β*b* + *b*β*a*)α*a*) = F(*a*)αθ(*a*β*b* + *b*β*a*) + F(*a*β*b* + *b*β*a*)αθ(*a*)

= F(*a*)αθ(*a*)βθ(*b*)) + F(*a*)αθ(*b*)βθ(*a*)) + F(*a*)βθ(*b*)αθ(*a*) + F(*b*)βθ(*a*)αθ(*a*)

= F(*a*)αθ(*a*)βθ(*b*) + F(*a*)αθ(*b*)βθ(*a*) + F(*a*)βθ(*b*)αθ(*a*) + F(*b*)βθ(*a*)αθ(*a*) …(1)

On the other hand

F(*a*α(*a*β*b* + *b*β*a*)+(*a*β*b* + *b*β*a*)α*a*)=F(*a*α*a*β*b* + *a*α*b*β*a* + *a*β*b*α*a* + *b*β*a*α*a*)

= F(*a*)αθ(*a*)βθ(*b*)+F(*b*)βθ(*a*)αθ(*a*) + F(*a*α*b*β*a*+*a*β*b*α*a*) …(2)

Compare (1) and (2), we get:

F(*a*α*b*β*a*+ *a*β*b*α*a*)=F(*a*)αθ(*b*)βθ(*a*) + F(*a*)βθ(*b*)αθ(*a*)

(ii) Replace *a* + *c* for *a* in Definition (3.1) , we get:

F((*a* + *c*)α*b*β(*a* + *c*)) = F(*a* + *c*)αθ(*b*)βθ(*a* + *c*)

= F(*a*)αθ(*b*)βθ(*a*) + F(*a*)αθ(*b*)βθ(*c*) + F(*c*)αθ(*b*)βθ(*a*) + F(*c*)αθ(*b*)βθ(*c*) …(1)

On the other hand

F((*a* + *c*)α*b*β(*a* + *c*)) = F(*a*α*b*β*a* + *a*α*b*β*c* + *c*α*b*β*a* + *c*α*b*β*c*)

$=F(a)\alpha\theta(b)\beta\theta(a)+F(c)\alpha\theta(b)\beta\theta(c)+F(a\alpha b\beta c+c\alpha b\beta a)$ …(2)

Compare (1) and (2), we get:

$F(a\alpha b\beta c+c\alpha b\beta a) = F(a)\alpha\theta(b)\beta\theta(c) + F(c)\alpha\theta(b)\beta\theta(a)$

(iii) By (ii) and since M, M' be two commutative Γ-rings and M' is a 2-torsion free Γ-ring

$F(a\alpha b\beta c + a\alpha b\beta c) = 2F(a\alpha b\beta c) = F(a)\alpha\theta(b)\beta\theta(c)$

(iv) Replace α for β in (ii), we get:

$F(a\alpha b\alpha c + c\alpha b\alpha a) = F(a)\alpha\theta(b)\alpha\theta(c) + F(c)\alpha\theta(b)\alpha\theta(a)$

Definition (3.4):

Let F be a generalized Jordan homomorphism of a Γ-ring M into a Γ-ring M',then for all $a, b \in M$ and $\alpha \in \Gamma$, we define

$\delta(a,b,a)_{\alpha,\beta} = F(a\alpha b\beta a) - F(a)\alpha\theta(b)\beta\theta(a)$.

Lemma (3.5):

If F be a generalized Jordan triple homomorphism of a Γ-ring M into a Γ-ring M',then for all $a, b, c, d \in M$ and $\alpha, \beta \in \Gamma$

(i) $\delta((a + b),c,d)_{\alpha,\beta} = \delta(a,c,d)_{\alpha,\beta} + \delta(b,c,d)_{\alpha,\beta}$

(ii) $\delta(a,(b + c),d)_{\alpha,\beta} = \delta(a,b,d)_{\alpha,\beta} + \delta(a,c,d)_{\alpha,\beta}$

(iii) $\delta(a,b,(c+ d))_{\alpha,\beta} = \delta(a,b,c)_{\alpha,\beta} + \delta(a,b,d)_{\alpha,\beta}$

Proof:

(i) $\delta((a + b),c,d)_{\alpha,\beta} = F((a + b)\alpha c\beta d) - F(a + b)\alpha\theta(c)\beta\theta(d)$

$= F(a\alpha c\beta d) - F(a)\alpha\theta(c)\beta\theta(d) + F(b\alpha c\beta d) - F(b)\alpha\theta(c)\beta\theta(d)$

$= \delta(a,c,d)_{\alpha,\beta} + \delta(b,c,d)_{\alpha,\beta}$

(ii) $\delta(a,(b + c),d)_{\alpha,\beta} = F(a\alpha(b + c)\beta d) - F(a)\alpha\theta(b + c)\beta\theta(d)$

$= F(a\alpha b\beta d) - F(a)\alpha\theta(b)\beta\theta(d) + F(a\alpha c\beta d) - F(a)\alpha\theta(c)\beta\theta(d)$

$= \delta(a,b,d)_{\alpha,\beta} + \delta(a,c,d)_{\alpha,\beta}$

(iii) $\delta(a,b,(c+ d))_{\alpha,\beta} = F(a\alpha b\beta(c + d)) - F(a)\alpha\theta(b)\beta\theta(c + d)$

$= F(a\alpha b\beta c) - F(a)\alpha\theta(b)\beta\theta(c) + F(a\alpha b\beta d) - F(a)\alpha\theta(b)\beta\theta(d)$

$= \delta(a,b,c)_{\alpha,\beta} + \delta(a,b,d)_{\alpha,\beta}$

Proposition (3.6):

Let F be a generalized Jordan homomorphism from a Γ-ring M into a 2-torsion free Γ-ring M', such that $a\alpha b\beta a = a\beta b\alpha a$, for all $a, b \in M$ and $\alpha, \beta \in \Gamma$,

$a'\alpha b'\beta a' = a'\beta b'\alpha a'$, for all $a', b' \in M'$ and $\alpha, \beta \in \Gamma$.Then F is a generalized Jordan triple homomorphism.

Proof:

Replace b by $a\beta b + b\beta a$ in Definition of generalized Jordan homomorphism , we get:

$F(a\alpha(a\beta b + b\beta a) + (a\beta b + b\beta a)\alpha a) = F(a)\alpha\theta(a\beta b + b\beta a) + F(a\beta b + b\beta a)\alpha\theta(a)$

$= F(a)\alpha\theta(a)\beta\theta(b) + F(a)\alpha\theta(b)\beta\theta(a) + F(a)\beta\theta(b)\alpha\theta(a) + F(b)\beta\theta(a)\alpha\theta(a)$

Since $a'\alpha b'\beta a' = a'\beta b'\alpha a'$, for all $a', b' \in M'$ and $\alpha, \beta \in \Gamma$, we get:

$=F(a)\alpha\theta(a)\beta\theta(b) + 2F(a)\alpha\theta(b)\beta\theta(a) + F(b)\beta\theta(a)\alpha\theta(a)$ …(1)

On the other hand:

$F(a\alpha(a\beta b + b\beta a) + (a\beta b + b\beta a)\alpha a) = F(a\alpha a\beta b + a\alpha b\beta a + a\beta b\alpha a + b\beta a\alpha a)$

Since $a\alpha b\beta a = a\beta b\alpha a$, for all $a, b \in M$ and $\alpha, \beta \in \Gamma$, we get:

$= F(a)\alpha\theta(a)\beta\theta(b) + F(b)\beta\theta(a)\alpha\theta(a) + 2F(a\alpha b\beta a)$ …(2)

Compare (1) and (2), we get:

$2F(a\alpha b\beta a) = 2F(a)\alpha\theta(\mathrm{b})\beta\theta(a)$.

Since M' is 2-torsion free Γ-ring , we get F is a generalized Jordan triple homomorphism.

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