P, P-L. compact topological ring

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Abstract

In this paper, we introduced some new definitions on P-compact topological ring and PL-compact topological ring for the compactification in topological space and rings, we obtain some results related to P-compact and P-L compact topological ring.

Keywords: rings, ideals, topological ring, D-cover groups, isomorpnism, direct product, D-compact group.

Introduction

A topological ring is a ring R which is also a topological space such that both the addition and the multiplication are continuous as maps

$R \times R \rightarrow R$ where $R \times R$ carries the product topology. A topological ring (R,∗, . ,τ) is said to be compact, if a topological space (R, τ) is compact space, for details see [1] . In [2] D.G. Salih gave the concept of D–cover groups and D–compact groups. In this paper, we shall generalize this concept to topological rings so we investigated the P–compact and the P-L. compact topological rings, in particular case we deal with the proper ideals of a topological rings, so we introduce the PI-compact and PI-L. compact topological rings, we obtain some good results related these concepts above. We mean throughout this paper a topological rings is just ring as a set with topology.

2. Definitions and examples .

Definition 1

Let (R,∗, .,τ) be a topological ring and I be an index set, we say that

1. The family $\{Ri \in \tau; (R_i,*,.) is\ a\ proper\ subrings\ of\ (R,*\,.),\ \forall i \in I\}$ is a P-cover topological rings of (R,∗, . ,τ) if $R = \cup_{i \in I} R_i$

Definition 2

Let $(R,*,.,\tau)$ be a topological ring we say that;

1- $(R,*,.,\tau)$ is weakly P–compact topological ring if there is a finite P– cover topological rings of $(R,*,.,\tau)$.

2- $(R,*,.,\tau)$ is P-compact topological ring if for any P–cover topological rings of $(R,*,.,\tau)$, there is a finite sub P–cover topological rings of $(R,*,.,\tau)$.

3- $(R,*,.,\tau)$is weakly P–L. compact topological ring if there exists a countable P–cover topological rings of $(R,*,.,\tau)$.

4- $(R,*,.,\tau)$ is P–L. compact topological ring if for any P–cover topological ring of $(R,*,.,\tau)$, there is a countable sub P–cover topological rings of $(R,*,.,\tau)$.

Definition 3

Let $(R,*,.,\tau)$be a topological ring and $(H,*,.)$ be a subring of $(R,*,.)$. The topological subring $(H,*,.,\tau_H)$ $[\tau_H = \tau \cap H]$ is said to be:

P–compact topological subring (weakly P–compact topological subring, P–L. compact topological subring, weakly P.L. compact topological subring) if $(H,*,.,\tau_H)$ is P–compact (weakly P–compact, P–L. compact and weakly P–L. compact) topological ring respectively.

Definition 4

1- Let $(R,*,.,\tau)$ and $(\bar{R},\bar{*},\bar{.},\bar{\tau})$ be two topological rings then,

(i) $f:(R,*,.,\tau) \to (\bar{R},\bar{*},\bar{.},\bar{\tau})$ is a homomorphism topological rings if $f:(R,\tau) \to (\bar{R},\bar{\tau})$ is continuous such that $f(x * y) = f(x) \bar{*} f(y)$ and $f(x.y) = f(x)\bar{.}f(y)$ for each pair of elements $x, y \in R$

(ii) f : $(R,*,.,\tau) \to (\bar{R},\bar{*},\bar{.},\bar{\tau})$ is a topological isomorphism if it is topological homeomorphism and ring isomorphism.

2- Suppose that Λ is a non-empty set and $(R_\lambda,*_\lambda,._\lambda,\tau_\lambda)$ is a topological rings for each $\lambda \epsilon \Lambda$, their product is $\pi_{\lambda \epsilon \Lambda} R_\lambda$ equipped with the usual product topology $\tau_{\pi_{\lambda\in\Lambda}R_\lambda}$ and with multiplication given by $(x \otimes y) = x_\lambda \otimes y_\lambda$ for each $x_\lambda, y_\lambda \epsilon R_\lambda$ and $\lambda \in \Lambda$.

3- If $R_\lambda = R$ and $\tau_\lambda = \tau, \forall \lambda \epsilon \Lambda$ then we denoted that

$R^\Lambda = \pi_{\lambda\epsilon\Lambda} R_\lambda$ and $\tau^\Lambda = \pi_{\lambda\in\Lambda}\, \tau_\lambda$

Example 1.

Let $X = \{a, b, c, d\}$ and P(X) the power set of X i.e.

$P(X) =$
$\{\emptyset, X, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\},$
$\{a, b, d\}, \{b, c, d\}, \{a, c, d\}\}$

One can show easily that $R = (P(X), \Delta, \cap)$ where $(A \vartriangle B = A \cup B - A \cap B)$, is a ring

Let τ be the discrete topology defined on $P(X)$ and let P_i , $1 \le i \le 12$ be the following sets :

$P_1 = \{\emptyset, X\}$

$P_2 = \{\emptyset, \{a\}\}$

$p_3 = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

$P_4 = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

$P_5 = \{\emptyset, \{d\}\}$

$P_6 = \{\emptyset, \{d\}, \{a\}, \{a, d\}\}$

$P_7 = \{\emptyset, \{d\}, \{a\}, \{c\}, \{a, d\}, \{a, c\}, \{d, c\}, \{a, d, c\}\}$

$P_8 = \{\emptyset, \{c\}, \{a, b\}, \{a, b, c\}\}$

$P_9 = \{\emptyset, \{b\}, \{a, d\}, \{a, b, d\}\}$

$P_{10} = \{\emptyset, \{a\}, \{c, d\}, \{a, c, d\}\}$

$P_{11} = \{\emptyset, \{d\}, \{b, c\}, \{b, c, d\}\}$

$P_{12} = \{\emptyset, X, \{a, c\}, \{b, d\}\}$

It is clear that the family $\{P_i \in \tau; (P_i, \Delta, \cap)\}$ is a P–cover topological rings of $(P(X), \Delta)$, which has a finite sub P-cover $\{p_8, p_9, p_{10}, p_{11}, p_{12}\}$ so $(P(X), \Delta, \cap)$ is weakly P–compact in fact it is P–compact .

Recall that see[3] , a sub ring I of the ring R is said to be a two side ideal of R if and only if r ∈ R and $a \in I$ imply both $ra \in I$ and $ar \in I$.

For rings with identity 1, it is clear that if $1 \in I$ then $I \equiv R$, so we give the following modification for the preceding definitions.

Definition 5.

Let $(R,*,.,\tau)$ be a topological rings and let Λ be an index we say that the family $\{I_i \in \tau; (I_i,*,.)\}$ is a proper ideal of $\{(R,*,.), \forall i \in A\} \cup \{1\}$ is PI-cover topological ideals of $(R,*,.,\tau)$, if $R = \cup_{i \in \Lambda} I_i \cup \{1\}$

Definition 6.

Let $(R,*,.,\tau)$ be a topological ring, we say that

1- $(R,*,.,\tau)$ is weakly PI – compact topological ring if there exists a finite PI – cover topological ideals of $(R,*,.,\tau)$
2- $(R,*,.,\tau)$ is PI–compact topological ring if for any PI–cover topological ideals of $(R,*,.,\tau)$, there is a finite sub PI-cover topological ideals of $(R,*,.,\tau)$
3- $(R,*,.,\tau)$ is weakly PI–L. compact topological ring if ther exists a countable PI – cover topological of $(R,*,.,\tau)$
4- $(R,*,.,\tau)$ is PI–L. compact topological ring if for any PI- cover topological ideals of $(R,*,.,\tau)$, there is a countable sub PI- cover topological rings of $(R,*,.,\tau)$.

Example2.

Let R be the ring $(Z, +,.,\tau)$ where τ is the discrete topology defined on Z. Note that the following:

$2Z = \{0, \mp 2, \mp 4, \dots\}$

$3Z = \{0, \mp 3, \mp 6, \dots\}$

$4Z = \{0, \mp 4, \mp 6, \dots\}$

$5Z = \{0, \mp 5, \mp 10, \dots\}$

$6Z = \{0, \mp 6, \mp 12, \dots\}$ …….etc.

are all proper ideals of $(Z, +, ., \tau)$. Now it is easy to show that $I = \{kZ / k \in Z^{+}\} \cup \{1\}$ is a countable PI – cover toplological ideals of $(Z, +, ., \tau)$, that's mean $Z = \cup_{k \in z^{+}} I_k$

Which has been a countable sub PI-cover since for example $4Z \subseteq 2Z$ and $6Z \subseteq 3Z$, …. etc. Hence $(Z, +, ., \tau)$ is PI–L. compact which is not PI-compact because the prime numbers are infinite see [4].

Main results .

It is easy to prove direct from definitions the following Lemmas

Lemma 1.

1. Any P –compact topological ring is weakly P–compact.
2. Any P – compact topological ring is P–L. compact .

Lemma 2.

1. Any PI – compact topological ring is weakly PI–compact .
2. Any PI – compact topological ring is PI–L. compact .

Also we can prove directly by Lemma (1) , the following theorem

Theorem 1.

Let (R , ∗ , . , τ) be a topological ring such that R is finite set, then the following are equivalents :

1. (R , ∗ , . , τ) is P–compact topological ring .
2. (R , ∗ , . , τ) is P–L. compact topological ring .

If we replace P–(P–L.) compact topological with PI–(PI–L.) compact topological ring respectively , the result is true .

Example 3.

Let R = $Z_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ and

$\tau = \{\{\bar{0}\},\{\bar{0},\bar{3}\},\{\bar{0},\bar{2},\bar{4}\},\{\bar{1},\bar{5}\},\{\bar{2},\bar{3},\bar{4}\},\{\bar{0},\bar{1},\bar{3},\bar{5}\},\emptyset,R\}$

Although (R , + , . , τ) is finite topological ring but it is not P-compact and not PI-compact since $\{\bar{0}\},\{\bar{0},\bar{3}\},\{\bar{0},\bar{2},\bar{4}\}$ are the only proper sub rings which are not cover (R , + , . , $\bar{\tau}$).

Recall that a ring (R , ∗ , .) is said to be a field provided that the set R/{0} is commutative group under the multiplication of R. It is known that if R is a field then R has no non trivial ideals see [3], so we have the following theorem .

Theorem 2.

Any infinite ring (not field) can be PI–compact.

Proof.

Suppose first (R,*,.,τ) is a ring without identity . Let I be a set (finite or infinite) defined $\tau = \{A_i \subseteq R, A_i^C \text{ } is\ finite\ set, (A_i,*,.)\ proper\ ideal\ \forall_i \in I\ and\ A_{i_1} \subseteq A_{i_2}\ for\ i_1 \leq i_2\} \cup \{\emptyset\}$.

Now since any arbitrary set { na / a ∈ R , $n \in Z^+$} is ideal see [3] hence Z ∈ ∅ . Clear that (R , ∗ , . , τ) is topological ring since .

(1) $\emptyset \in \tau$ and $R^c = \emptyset$ is finite i.e. $R \in \tau$.

(2) Let A_1 , $A_2 \in \tau$, so A_1^c, A_2^c are finite but $(A_1 \cap A_2)^c = A_1^c \cup A_2^c$ implies $(A_1 \cap A_2)^c$ is finite and since $(A_1 \cap A_2,*,.)$ is ideal see[3] hence $A_1 \cap A_2 \in \tau$.

(3) Let Λ be any index and let $A_s \in \tau$, $\forall_s \in \Lambda$, hence A_s^c is finite for each s ∈ Λ which leads to $\bigcap_{s\in\Lambda} A_s^c$ is finite . Now since $\left(\bigcup_{s\in\Lambda} A_s\right)^c = \bigcap_{s\in\Lambda} A_s^c$, but $\bigcup_{s\in\Lambda} A_s = A_t$, for each s ∈ Λ . So $(\bigcup_{s\in\Lambda}, A_i, *, .)$ is an ideal , hence (R , ∗ , . , τ) is topological ring.

Now let $\{A_\lambda \epsilon\, \tau, \lambda \in \Lambda\}$ be any PI–cover topological rings of (R,∗,. , τ), that is R = $\bigcup_{\lambda \in \Lambda} A_\lambda$.

Let $A_0 \epsilon \{A_\lambda\}_{\lambda \epsilon \hat{\Lambda}}$ implies $(A_0,*,.)$ is an ideal and A_0^c is finite set. Suppose that $A_0^c = \{a_1, a_2, \dots\dots, a_n\}$ where a_j ∈ R for each 1 ≤ j ≤ n, but $\{A_\lambda \in \tau, \lambda \in \Lambda\}$ is

PI–cover of (R,∗,.,τ) so there is $A_{\lambda_j} \in \{A_\lambda \in \tau, \lambda \in \Lambda\}$ such that $a_j \in A_{\lambda j}$ for each j implies $A_0^c \subseteq \bigcup_{j=1}^{n} A_{\lambda_j}$.

Thus $R \subseteq \bigcup_{j=1}^{n} A_{\lambda_j} \cup A_0$ (of course R = $A_0 \cup A_0^c$) that means there is a finite sub PI – cover topological rings $\{A_0, A_{\lambda 1}, \dots, A_{\lambda n}\}$ each of which is ideal , therefore (R , ∗ , . , τ) is PI–compact topological ring . If (R , ∗ , . , τ) is a ring with identity we take $\grave{\tau} = \tau \cup \{1\}$ and the prove is similar

Corollary 1 .

Any infinite ring (not field) can be a weakly PI–compact .

For product P – compact rings we have the following two theorems .

Theorem 3.

Let (R , ∗ , . , τ) and $(\bar{R}, \bar{*}, \bar{.}, \bar{\tau})$ be two topological rings if $(\bar{R}, \bar{*}, \bar{.}, \bar{\tau})$ is a P–compact topological ring . Then (R × $\bar{R}$, ⊗ , ⊙, τ × $\bar{\tau}$) is P–compact topological ring.

Proof.

Let $\{(R \times \bar{R}_i, \otimes, \odot); \bar{R}_i \in \bar{\tau}\ and\ (\bar{R}_i, \bar{*}, \bar{.})\ rings\ \forall i \in I\}$ be any P–cover topological rings of R × $\bar{R}$, i.e. R × $\bar{R} = \bigcup_{i \in I}(R \times \bar{R}_i) = R \times (\bigcup_{i \in I} \bar{R}_i)$ implies $\bar{R} = \bigcup_{i \in I} \bar{R}_i$, but $(\bar{R}, \bar{*}, \bar{.}, \bar{\tau})$ is P–compact topological ring so there is finite subset J ⊆ I such that $\bar{R} = \bigcup_{j \in J} \bar{R}_j \Longrightarrow$ R × $\bar{R}$ = R × $\left(\bigcup_{j \in J} \bar{R}_j\right) = \bigcup_{j \in J}\left(R \times \bar{R}_j\right)$, where $(R \times \bar{R}_j, \otimes, \odot)$ is a ring for each j ∈ J. Therefore (R × $\bar{R}$, ⊗ , ⊙, τ × $\bar{\tau}$) is P-compact topological ring.

Theorem 4.

Let (R , ∗ , . , τ) and $(\bar{R}, \bar{*}, \bar{.}, \bar{\tau})$ be two P–compact topological rings then (R × $\bar{R}$, ⊗ , ⊙, τ × $\bar{\tau}$) is P–compact topological ring.

Proof.

Let (R,∗,., τ) and $(\bar{R}, \bar{\otimes}, \bar{\odot}, \bar{\tau})$ be any two P–compact topological rings. Then there exists a P–cover topological rings $\{R_a\}_{a \in A}$ and $\{R_b\}_{b \in B}$ of R and $\bar{R}$ respectively (A , B any index), that's mean

R $\times \bar{R}$ = $(\bigcup_{a\in A} R_a) \times (\bigcup_{b\in B} \bar{R}_b) = \bigcup_{a\in A,b\in B}(R_a \times \bar{R}_b)$ implies $\{R_a \times \bar{R}_b\}_{a\in A,b\in B}$ is a P–cover topological rings of (R $\times \bar{R}$, $\otimes$, $\odot$, $\tau \times \bar{\tau}$).

Let $\{W_i\}_{i\in\Lambda}$ be any P – cover topological rings of (R $\times \bar{R}$, $\otimes$, $\odot$, $\tau \times \bar{\tau}$) then R $\times \bar{R} = \bigcup_{i\in\Lambda} W_i$ such that $W_i = U_i \times V_i$, where $U_i \in \tau$ and $V_i \in \bar{\tau}$ for each $i \in \Lambda$. But (R , $*$, . , τ) is P–compact ring , so there is a finite sub set $J \subseteq \Lambda$ such that R = $\bigcup_{j\in J} U_j$ and $(U_j ,*,.)$ is a ring for each $j \in J$. Let $U_{j_1} \in \{U_j\}_{j\in J}$ implies $\{U_{j_1} \times V_i\}_{i\in\Lambda}$ is a P–cover topological rings of $(U_{j_1} \times \bar{R}, ,\otimes,\odot)$ hence $U_{j_1} \times \bar{R} = \bigcup_{i\in\Lambda}(U_{j_1} \times V_i)$, but $(U_{j_1} \times \bar{R} ,\otimes,\odot)$ is P–compact topological ring since $(U_{j_1} ,*,.)$ is a ring and $(\bar{R} ,\bar{*},\bar{.})$ is P-compact topological ring (theorem 3) so there is a finite set S $\subset \Lambda$ such that $\{U_{j_1} \times V_s\}_{s\in S}$ is a ring , $\forall\, s \in S$. Now $U_{j_1} \times \bar{R} = \bigcup_{s\in S}(U_{j_1} \times V_s)$ hence $U_{j_1} \times \bar{R} = U_{j_1} \times (\bigcup_{s\in S} V_s)$ [see5] and hence

R $\times \bar{R} = \left(\bigcup_{j\in J} U_j\right) \times (\bigcup_{s\in S} V_s) = \bigcup_{j\in J, s\in S}(U_j \times V_s)$, where

$(U_j \times V_s ,\otimes,\odot)$ are rings for each $j \in J$, $s \in S$. Therefore

(R $\times \bar{R}$, $\otimes$, $\odot$, $\tau \times \bar{\tau}$) is P–compact topological rings.

If we replace P–compact topological ring with PI–compact in theorems 3,4 the result is true since the product of ideals is also ideal (for instance see [3]) .

Theorem 5 . [1]

Let $\{R_i , i \in I\}$ be a family of topological rings . Then the direct product $R = \prod_{i\in I} R_i$, equipped with the product topology is topological rings.

From theorem 4 and theorem 5 , respectively , and by induction we can prove the following theorem

Theorem 6

The product of any finite collection of P–compact topological rings is P–compact topological ring.

If we replace P–compact topological ring with P–L. compact topological ring , the result is true .

Corollary 2.

If (R , ∗, . , τ) is a P – compact topological ring . Then (R^n , ⊗ , ⊙ , τ^n) is P – compact topological ring , where

$R^n = \frac{R \times R \times .....\times R}{n-time}$ and $\tau^n = \frac{\tau \times \tau \times .....\times \tau}{n-time}$

Theorem 7 .

Let (R , ∗ , . , τ) and ($\bar{R}$,$\bar{*}$,$\bar{.}$,$\bar{\tau}$) be two topological rings , and let f : (R , ∗ , . , τ) → ($\bar{R}$,$\bar{*}$,$\bar{.}$,$\bar{\tau}$) be a homomorphism. Then

1. If S is a P–compact topological subring in (R , ∗ , . , τ) , then f(S) is P–compact topological subring in ($\bar{R}$,$\bar{*}$,.,$\bar{\tau}$) .
2. If T is a P – compact topological subring in ($\bar{R}$,$\bar{*}$,$\bar{.}$,$\bar{\tau}$) and f is an isomorphism then f^{-1}(T) is P–compact topological subring in (R , ∗ , . , τ).

Proof .

Let $\{\bar{R}_i\}_{i \in I}$ be any P– cover topological rings of f(S) in ($\bar{R}$,$\bar{*}$,$\bar{.}$,$\bar{\tau}$) that is f(S) = $\cup_{i \in I} \bar{R}_i$. Now since S ⊆ $f^{-1}\left(f(S)\right)$ see [6] , implies S ⊆ $f^{-1}\,(\cup_{i \in I} \bar{R}_i)$ but $f^{-1}\,(\cup_{i \in I} \bar{R}_i) = \cup_{i \in I} f^{-1}\,(\bar{R}_i)$, see also [6], hence S ⊆ $\cup_{i \in I} f^{-1}\,(\bar{R}_i)$ on the other hand $f^{-1}\,(R_i)$ for each i ∈ I is a sub ring in R for enstance see [3, p. 186], and since S is P-compact and f continuous hence there exists a finite set J ⊂ I, such that

$S = \cup_{j \in J} f^{-1}\left(R_j\right) = f^{-1}(\cup_{j \in J} R_j)$ implies $f(S) = f\left(f^{-1}(\cup_{j \in J} R_j)\right)$. But $f\left(f^{-1}(\cup_{j \in J} R_j)\right) \subseteq \cup_{j \in J} R_j$ see [6] , i.e. $f(S) \subseteq U_{j \in J} R_j$.Thus $f(S)$ is P-compact topological sub ring in ($\bar{R}$,$\bar{*}$, $\bar{.}$, $\bar{\tau}$)

2- Let $\{R_i\}_{i \in I}$ be any P-cover topological rings of $f^{-1}\,(T)$ in (R , ∗ , . , τ) that is $f^{-1}\,(T) = \cup_{j \in I} R_i$, $R_i \in \tau$, $\forall\, i \in I$ implies T = $f(\cup_{i \in I} R_i) = f(\cup_{i \in I} R_i)$. It is clear that f (R_i) $\in \bar{\tau}$, $\forall\, i \in I$ since f is isomorphism (definition 4), but T is a P-compact topological subring of $(\bar{R}, \bar{*}, \bar{.}, \bar{\tau})$, so there is a finite subset $J \subseteq I$ such that $T = \cup_{j\in J} f(R_j)$ where $(f(R_j), \bar{*}, \bar{.})$ is a ring, $\forall j \in J$ see [3 , p. 186]. Thus $T = f(\cup_{j\in J} R_j)$ hence

$f^{-1}(T) = U_{j\in J} R_j$ and hence $f^{-1}(T)$ is P-compact topological subring of (R, ∗,.,τ).

For PI-compact ring we have the following theorem.

Theorem 8

Let (R , ∗ , . , τ) → $(\bar{R}, \bar{*}, \bar{.}, \bar{\tau})$ be an isomorphism, then

1. If S is a PI-compact topological ideal in (R ,∗ ,. , τ), then f (S) is PI-compact topological ideal in $(\bar{R}, \bar{*}, \bar{.}, \bar{\tau})$.
2. If T is a PI-compact topological ideal in $(\bar{R}, \bar{*}, \bar{.}, \bar{\tau})$, then $f^{-1}(T)$ is PI-compact topological ideal in (R, ∗ , . , τ).

Proof.

Let $\{I_i\}_{i\in\wedge}$ be any PI-cover topological ideal of f(S) in $(\bar{R}, \bar{*}, \bar{.}, \bar{\tau})$, that is $f(S) = \cup_{i\in\wedge} I_i \cup \{1\}, hence\ S = f^{-1}(\cup_{i\in\wedge} I_i) \cup \{1\}$, of course $f^{-1}(I_i)$, $\forall i$ are ideals see [3].

Also $f^{-1}(I_i) \in \tau$, $\forall i$ since f is isomorphism. Now S is PI compact ideal in (R , ∗ , . , τ), hence there exists a finite set J⊆∧ such that

$S = \cup_{j\in J} f^{-1}(I_j) \cup \{1\}$ implies $f(S) = f\left(f^{-1} \cup_{j\in J} I_j\right) \cup f\{1\}$, hence

$f(S) = \cup_{j\in J} I_j \cup \{1\}$, that means $f(S)$ is PI- compact ideal in $(\bar{R}, \bar{*}, \bar{.}, \bar{\tau})$.

2. Let $\{I_i\}_{i\in\wedge} \cup \{1\}$ be any PI-cover topological ideal of $f^{-1}(T)$ that is

$f^{-1}(T) = (\cup_{i\in\wedge} I_i) \cup \{1\}$, $\{I_i \in \tau, \forall i \in \wedge\}$ implies

$T = f(\cup_{i\in\wedge} I_i) \cup \{1\}$

$= \cup_{i\in\wedge} (f(I_i)) \cup \{1\}$

It is clear that $f(I_i) \in \bar{\tau}$, $\forall i \in \wedge$ since f is an isomorphism (definition 4), but T is PI-compact topological ideal in $(\bar{R}, \bar{*}, \bar{.}, \bar{\tau})$, so there is a finite subset $J \subseteq \wedge$ such that $T = \cup_{j\in J} f(I_j) \cup \{1\}$ where $(f(I_j), \bar{*}, \bar{.})$ are ideals for each j∈J see

[3, p.198] . Now $T = f(\cup_{j\in J}\ I_j) \cup \{1\}$, hence $f^{-1}\ (T) = U_{j\in J}\, I_j\ U\ \{1\}$ means $f^{-1}\ (T)$ is PI-compact topological ideal and we have done.

The following theorem show that the P-compact is topological property.

Theorem 9.

Let (R , ∗ , . , τ) and ($\bar{R}$, $\bar{*}$, $\bar{.}$, $\bar{\tau}$) be two topological rings and

f: (R , ∗ , . , τ) → ($\bar{R}$, $\bar{*}$, $\bar{.}$, $\bar{\tau}$) be an isomorphism, then the following are equivalents:

1- (R , ∗ , . , τ) is P-compact topological ring.
2- ($\bar{R}$, $\bar{*}$, $\bar{.}$, $\bar{\tau}$) is P-compact topological ring.

Proof.

(⇒) suppose that (R , ∗ , . , τ) is P-compact topological ring, let $\{R_i\}_{i\in I}$ be any P-cover topological rings of ($\bar{R}$, $\bar{*}$, $\bar{.}$, $\bar{\tau}$) , that is $\bar{R} = \cup_{i\in\wedge} \bar{R}_i$ gives

$R = f^{-1}(\bar{R}) = f^{-1}\ (\cup_{i\in\wedge} \bar{R}_i) = \cup_{i\in\wedge} f^{-1}(\bar{R}_i)$. But (R , ∗ , . , τ) is P-compact topological ring, so there is a finite subset J∈Λ , such that $R = \cup_{j\in J}\ f^{-1}(\bar{R}_j)$. Clear that $\left(f^{-1}(\bar{R}_j), *, .\right)$ is subring $\forall\, j \in J$, hence $R = f^{-1}(\cup_{j\in J} \bar{R}_j)$ implies $\bar{R} = f(R) = f\left(f^{-1}\left(\cup_{j\in J} \bar{R}_j\right)\right) = \cup_{j\in J} \bar{R}_j$

therefore ($\bar{R}$, $\bar{*}$, $\bar{.}$, $\bar{\tau}$) is P-compact topological ring.

(⇐) suppose that ($\bar{R}$, $\bar{*}$, $\bar{.}$, $\bar{\tau}$) is a P-compact topological ring, let $\{R_i\}_{i\in\wedge}$ be any P-cover topological rings of (R , ∗ , . , τ), i.e. $R = \cup_{i\in\wedge} R_i$.

Clear that $\bar{R} = f(R) = f(\cup_{i\in\wedge} R_i) = \cup_{i\in\wedge} f(R_i)$ where $(f(R_i), \bar{*}, \bar{.})$ is a ring $\forall i \in\wedge$ see [3, p. 198], and since f is isomorphism (definition 4) implies $f(R_i) \in \bar{\tau}, \forall i \in \wedge$. But ($\bar{R}$, $\bar{*}$, $\bar{.}$, $\bar{\tau}$) is P-compact so there is a finite subset $J \subseteq \wedge$ such that $\bar{R} = \cup_{j\in J} f(R_j)$.

Now

$$R = f^{-1}(\bar{R}) = f^{-1}\left(\cup_{j\in J} f(R_j)\right) = f^{-1}\left(f\left(\cup_{j\in J} R_j\right)\right) = \cup_{j\in J} R_j\ .$$

Thus (R , ∗ , . , τ) is P-compact which complete the proof.

We can prove by the similar way the following theorem.

Theorem 10.

Let (R , $*$, . , τ) and ($\bar{R}$, $\bar{*}$, $\bar{.}$, $\bar{\tau}$) be two topological rings and

f : (R , $*$, . , τ) $\rightarrow$ ($\bar{R}$, $\bar{*}$, $\bar{.}$, $\bar{\tau}$) be an isomorphism. Then the following are equivalents

1- (R , $*$, . , τ) is PI-compact topological ring.
2- ($\bar{R}$, $\bar{*}$, $\bar{.}$, $\bar{\tau}$) is PI-compact topological ring.

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