Variational Iteration Method for Mixed Type Integro-Differential Equations

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Abstract: In this paper, we approximate Lagrange multipliers to solve integro differential equations of mixed type those are linear first and second order. It is observed that use of approximate Lagrange multipliers reduces the iteration and give faster results as compare to other techniques. Numerical examples support this idea.

Keywords Approximate Lagrange Multiplier, Variational Iteration Method, Mixed Type Integro-Differential Equations.

1. Introduction

Variational iteration method (VIM) was first developed by J.H He in 1999. VIM rapidly converge the approximations and solves linear and non linear problems in same way. The variational iteration method was successfully applied to seventh order Sawada-Kotera equations [1], to Schruodinger-KdV, generalized KdV and shallow water equations [2].it also applies on PDEs and IEs. The VIM basically consisted restricted variations and correct functional which has found a wide application for the solution of nonlinear problems [3-8].

In present work we formulate the approximate Lagrange multiplier and analysis of VIM and also we apply VIM using approximate Lagrange multiplier on mixed type integro differential equations of first order and second order.

2. Analysis of Variational Iteration method (VIM)

We consider the following general differential equation

$$Lu + Nu = g(t) \qquad (1)$$

Where L is the linear differential operator N is non linear operator and *g(t)* is the known function. We can construct the correction functional according to variational iteration method as

$$u_n(t) = u_n(t) + \int_0^t \lambda(\mathrm{s})\{\mathrm{Lu_n(s)} + \mathrm{\tilde{N}u_n(s)} - \mathrm{g(s)}\}\mathrm{ds} \tag{2}$$

The above Eq. (2) is called the correction functional of the VIM. Where λ is the Lagrange multiplier which can be determine optimally.

3. Formulation of Approximate Lagrange Multiplier

The above Eq. (2) takes the form

$$u_{n+1}(t) = u_n(n) + \int_0^t \lambda s\{Lu_n(s)\} + \tilde{N}u_n(s) - \tilde{g}(t)\}ds \tag{3}$$

Applying restriction on both sides of equation (3) we get

$$\delta u_{n+1}(t) = \delta u_n(t) + \delta \int_0^t \lambda(s)\{Lu_n(s) + \tilde{N}u_n(s) - \tilde{g}(s)\}\, ds$$

Now we apply δ and restrict the non linear term and the forcing function. Then solve the linear term by parts integration to fine out the value ofλ

$$\delta u_{n+1}(t) = \delta u_n(t) + \delta \int_0^t \lambda(s) Lu_n(s) ds$$

$$\delta u_{n+1}(t) = \delta u_n(t) + \delta \int_0^t \lambda u_n'(s)\, ds$$

$$\delta u_{n+1}(t) = u_n(t) + \delta[\lambda(s)u_n(s) - \int_0^t u_n(s)\lambda'(s)ds]$$

$$\delta u_{n+1}(t) = \delta u_n(t) + \delta\lambda(s)u_n(s) - \int_0^t u_n(s)\lambda'(s)ds$$

$$0 = \delta u_n(t) + \delta\lambda(t)u_n(t) - \delta \int_0^y u_n(s)\lambda'(s)ds$$

$$0 = \big(1 + \lambda(s)\big)\delta u_n(t) - \delta \int_0^t u_n(s)\lambda'(s)\, ds.$$

Equating the terms on both sides

$$\begin{Bmatrix} \delta u_n(t); & 1 + \lambda(s) = 0 \\ \delta u_n(t); & 0 = \lambda'(s) \end{Bmatrix} \tag{4}$$

Equation (4) is the stationary conditions. By using first stationary condition

$$1 + \lambda(s) = 0, \lambda(s) = 1.$$

By using second stationary condition

$$\lambda(s) = -1$$

So the value of $\lambda(s)$ for first order differential equation is always -1, for second order $\lambda(s) = s - t$, for third order equation $\lambda(s) = -\frac{1}{2}(s-t)^2$ and so on.

4. Numerical Applications

In this section we have applied the method presented in this paper to three examples to show the efficiency of the approach.

Example 4.1 Consider the following Volterra- Fredholm integro- differential equation [9-10,16]

$$u'(x) = 6 - 2x - {}^1\!/_2\, x^2 - x^3 + \int_0^x (x-t)u(t)dt + \int_{-1}^1 xu(t)dt \quad (5)$$

u (0) = 1

Eq. (5) can be written as

$$u'(x) = 6 - 2x - {}^1\!/_2\, x^2 - x^3 + \int_0^x (x-t)u(t)dt + \alpha x \quad (6)$$

Where

$$\alpha = \int_{-1}^1 u(t)dt \quad (7)$$

The correction functional for the Volterra- Fredholm integro-differential equation is given by

$$u_{n+1}(x) = u_n(x) - \int_0^x [u_n'(t) - 6 + 2t + {}^1\!/_2\, t^2 + t^3 - \int_0^t (t-r)u(r)dr - \alpha t]dt$$

(8)

Select as initial approximate solution as

$$u_0(x) = 1,$$

Consequently, we have

$$u_1 = 1 + 6x - x^2 + \alpha\, {}^{x^2}\!/_2 - {}^{x^4}\!/_4,$$

$$u_2(x) = 1 + 6x - x^2 + {\alpha x^2}/_2 - {x^5}/_{60} + {\alpha x^5}/_{120} - {x^7}/_{840}, \quad (9)$$

and so on. We Eq. (9) and (7), $\alpha = 2$

From Eq. (9), the solution is

$$u(x) = 1 + 6x.$$

Example 4.2 Consider the following Volterra- Fredholm integro- differential equation [9-10,16]

$$u''(x) = -x - {1}/_{6} x^3 + \int_0^x (x-t)u(t)dt + \int_{-\pi}^{\pi} xu(t)dt, \quad (10)$$

$$u(0) = 0, u'(0) = 2$$

It can be written as

$$u''(x) = -x - \frac{1}{6}x^3 + \int_0^t (x-t)u(t)dt + \alpha t, \quad (11)$$

Where

$$\alpha = \int_{-\pi}^{\pi} u(t)dt \quad (12)$$

The correction functional for Volterra Fredholm integro- differential equation is given by

$$u_{n+1}(x) = u_n(x) + \int_0^t (t-x)\{u_n''(t) + t + \frac{1}{6}t^3 - \int_{-\pi}^{\pi}(t-r)u_n(r)dr - \alpha t\}\, dt, \quad (13)$$

Here we use $\lambda = (t-x)$ for second order integro-differential equation

and we select $u_0(x) = u(0) + u'(0)x = o + 2x = 2x$ to find out the approximations.

Consequently, we have

$$u_0(x) = 2x$$

$$u_1(x) = u_0(x) + \int_0^x (t-x)\{u_0''(t) + t + \frac{1}{6}t^3 - \alpha t - \int_{-\pi}^{\pi}(t-r)u_0(r)dr\}dt$$

$$u_1(x) = 2x - \frac{x^3}{6} + \frac{\alpha x^3}{6} + \frac{x^5}{120}$$

$$u_2(x) = 2x - \frac{x^3}{3!} + \frac{\alpha x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{\alpha x^7}{7!} + \frac{x^9}{9!} \quad (14)$$

and so on. From Eq. (14) and (13) $\alpha = 0$

The closed solution from Eq. (14) is

$$u(x) = x + \sin x.$$

Example 4.3 Consider the following Volterra- Fredholm integro- differential equation [9-10,16]

$$u'(x) = 6 + 4x - x^3 + \int_0^x (x-t)u(t)dt + \int_{-1}^1 (1-xt)u(t)dt, \quad (15)$$

$$u(0) = 0$$

Eq. (15) can be written as

$$u'(x) = 6 + 4x - x^3 + \int_0^x (x-t)u(t)dt + \alpha - \beta x, \quad (16)$$

where

$$\alpha = \int_{-1}^1 u(t)dt\,, \qquad \beta = \int_{-1}^1 tu(t)dt \quad (17)$$

The correction functional for Volterra Fredholm integro- differential equation is given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \{u_n'(t) - 6 - 4t + t^3 - \int_0^t (t-r)u_n(r)dr - \alpha + \beta t\}dt, \quad (18)$$

Here $\lambda = -1$ for first order integro-differential equation.

We select $u_0(x) = u(0) = 0$,

Consequently, we have

$$u_1(x) = 6x + \alpha x + 2x^2 - \beta\frac{x^2}{2} - \frac{x^4}{4},$$

$$u_2(x) = 6x + \alpha x + 2x^2 - \beta\frac{x^2}{2} + \alpha\frac{x^4}{24} + \frac{x^5}{30} - \beta\frac{x^5}{120} - \frac{x^7}{840}, \quad (19)$$

and so on. One have

$$\alpha = 0,\ \beta = 4$$

Using the values of α and β in Eq. (19), the closed form solution is

$$u(x) = 6x.$$

5. Conclusion

In this work, we have successfully applied the proposed VIM on three problems. The obtained numerical results proved that the VIM gives more rapidly convergence to the exact solution. By appropriate choosing initial approximate solution we have gotten approximate of the exact solution of the problem with little iteration.

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