

BOUNDS OF SOLUTIONS OF DUFFING'S EQUATION

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Abstract.

In this paper, a sufficient criteria is given to determine the bounds of solutions for a Duffing's type equation using fixed point theorem of Schauder and augmented by Schaefer's Lemma.Our results include and improve some well-known results in literature

1. INTRODUCTION

The existence of Solutions to ordinary differential equations is very much connected with fixed point theorem.A good number of cases arise in literature where the Schauder fixed point theorem has been used to generalized the cases of solution to non-linear differential equations, for instances Ezeilo [1,2]Ressig[3,4]Ressig, et al [5] Villari [6]. Stopelli [7] was the first to use Schauder fixed point technique for the second order differential equation. The application of Schauder fixed point for third order differential equation and above was due to Ezeilo [8] which was concerned with the consequence rather than the topological nature of the result of Schaefer [9]. On the use of Schauder fixed point and integrated equation as a mode to establish aprior bounds for non-linear differential equation see Ogbu [10] and Ezeilo[8]

One of the reasons for the search of such bounds of solutions to any differential equation is to achieve global optimum of the solution in a closed and bounded domain. Many results using different techniques on bounds of solutions of Duffing's equation have been obtained by many authors. For instance see Ding [11], Littlewood [12], Markus [13], Moser [14], Voitovich [15], Lim and Kazda[16],Zanolin and Liu [17] and Daxiong [18]. Morris [19] proved that each solution of special Duffing's equation of the form:

$$\ddot{x} + 2x^3 = p(t) \tag{1.1}$$

is bounded for $t \in \mathbb{R}$. In this paper, we will introduce damping and stiffness terms into (1.1) using different approaches to investigate the bounds of solutions. The Duffing's equation is a Hamiltonian equation of motion that is characterized with multiple periodic solutions. It is suggested in Tamas [20] that the presence of the cubic non-linear term is responsible for the presence of several periodic solutions. Therefore trying to investigate the bounds of the solution of such equation is somehow complicated but possible.

Motivated by the potential application in Physics, Engineering, Biology and Communication theory, the purpose of this paper is to investigate bounds of solutions of Duffing's equation of the form:

(1.2) $$\ddot{x} + c\dot{x} + ax + bx^2 + 2x^3 = p(t)$$

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wherea, b, c are real constants and $p : [0, 2\pi] \to R^n$ is continuous with

(1.3) $$x(0) = x(2\pi)$$

$\dot{x}(0) = \dot{x}(2\pi)$

Using Schauder theorem and Scheafer's Lemma.

In equation (1.2) a, is the stiffness constant, c is the coefficient of viscous damping and $bx^2 + 2x^3$ represent the non-linearity in the restoring force acting like a hard spring.

Equation (1.2) has received wide interest in neurology, ecology, secure communications, cryptography, chaotic synchronization and so on. Due to the rich behavior of this equation, recently there have been also several studies on the synchronization of two double Duffing equation.Njah and Vincent [21]. Equation (1.2) is a model arising in many branches of Physics and Engineering such as oscillation of rigid pendulum using moderately large amplitude motion, see Jordan and Smith [22]. Vibration of buckled beam, for instance see Thompson and Stewart [23], Rezeshki and Dowell [24]. This oscillation involves an electromagnetized vibrating beam analyzed as exhibiting cusp catastrophic behavior for certain parameter values, Zeeman [25] as cited in Guastello [26]. This equation together with Van der Pol's equation, has become one of the most common examples of nonlinear oscillation in textbooks and research articles, Puu [27], Ueda [28] and Zhang [29]

2. PRELIMINARIES

Definition 2.1. A functional series of the form

(2.1) $$\frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$$

or more compactly, a series of the form

(2.2) $$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is called a Fourier series. The constant $a_0, a_n$and $b_n (n = 1, 2, \dots)$ are called coefficient of the Fourier series. If series (2.2) converges, then its sum is a periodic function $f(x)$ with period 2π, since $\sin nx$ and $\cos nx$ are periodic functions with period 2π. Thus $f(x) = f(x + 2\pi)$

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Definition 2.2. Let$x_i, y_i \in \Omega$, $i = 1, 2, \dots, n$, then Cauchy Schwartz's inequality for finite sum is given as

(2.3) $$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum |x_i|^2\right)^{1/2} \left(\sum |y_i|^2\right)^{1/2}$$

Theorem 2.3. Suppose there exist $a > 0, b > 0$ and $\beta > 0$ such that

(i) $h^I(x) < b, \beta^2 = b$

(ii) $|h(x) - x| > 0$for all x

(iii) $|h(x)| \to \infty$ as$|x| \to \infty$

(iv) $x^2 + y^2 \to \infty$as$|x| \to \infty,\ |y| \to \infty$ then (1.1) through (1.2) has stable bounded and periodic solution when $p(t) = 0$

Theorem 2.4. Suppose further in theorem (2.3) that condition (i) is replaced by

1. $h^I(x) < b, \beta^2 \neq band|a\dot{x} - p(t)| > 0$ then (1.1) and (1.2) have stable bounded and periodic solution when $p(t) \neq 0$.

Theorem 2.5.(Lerary Schauder fixed point theorem) Let C be a closed convex subset of the Banach space X. Suppose$f: C \to C$ and f is compact i.e. (bounded set in C are mapped into relatively compact sets). Then f has a fixed point in C.

Proof. $f(C)$is relatively compact, so $k = \overline{f(C)}$ is compact. For each $\varepsilon > 0$, there exist a finite $\in -$ net for k. Let F $= \{x_1, \dots, x_n\}$ be this finite $\in -$ net (note that n is dependent on ε). We show that the equation $f(z) = z$ is approximately solvable in C. That is, we show there exists $x_0 \in C$with$|x_0 - f(x_0)| < \varepsilon$. Consider the mapping $g := PoFg$ maps C into con$F$ and so if we restrict g to con$F$, then since C is convex, con$F \subset C$, we have that g: con$F \to$ con$F$. By Brouwer's theorem and its corollaries (con$F$ is compact, convex and finite dimensional), there exists a $x_0 \in$ con$F$ with $g(x_0) = x_0$. But then

(2.4) $$|x_0 - f(x_0)| = |g(x_0) - f(x_0)| = \left|p\big(f(x_0)\big) - f(x_0)\right| < \varepsilon$$

where the last step is because of proposition (2.6).

Thus $f(z) = z$ is approximately solvable in C.So, we know that there exist a fixed point $\hat{x} \in C$ with $\hat{x} = f(\hat{x})$.

Proposition 2.6. Let k be a compact subset of the Banach space X. Then given $\varepsilon > 0$, there exists a finite subset $F \subset X$ and a mapping $P : K \to$ con$F$ such that for any $x \in K$ we have $\big(x, P(x)\big) = \|x - P(x)\| < \varepsilon$.

Proof. See Bockelman [30].

Lemma 2.7. (The Banach fixed point theorem) Let E be a Banach space and $f : E \to E$ is a contraction mapping, then f has a unique fixed point in E, i.e. there exist a unique$x \in E$ such that $f(x) = x$.

Theorem 2.8. Let$X$ be a Banach space and $f: X \to X$ be completely continuous then either there exists for each $\lambda \in [0, 1]$ one small $x \in X$ such that $x = \lambda f(x)$ or the set $\{x \in X: x = \lambda f(x), 0 < \lambda < 1\}$ is bounded in X.

Proof. See Scheafer [9].

Proposition 2.9. Let $G \subset \mathbb{R}^2$ be a connected open set and let $z_0 \in G$ be a fixed point of Φ. Let $r > |z_0|$ and assume that

(2.5)$\Phi(\bar{G}) \cap S_r \neq \emptyset$

where$\bar{G}$ denotes the closure of G. Then we have $\partial G_r^I \cap S_r \neq \emptyset$.

Proof. See [12].

3. RESULTS

We consider the more general form of Duffing's equation (1.2) as a parameter λ dependent equation of the form

(3.1) $$\ddot{x} + c\dot{x} + h_\lambda(x) = \lambda p(t)$$

where$h_\lambda(x) = (1-\lambda)ax + \lambda h(x)$ and $\lambda h(x) = bx^2 + 2x^3$

λassociated with the generic forcing term p measures the strength of the force. $\lambda$when associated with the non-linearity is the perturbed term. $\lambda$is the range of $0 \leq \lambda \leq 1$ and b is a constant satisfying c$> 0, b > 0$. When λ associated with the generic forcing term is sufficiently large, equation (3.1) has a T periodic solutions, see Katriel [31]. Using some classical critical assumption, the existence of solution of all λ is a well-known application of Degree theorem or Schauder fixed point theorem, see Habets and Metzon [32].

The equivalent systems of (3.1) is given by

(3.2) $$\dot{x} = y$$

$$\dot{y} = -cy - h_\lambda(x) + \lambda p(t)$$

Let $X(t)$ be a possible 2π periodic solution of (3.1)

The main tool here is the verification of the function $W(x,y)$ defined by

(3.3) $$W(x\ y) = {}^1/_2 y^2 + H_\lambda(x) \text{ where } H_\lambda(x) = \int_0^x h_\lambda(s)ds$$

The time derivative $\dot{W}$ of (3.3) along the solution paths of (3.2) is

$$\dot{W} = y\dot{y} + h_\lambda(x)\dot{x}$$

$$= -cy^2 - h_\lambda(x)y + \lambda p(t)y + h_\lambda(x)y$$

(3.4) $$= -cy^2 + \lambda p(t)y$$

Integrating (3.4) with respect to t from $t = 0$ to $t = 2\pi$ we have

(3.5) $$\int_0^{2\pi} \dot{W}\, dt = \int_0^{2\pi} -c\dot{x}^2\, dt + \int_0^{2\pi} \lambda p(t)\dot{x}\, dt$$

$$[W(t)]_0^{2\pi} = -\int_0^{2\pi} c\dot{x}^2\, dt + \lambda \int_0^{2\pi} p(t)\dot{x}\, dt$$

Since $W(0) = W(2\pi)$, it implies that $[W(t)]_0^{2\pi} = 0$ which is 2π periodic solution.

Equation (3.4) and (3.5) shows that the behaviour of Duffing's equation is stable, asymptotically stable, bounded and periodic.

Thus $$0 = -\int_0^{2\pi} c\dot{x}^2\, dt + \lambda \int_0^{2\pi} p(t)\dot{x}\, dt$$

(3.6) $$\int_0^{2\pi} c\dot{x}^2\, dt \leq |\lambda||p(t)| \int_0^{2\pi} \dot{x}\, dt$$

Since $|\lambda| \leq 1$ and $p(t)$ is continuous, then

(3.7) $$\int_0^{2\pi} \dot{x}^2\, dt \leq C_1(2\pi)^{1/2}\left(\int_0^{2\pi} \dot{x}^2\, dt\right)^{1/2}$$

By the hypothesis of Schwartz's inequality we have

$$\left(\int_0^{2\pi} \dot{x}^2 dt\right)^{1/2} \leq C_1(2\pi)^{1/2} \equiv C_2$$

(3.8) $$\left(\int_0^{2\pi} \dot{x}^2 dt\right)^{1/2} \leq C_2$$

Now since $x(0) = x(2\pi)$, it is clear that there exists $\dot{x}(T) = 0$ for $T \in [0, 2\pi]$

Thus using the identity $\dot{x}(t) = \dot{x}(T) + \int_0^{2\pi} \ddot{x}^2\,(s)ds = \int_0^{2\pi} \ddot{x}^2\,(s)ds$

By Schwartz's inequality we have

(3.9) $$Max_{0\leq t\leq 2\pi}|\dot{x}(t)| \leq \int_0^{2\pi}|\ddot{x}(t)|\, dt \leq (2\pi)^{1/2}(\int_0^{2\pi} \ddot{x}^2\,(s)ds)^{1/2}$$

We hereby invoke the Fourier expansion of X$\sim \sum_{r=0}^{\infty}(ar\cos 2\pi l + br\sin 2\pi l)$

For the derivative of Fourier expansion see Ezeilo and Onyia [3].

From (3.6) we obtain $\int_0^{2\pi} \dot{x}^2 dt \leq |\lambda||p(t)| \int_0^{2\pi} \dot{x}\, dt$

which is

(3.10) $$\int_0^{2\pi} \ddot{x}^2 dt \leq C_1(2\pi)^{1/2}(\int_0^{2\pi} \ddot{x}^2 dt)^{1/2}$$

Therefore we have $\left(\int_0^{2\pi} \dot{x}^2 dt\right)^{1/2} \leq C_1(2\pi)^{1/2} \equiv C_2$

$$Max_{0\leq t\leq 2\pi}|\dot{x}(t)| \leq (2\pi)^{1/2} . C_1(2\pi)^{1/2} \equiv C_3$$

(3.11) $$|\dot{x}|_\infty \leq C_3$$

Now integrating (3.1) with respect to t from $t = 0$ to $t = 2\pi$, we obtain

(3.12) $\int_0^{2\pi} \ddot{x}\, dt + \int_0^{2\pi} c\dot{x}\, dt + \int_0^{2\pi} h_\lambda(x)\, dt = \int_0^{2\pi} \lambda p(t)\, dt$

Substituting for $h_\lambda(x) = (1-\lambda)ax + \lambda h(x)$in (3.12) we have

$$\int_0^{2\pi} \ddot{x}\, dt + \int_0^{2\pi} c\dot{x}\, dt + \int_0^{2\pi} (1-\lambda)ax\, dt + \int_0^{2\pi} \lambda h(x)\, dt = \int_0^{2\pi} \lambda p(t)\, dt$$

(3.13) $\int_0^{2\pi} (1-\lambda)ax\, dt + \int_0^{2\pi} \lambda h(x)\, dt = \int_0^{2\pi} \lambda p(t)\, dt$

The continuity of $p(t)$ assures us of boundedness and the fact that $0 \le \lambda \le 1$, the right hand side

of(3.13) is bounded. ie

(3.14) $\left|\int_0^{2\pi} \lambda p(t)\, dt\right| \le C_4$

$$\left|\int_0^{2\pi} (1-\lambda)ax\, dt + \int_0^{2\pi} \lambda h(x)\, dt\right| \le C_4$$

Therefore given $\alpha > 0$ there exist $\mu > 0$ such that for $T \in [0, 2\pi]$

(3.15) $|x(T)| \le C_5$

Substituting for $T = 0$in (3.15) shows that Duffing's equation is bounded

Suppose $x(T) \neq 0$for any T then (3.14) yields

(3.16) $\int_0^{2\pi} (1-\lambda)a|x|\, dt + \int_0^{2\pi} |\lambda||h(x)|\, dt > \int_0^{2\pi} (1-\lambda)a\mu\, dt + \int_0^{2\pi} \lambda\alpha(x)\, dt$

$> 2\pi(1-\lambda)a\mu + 2\pi\lambda\alpha$

But equation (3.16) implies that $\int_0^{2\pi} (1-\lambda)ax\, dt + \int_0^{2\pi} \lambda h(x)\, dt$is no more bounded which is a negation of (3.14). Thus $|h(x)| \to \infty$as $|x| \to \infty$, equation (3.15) and the identity

$X(t) = X(T) + \int_T^t \dot{x}\, dt$holds.

Thus$Max_{0 \le t \le 2\pi} |\dot{x}(t)| \le |x(T)| + \int_0^{2\pi} |\dot{x}(t)| dt \le C_5 + (2\pi)^{1/2} (\int_0^{2\pi} \ddot{x}^2 dt)^{1/2}$

$\le C_5 + (2\pi)^{1/2} . C_2$ (by Schwartz's inequality and equation (3.7))

$|x|_\infty \le C_5 + (2\pi)^{1/2} \cdot C_2 \equiv C_6$and we finally obtain

(3.17)$|x|_\infty \le C_6$

Equation (3.11) and (3.17) establishes the bounds of solution. This bounds show the limit of intervals where the solution of our differential system will lie. Using the Scheafer's lemma, the actual lower limit point and upper limit point which form the bounds of the solution were determined. One of the advantages of this method is that it helps to determine the unique point within these bounds the solution of our differential equation will exist. This unique point is the optimal point which coincides with the solution of our Duffing's equation.

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