Reliability Estimation for a Component Exposed Two, Three Independent Stresses Based on Weibull and InversLindley Distribution

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Abstract
The reliability function for a component which has strength independently exposed two stresses ;R_1, also when exposed three stresses; R_2 using Weibull distribution with unknown scale and known shape parameters ,and using Invers Lindley distribution . Estimate the reliability R_1, R_2 for Weibull distribution by four methods (MLE,MOM,LSE and WLSE) and also in the numerical simulation study a comparison between the four estimates by MES ,MAPE are introduced.

Keywords: Weibull , Invers Lindley distribution , stress-strength, reliability estimation, MLE, MOM , LSE and WLS estimation.

1. Introduction
Acquire term stress especially in the contemporary human life the importanceahead of the second half of the twentieth century, we are all exposed in our daily lives to pressures or stresses constant psychological and variable On the other hand We do not have, enough strength or durability (Strength) to overcome these stresses or psychological stress, from this point has become a term stress - durability subject of study and research in the it humanities and psychology and genetics by trying to researchers give an explanation of the nature of the relationship between stress and the ability to afford it(AL- Badran 2014).a component which has strength independently exposed to two stresses studied by Hanagal & Karaday et al (Hanagal 1999), (Karaday et 2011).
The Weibull distribution is attributed to the Swedish physicist Waloddi Weibull which is derived and used this distribution in (1939) to study the properties of the industrially produced number (Ghanim 2015) it's also considered as one of the distributions that applied in many fields such as industrial engineering to represent replaced and manufacturing time (Laazm 2011)
The cdf of W(θ, α)is : $F(x)$ =1$-e^{-\frac{x^\theta}{\alpha}}$ $x > 0; \theta, \alpha > 0$ Where θ, α are shape and scal parameters respectively.Its PDF is:$f(x) = \frac{\theta}{\alpha}\, x^{\theta-1}\, e^{-\frac{x^\theta}{\alpha}} x > 0; \theta, \alpha > 0$.
The inverse Lindely is continuous distribution considering the fact that all inverse distribution possess the upside-down bathtub shape for their hazard rates, we ,in this article,proposed a inverted version of the Lindely distribution that can be effectively used to model the upside- down bathtub shape hazard rates data. if random variable Y has a Lindely distribution LD(τ),then the random variable X=(1/Y) is said to be follow the inverse Lindely distribution having a scale parameter with its probability density function (Pdf), denoted by$f(x) = \frac{\tau^2}{1+\tau}\left(\frac{1+x}{x^3}\right) e^{-\frac{\tau}{x}}; x > 0, \tau > 0$ andthe cumulative distribution function, cdf is: $F(x) = \left(1 + \frac{\tau}{1+\tau}\frac{1}{x}\right) e^{-\frac{\tau}{x}}$ $x > 0, \tau > 0$ (Sharma et al 2014).
The main aim of this article is to discuss the derivation of the mathematical formula of reliability in case component has one strength and exposed two independent stress $\boldsymbol{R_1}$**,** for weibull, inverse Lindely distribution also when case component has one strength its exposed three independent stress $\boldsymbol{R_2}$ for weibull, inverse Lindely distribution then estimation $\boldsymbol{R_1}$ **,** $\boldsymbol{R_2}$ for weibull distribution by using MLE,MOM,LSE and WLSE methods, and comparison among the results of the estimation methods by using mean square error (MSE) and mean absolute percentage error (MAPE), that will get from a simulation study.

2.Two Stress- one Strength Component Reliability
when a component exposed to two independent Y_i,i=1,2 stresses then the stress-strength reliabilit is $R_1 = P(Max(Y_1, Y_2) < X)$,in section we will find Theoretical Expression of R_1 for weibull and inverse Lindely distribution.

2.1 For weibull distribution

Let the strength random variable of the componen represented by X as a $W(\theta,\alpha)$, and the component subjected to two stress random variables are represented by Y_i , i = 1,2 following weibull distribution with the parameters $W(\theta,\alpha_i);i = 1,2$.Probability density functions (pdf) and cumulative distribution functions cdf of the random variables are given as:

$$f(x)=\frac{\theta}{\alpha}\,x^{\theta-1}\,e^{-\frac{x^{\theta}}{\alpha}}\quad x>0;\theta,\alpha>0 \tag{1}$$

$$F_1(y_1)=1-e^{-\frac{y_1{}^{\theta}}{\alpha_1}}\quad y_1\geq 0,\theta,\alpha_1>0 \tag{2}$$

$$F_2(y_2)=1-e^{-\frac{y_2{}^{\theta}}{\alpha_2}}\quad y_2\geq 0,\theta,\alpha_2>0 \tag{3}$$

Hence , the model reliability of such a component, R_1,isgivenby

$$R_1=\int_{x=0}^{\infty}\int_{y_1=0}^{\infty}\int_{y_2=0}^{\infty}f(y_1,y_2,x)\,dy_1dy_2dx$$

Since the r. vs are non- identical independently distributed, then:

$$R_1=\int_{x=0}^{\infty}F_{1y_1}(x)F_{2y_2}(x)f(x)dx \tag{4}$$

the stress- strength reliabilityR_{1w} of weibull can be obtained by substitution (1) , (2) and(3) in (4), as:

$$R_{1w}=\int_{x=0}^{\infty}\left(1-e^{-\frac{x^{\theta}}{\alpha_1}}\right)\left(1-e^{-\frac{x^{\theta}}{\alpha_2}}\right)\frac{\theta}{\alpha}\,x^{\theta-1}\,e^{-\frac{x^{\theta}}{\alpha}}$$

$$=1-\int_{x=0}^{\infty}\frac{\theta}{\alpha}\,x^{\theta-1}e^{-\left(\frac{1}{\alpha_1}+\frac{1}{\alpha}\right)x^{\theta}}dx-\int_{x=0}^{\infty}\frac{\theta}{\alpha}\,x^{\theta-1}e^{-\left(\frac{1}{\alpha_2}+\frac{1}{\alpha}\right)x^{\theta}}dx+\int_{x=0}^{\infty}\frac{\theta}{\alpha}\,x^{\theta-1}\,e^{-\left(\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\frac{1}{\alpha}\right)x^{\theta}}dx$$

By transformation:, we get the final Expression of R_1as:

$$\boldsymbol{R_{1W}}=1-\frac{\alpha_1}{\alpha_1+\alpha}-\frac{\alpha_2}{\alpha_2+\alpha}+\frac{\alpha_1\alpha_2}{\alpha_2\alpha+\alpha_1\alpha+\alpha_1\alpha_2} \tag{5}$$

2.2 For Inverse Lindely distribution

Let X the strength random variable of inverse Lindely with parameter (τ), and Y_i , i = 1,2 the stress random variables following inverse Lindely with the parameters (τ_i); $i = 1,2$. the Probability density functions (pdf) and cumulative distribution functions cdf of the random variables are given

$$f(x)=\frac{\tau^2}{1+\tau}\left(\frac{1+x}{x^3}\right)e^{-\frac{\tau}{x}};x>0,\tau>0 \tag{6}$$

$$F_1(y_1)=\left(1+\frac{\tau_1}{1+\tau_1}\frac{1}{y_1}\right)e^{-\frac{\tau_1}{y_1}},y_1>0,\tau_1>0 \tag{7}$$

$$F_2(y_2)=\left(1+\frac{\tau_2}{1+\tau_2}\frac{1}{y_2}\right)e^{-\frac{\tau_2}{y_2}},y_2>0,\tau_2>0 \tag{8}$$

the stress- strength reliability R_{1ILD} of Invers Lindley can be obtained by substitution (6) , (7) and(8) in (4), as:

$$R_{1IL}=\int_{x=0}^{\infty}\left[\left(1+\frac{\tau_1}{1+\tau_1}\frac{1}{x}\right)e^{-\frac{\tau_1}{x}}\right]\left[\left(1+\frac{\tau_2}{1+\tau_2}\frac{1}{x}\right)e^{-\frac{\tau_2}{x}}\right]\frac{\tau^2}{1+\tau}\left(\frac{1+x}{x^3}\right)e^{-\frac{\tau}{x}}\,dx$$

$$=\int_{x=0}^{\infty}\frac{\tau^2}{1+\tau}\left(\frac{1+x}{x^3}\right)e^{\frac{-(\tau_1+\tau_2+\tau)}{x}}+\int_{x=0}^{\infty}\frac{\tau_1\,\tau_2\tau^2}{(1+\tau_1)(1+\tau_2)(1+\tau)}\frac{1}{x^2}\frac{(1+x)}{x^3}\quad e^{\frac{-(\tau_1+\tau_2+\tau)}{x}}dx$$

$$\int_{x=0}^{\infty}\frac{\tau_1\tau^2}{(1+\tau_1)(1+\tau)}\frac{1}{x}\frac{(1+x)}{x^3}e^{\frac{-(\tau_1+\tau_2+\tau)}{x}}dx+\int_{x=0}^{\infty}+\frac{\tau_2\tau^2}{(1+\tau_2)(1+\tau)}\frac{1}{x}\frac{(1+x)}{x^3}e^{\frac{-(\tau_1+\tau_2+\tau)}{x}}dx$$

Let , $\int_{x=0}^{\infty}\frac{\tau^2}{1+\tau}\left(\frac{1+x}{x^3}\right)e^{\frac{-(\tau_1+\tau_2+\tau)}{x}}=\frac{\tau^2}{1+\tau}\;\frac{1+(\tau_1+\tau_2+\tau)}{(\tau_1+\tau_2+\tau)^2}$

$$\int_{x=0}^{\infty}\frac{\tau_1\,\tau_2\tau^2}{(1+\tau_1)(1+\tau_2)(1+\tau)}\frac{1}{x^2}\frac{(1+x)}{x^3}\quad e^{\frac{-(\tau_1+\tau_2+\tau)}{x}}=\frac{\tau_1\,\tau_2\tau^2}{(1+\tau_1)(1+\tau_2)(1+\tau)}\left[\frac{\Gamma 4}{(\tau_1+\tau_2+\tau)^4}+\frac{\Gamma 3}{(\tau_1+\tau_2+\tau)^3}\right]$$

$$\int_{x=0}^{\infty}\frac{\tau_1\tau^2}{(1+\tau_1)(1+\tau)}\frac{1}{x}\frac{(1+x)}{x^3}e^{\frac{-(\tau_1+\tau_2+\tau)}{x}}=\frac{\tau_1\,\tau^2}{(1+\tau_1)(1+\tau)}\left[\frac{\Gamma 3}{(\tau_1+\tau_2+\tau)^3}+\frac{\Gamma 2}{(\tau_1+\tau_2+\tau)^2}\right]$$

$$\int_{x=0}^{\infty}\frac{\tau_2\tau^2}{(1+\tau_2)(1+\tau)}\frac{1}{x}\frac{(1+x)}{x^3}e^{\frac{-(\tau_1+\tau_2+\tau)}{x}}=\frac{\tau_2\,\tau^2}{(1+\tau_2)(1+\tau)}\left[\frac{\Gamma 3}{(\tau_1+\tau_2+\tau)^3}+\frac{\Gamma 2}{(\tau_1+\tau_2+\tau)^2}\right]$$

then by transformation, we get the reliability R_{1ILD} as:

$$\mathbf{R_{1ILD}}=\frac{\boldsymbol{\tau}^2}{1+\boldsymbol{\tau}}\;\frac{1+(\boldsymbol{\tau}_1+\boldsymbol{\tau}_2+\boldsymbol{\tau})}{(\boldsymbol{\tau}_1+\boldsymbol{\tau}_2+\boldsymbol{\tau})^2}+\frac{\tau_1\,\tau^2}{(1+\tau_1)(1+\tau)}\left[\frac{\Gamma 3}{(\tau_1+\tau_2+\tau)^3}+\frac{\Gamma 2}{(\tau_1+\tau_2+\tau)^2}\right]+\frac{\tau_2\,\tau^2}{(1+\tau_2)(1+\tau)}\left[\frac{\Gamma 3}{(\tau_1+\tau_2+\tau)^3}+\frac{\Gamma 2}{(\tau_1+\tau_2+\tau)^2}\right]+\frac{\boldsymbol{\tau}_1\,\boldsymbol{\tau}_2\boldsymbol{\tau}^2}{(1+\boldsymbol{\tau}_1)(1+\boldsymbol{\tau}_2)(1+\boldsymbol{\tau})}\left[\frac{\Gamma 4}{(\boldsymbol{\tau}_1+\boldsymbol{\tau}_2+\boldsymbol{\tau})^4}+\frac{\Gamma 3}{(\boldsymbol{\tau}_1+\boldsymbol{\tau}_2+\boldsymbol{\tau})^3}\right]$$

3.Three Stress- one Strength Component Reliability

when a component exposed to three independent Y_i , i = 1,2,3 stresses we get the stress-strength reliability as

$$R_2=\int_{x=0}^{\infty}F_{1y_1}(x)\,F_{2y_2}(x)\,F_{3y_3}(x)f(x)dx \tag{9}$$

in section we will find Theoretical Expression of R_2 for weibull and inverse Lindely distribution.

3.1 For weibull distribution

When the component have strength X of weibull distribution its exposed to three independent stress Y_i , i = 1,2,3 following weibull distribution with the parameters (θ,α_i); i = 1,2 ,3 then Probability density functions (pdf) and cumulative distribution functions cdf of the random variables are given in(1),(2),(3) and $F_3(y_3) = 1 - e^{-\frac{y_3{}^{\theta}}{\alpha_3}}$ $\quad y_3 \geq 0, \theta, \alpha_3 > 0$ (10)

the stress- strength reliability R_{2_w} of weibull can be obtained by substitution (1) , (2) ,(3)and(10) in (9), as:

$$R_{2_w} = \int_{x=0}^{\infty} \left(1 - e^{-\frac{x^\theta}{\alpha_1}}\right)\left(1 - e^{-\frac{x^\theta}{\alpha_2}}\right)\left(1 - e^{-\frac{x^\theta}{\alpha_3}}\right) \frac{\theta}{\alpha}\, x^{\theta-1}\, e^{-\frac{x^\theta}{\alpha}} dx$$

$$= - \int_{x=0}^{\infty} \frac{\theta}{\alpha}\, x^{\theta-1} e^{-\left(\frac{1}{\alpha_1}+\frac{1}{\alpha}\right)x^\theta} dx - \int_{x=0}^{\infty} \frac{\theta}{\alpha}\, x^{\theta-1} e^{-\left(\frac{1}{\alpha_2}+\frac{1}{\alpha}\right)x^\theta} dx + \int_{x=0}^{\infty} \frac{\theta}{\alpha}\, x^{\theta-1} e^{-\left(\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\frac{1}{\alpha}\right)x^\theta} dx - \int_{x=0}^{\infty} \frac{\theta}{\alpha}\, x^{\theta-1} e^{-\left(\frac{1}{\alpha_3}+\frac{1}{\alpha}\right)x^\theta} dx + \int_{x=0}^{\infty} \frac{\theta}{\alpha}\, x^{\theta-1} e^{-\left(\frac{1}{\alpha_1}+\frac{1}{\alpha_3}+\frac{1}{\alpha}\right)x^\theta} dx + \int_{x=0}^{\infty} \frac{\theta}{\alpha}\, x^{\theta-1} e^{-\left(\frac{1}{\alpha_2}+\frac{1}{\alpha_3}+\frac{1}{\alpha}\right)x^\theta} dx - \int_{x=0}^{\infty} \frac{\theta}{\alpha}\, x^{\theta-1} e^{-\left(\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\frac{1}{\alpha_3}+\frac{1}{\alpha}\right)x^\theta}$$

Finally the R_{2_W} can be expressed as:

$$\boldsymbol{R_{2W} = 1 - \frac{\alpha_1}{\alpha+\alpha_1} - \frac{\alpha_2}{\alpha+\alpha_2} - \frac{\alpha_3}{\alpha+\alpha_3} + \frac{\alpha_1\alpha_2}{\alpha_2\alpha+\alpha_1\alpha+\alpha_1\alpha_2} + \frac{\alpha_1\alpha_3}{\alpha_3\alpha+\alpha_1\alpha+\alpha_1\alpha_3} + \frac{\alpha_2\alpha_3}{\alpha_3\alpha+\alpha_2\alpha+\alpha_2\alpha_3} - \frac{\alpha_1\alpha_2\alpha_3}{\alpha_2\alpha_3\alpha+\alpha_1\alpha_3\alpha+\alpha_1\alpha_2\alpha+\alpha_1\alpha_2\alpha_3}}$$
(11)

3.2 For Inverse Lindely distribution

when the component have strength X of inverse Lindely distribution its exposed to three independent stress Y_i , i = 1,2,3 following inverse Lindely distribution with the parameters (θ,α_i); i = 1,2 ,3 then Probability density functions (pdf) and cumulative distribution functions cdf of the random variables are given in(6),(7),(8) and $F_3(x) = \left(1 + \frac{\tau_3}{1+\tau_3}\frac{1}{x}\right) e^{-\frac{\tau_3}{x}} \quad ; x > 0, \tau_3 > 0$ (12)

the stress- strength reliability R_{2ILD} of Invers Lindley can be obtained by substitution(6), (7),(8)and(12) in (9), as:

$$R_{2ILD} = \int_{x=0}^{\infty} \left[\left(1 + \frac{\tau_1}{1+\tau_1}\frac{1}{x}\right) e^{-\frac{\tau_1}{x}}\right] \left[\left(1 + \frac{\tau_2}{1+\tau_2}\frac{1}{x}\right) e^{-\frac{\tau_2}{x}}\right]\left[\left(1 + \frac{\tau_3}{1+\tau_3}\frac{1}{x}\right) e^{-\frac{\tau_3}{x}}\right] \frac{\tau^2}{1+\tau}\left(\frac{1+x}{x^3}\right) e^{-\frac{\tau}{x}} dx$$

$$R_{2ILD} = \frac{\tau^2}{1+\tau}\int_{x=0}^{\infty} \frac{1+x}{x^3}\, e^{\frac{-(\tau_1+\tau_2+\tau_3+\tau)}{x}} dx + \frac{\tau_1\tau_2\,\tau^2}{(1+\tau_1)(1+\tau_2)(1+\tau)}\int_{x=0}^{\infty} (1+x)x^{-5} e^{\frac{-(\tau_1+\tau_2+\tau_3+\tau)}{x}} dx$$
$$+ \frac{\tau_1\tau^2}{(1+\tau_1)(1+\tau)}\int_{x=0}^{\infty} (1+x)\; x^{-4} e^{\frac{-(\tau_1+\tau_2+\tau_3+\tau)}{x}} dx + \frac{\tau_2\tau^2}{(1+\tau_2)(1+\tau)}\int_{x=0}^{\infty} (1+x)x^{-4} e^{\frac{-(\tau_1+\tau_2+\tau_3+\tau)}{x}} dx$$
$$+ \frac{\tau_3\tau^2}{(1+\tau_3)(1+\tau)}\int_{x=0}^{\infty}(1+x)x^{-4}\, e^{\frac{-(\tau_1+\tau_2+\tau_3+\tau)}{x}} + \frac{\tau_1\,\tau_2\tau_3\tau^2}{(1+\tau_1)(1+\tau_2)(1+\tau_3)(1+\tau)}\int_{x=0}^{\infty} (1+x)\; x^{-6}\; e^{\frac{-(\tau_1+\tau_2+\tau_3+\tau)}{x}} dx$$
$$+ \frac{\tau_1\,\tau_3\tau^2}{(1+\tau_1)(1+\tau_3)(1+\tau)}\int_{x=0}^{\infty} (1+x)x^{-5} e^{\frac{-(\tau_1+\tau_2+\tau_3+\tau)}{x}} dx + \frac{\tau_2\,\tau_3\tau^2}{(1+\tau_1)(1+\tau_3)(1+\tau)}\int_{x=0}^{\infty} (1+x)x^{-5} e^{\frac{-(\tau_1+\tau_2+\tau_3+\tau)}{x}} dx$$

Let, $\frac{\tau^2}{1+\tau}\int_{x=0}^{\infty} \frac{1+x}{x^3}\, e^{\frac{-(\tau_1+\tau_2+\tau_3+\tau)}{x}} dx = \frac{\tau^2}{1+\tau}\, \frac{1+(\tau_1+\tau_2+\tau_3+\tau)}{(\tau_1+\tau_2+\tau_3+\tau)^2}$

$$\frac{\tau_1\,\tau_2\tau^2}{(1+\tau_1)(1+\tau_2)(1+\tau)}\int_{x=0}^{\infty} (1+x)x^{-5} e^{\frac{-(\tau_1+\tau_2+\tau_3+\tau)}{x}} dx\; \frac{\tau_1\,\tau_2\tau^2}{(1+\tau_1)(1+\tau_2)(1+\tau)}\left[\frac{\Gamma 4}{(\tau_1+\tau_2+\tau_3+\tau)^4} + \frac{\Gamma 3}{(\tau_1+\tau_2+\tau_3+\tau)^3}\right]$$

$$\frac{\tau_1\tau^2}{(1+\tau_1)(1+\tau)}\int_{x=0}^{\infty} (1+x)x^{-4} e^{\frac{-(\tau_1+\tau_2+\tau_3+\tau)}{x}} dx = \frac{\tau_1\tau^2}{(1+\tau_1)(1+\tau)}\left[\frac{\Gamma 3}{(\tau_1+\tau_2+\tau_3+\tau)^3} + \frac{\Gamma 2}{(\tau_1+\tau_2+\tau_3+\tau)^2}\right]$$

$$\frac{\tau_2\tau^2}{(1+\tau_2)(1+\tau)}\int_{x=0}^{\infty} (1+x)x^{-4} e^{\frac{-(\tau_1+\tau_2+\tau_3+\tau)}{x}} dx = \frac{\tau_2\tau^2}{(1+\tau_2)(1+\tau)}\left[\frac{\Gamma 3}{(\tau_1+\tau_2+\tau_3+\tau)^3} + \frac{\Gamma 2}{(\tau_1+\tau_2+\tau_3+\tau)^2}\right]$$

$$\frac{\tau_3\,\tau^2}{(1+\tau_3)(1+\tau)}\int_{x=0}^{\infty}(1+x)x^{-4} e^{\frac{-(\tau_1+\tau_2+\tau_3+\tau)}{x}} dx = \frac{\tau_3\,\tau^2}{(1+\tau_3)(1+\tau)}\left[\frac{\Gamma 3}{(\tau_1+\tau_2+\tau_3+\tau)^3} + \frac{\Gamma 2}{(\tau_1+\tau_2+\tau_3+\tau)^2}\right]$$

$$\frac{\tau_1\,\tau_2\tau_3\tau^2}{(1+\tau_1)(1+\tau_2)(1+\tau_3)(1+\tau)}\int_{x=0}^{\infty}(1+x)x^{-6} e^{\frac{-(\tau_1+\tau_2+\tau_3+\tau)}{x}} dx = \frac{\tau_1\,\tau_2\tau_3\tau^2}{(1+\tau_1)(1+\tau_2)(1+\tau_3)(1+\tau)}\left[\frac{\Gamma 5}{(\tau_1+\tau_2+\tau_3+\tau)^5} + \frac{\Gamma 4}{(\tau_1+\tau_2+\tau_3+\tau)^4}\right]$$

$$\frac{\tau_1\,\tau_3\tau^2}{(1+\tau_1)(1+\tau_3)(1+\tau)}\int_{x=0}^{\infty} (1+x)x^{-5} e^{\frac{-(\tau_1+\tau_2+\tau_3+\tau)}{x}} dx = \frac{\tau_1\,\tau_3\tau^2}{(1+\tau_1)(1+\tau_3)(1+\tau)}\left[\frac{\Gamma 4}{(\tau_1+\tau_2+\tau_3+\tau)^3} + \frac{\Gamma 3}{(\tau_1+\tau_2+\tau_3+\tau)^2}\right]$$

$$\frac{\tau_2\,\tau_3\tau^2}{(1+\tau_1)(1+\tau_3)(1+\tau)}\int_{x=0}^{\infty} (1+x)x^{-5} e^{\frac{-(\tau_1+\tau_2+\tau_3+\tau)}{x}} dx = \frac{\tau_2\,\tau_3\tau^2}{(1+\tau_1)(1+\tau_3)(1+\tau)}\left[\frac{\Gamma 4}{(\tau_1+\tau_2+\tau_3+\tau)^3} + \frac{\Gamma 3}{(\tau_1+\tau_2+\tau_3+\tau)^2}\right]$$

Finally the R_{2ILD} can be expressed as:

$$R_{2ILD} = \frac{\tau^2}{1+\tau}\,\frac{1+(\tau_1+\tau_2+\tau_3+\tau)}{(\tau_1+\tau_2+\tau_3+\tau)^2} + \frac{\tau_1\tau_2\,\tau^2}{(1+\tau_1)(1+\tau_2)(1+\tau)}\left[\frac{\Gamma 4}{(\tau_1+\tau_2+\tau_3+\tau)^4} + \frac{\Gamma 3}{(\tau_1+\tau_2+\tau_3+\tau)^3}\right] + \frac{\tau_1\tau^2}{(1+\tau_1)(1+\tau)}\left[\frac{\Gamma 3}{(\tau_1+\tau_2+\tau_3+\tau)^3} + \frac{\Gamma 2}{(\tau_1+\tau_2+\tau_3+\tau)^2}\right] + \frac{\tau_2\tau^2}{(1+\tau_2)(1+\tau)}\left[\frac{\Gamma 3}{(\tau_1+\tau_2+\tau_3+\tau)^3} + \frac{\Gamma 2}{(\tau_1+\tau_2+\tau_3+\tau)^2}\right] + \frac{\tau_3\tau^2}{(1+\tau_3)(1+\tau)}\left[\frac{\Gamma 3}{(\tau_1+\tau_2+\tau_3+\tau)^3} + \frac{\Gamma 2}{(\tau_1+\tau_2+\tau_3+\tau)^2}\right] + \frac{\tau_1\,\tau_2\tau_3\tau^2}{(1+\tau_1)(1+\tau_2)(1+\tau_3)(1+\tau)}\left[\frac{\Gamma 5}{(\tau_1+\tau_2+\tau_3+\tau)^5} + \frac{\Gamma 4}{(\tau_1+\tau_2+\tau_3+\tau)^4}\right] + \frac{\tau_1\,\tau_3\tau^2}{(1+\tau_1)(1+\tau_3)(1+\tau)}\left[\frac{\Gamma 4}{(\tau_1+\tau_2+\tau_3+\tau)^4} + \frac{\Gamma 3}{(\tau_1+\tau_2+\tau_3+\tau)^3}\right] + \frac{\tau_2\,\tau_3\tau^2}{(1+\tau_1)(1+\tau_3)(1+\tau)}\left[\frac{\Gamma 4}{(\tau_1+\tau_2+\tau_3+\tau)^4} + \frac{\Gamma 3}{(\tau_1+\tau_2+\tau_3+\tau)^3}\right]$$

4.Different method of estimation for weibull distribution

The unknown scale parameters of R_{1W}andR_{2W} for WD have been estimated by four methods of estimation; ML, MOM ,LS and WLS.

4.1 *Maximum likelihood function (MLE)*

The term maximum likelihood refers to a method of estimating parameters of a population from a random sample. It is applied when we know the general form of distribution of the population but when one or more parameters of this distribution are unknown. The method consists in choosing an estimator of unknown parameter whose values maximize the probability of obtaining the observed sample(Alwan 2015).let $x_1, x_2 \dots, x_n$strength random sample of size n from W(θ, α) where α is unknown parameter and where θ is known then the likelihood function using equation (1) as:-

$$L(x_1, x_2, \dots, x_n; \theta, \alpha) = \left(\frac{\theta}{\alpha}\right)^n \prod_{i=1}^{n} x_i^{\theta-1}\, e^{-\frac{\sum_{i=1}^{n} x_i^{\theta}}{\alpha}}$$

The partial derivative of log-likelihood function with respect to α is given by:

$$\frac{\partial \ln L}{\partial \alpha} = -\frac{n}{\alpha} + \frac{\sum_{i=1}^{n} x_i^{\theta}}{\alpha^2} \qquad (13)$$

Then by simplification equations (13), the ML's estimator for the unknown shape parametersα $\hat{\alpha}_{(MLE)}$is given by:

$$\hat{\alpha}_{(MLE)} = \frac{\sum_{i=1}^{n} x_i^{\theta}}{n}$$

In the same way, let Y_1, Y_2, Y_3 stress random variable have $W(\theta,\alpha_1), W(\theta,\alpha_2) W(\theta,\alpha_3)$, with sample size n_1, n_2, n_3 respectively and the ML estimator of unknown parameters $\alpha_1, \alpha_2, \alpha_3$ are :

$$\hat{\alpha}_{1(ML)} = \frac{\sum_{j_1=1}^{n_1} y_{j_1}^{\theta}}{n_1}$$

$$\hat{\alpha}_{2(ML)} = \frac{\sum_{j_2=1}^{n_2} y_{j_2}^{\theta}}{n_2}$$

$$\hat{\alpha}_{3(ML)} = \frac{\sum_{j_3=1}^{n_3} y_{j_3}^{\theta}}{n_3}$$

$\hat{\alpha}_{2(ML)}\hat{\alpha}_{(ML)}$ the MLestimator of R_{1W} say $\hat{R}_{1W(ML)}$ is obtained by substitute $\hat{\alpha}_{(ML)}, \hat{\alpha}_{1(ML)}$ in and $\hat{\alpha}_{2(ML)}$ equation(5) (by the invariant property of this method) as:

$$\hat{R}_{1W(ML)} = 1 - \frac{\hat{\alpha}_{1(ML)}}{\hat{\alpha}_{1(ML)}+\hat{\alpha}_{(ML)}} - \frac{\hat{\alpha}_{2(ML)}}{\hat{\alpha}_{2(ML)}+\hat{\alpha}_{(ML)}} + \frac{\hat{\alpha}_{1(ML)}\hat{\alpha}_{2(ML)}}{\hat{\alpha}_{2(ML)}\hat{\alpha}_{(ML)}+\hat{\alpha}_{1(ML)}\hat{\alpha}_{(ML)}+\hat{\alpha}_{1(ML)}\hat{\alpha}_{2(ML)}}$$

the MLestimator of R_{2W} say$\hat{R}_{2W(ML)}$is obtained by substitute$\hat{\alpha}_{(ML)}, \hat{\alpha}_{1(ML)}$,$\hat{\alpha}_{2(ML)}$ and $\hat{\alpha}_{3(ML)}$in equation(11) as:

$$\hat{R}_{2W} = 1 - \frac{\hat{\alpha}_{1(ML)}}{\hat{\alpha}_{(ML)}+\hat{\alpha}_{1(ML)}} - \frac{\hat{\alpha}_{2(ML)}}{\hat{\alpha}_{(ML)}+\hat{\alpha}_{2(ML)}} - \frac{\hat{\alpha}_{3(ML)}}{\hat{\alpha}_{(ML)}+\hat{\alpha}_{3(ML)}} + \frac{\hat{\alpha}_{1(ML)}\hat{\alpha}_{2(ML)}}{\hat{\alpha}_{2(ML)}\hat{\alpha}_{(ML)}+\hat{\alpha}_{1(ML)}\hat{\alpha}_{(ML)}+\hat{\alpha}_{1(ML)}\hat{\alpha}_{2(ML)}} + \frac{\hat{\alpha}_{1(ML)}\hat{\alpha}_{3(ML)}}{\hat{\alpha}_{3(ML)}\hat{\alpha}_{(ML)}+\hat{\alpha}_{1(ML)}\hat{\alpha}_{(ML)}+\hat{\alpha}_{1(ML)}\hat{\alpha}_{3(ML)}} + \frac{\hat{\alpha}_{2(ML)}\hat{\alpha}_{3(ML)}}{\hat{\alpha}_{3(ML)}\hat{\alpha}_{(ML)}+\hat{\alpha}_{2(ML)}\hat{\alpha}_{(ML)}+\hat{\alpha}_{2(ML)}\hat{\alpha}_{3(ML)}} - \frac{\hat{\alpha}_{1(ML)}\hat{\alpha}_{2(ML)}\hat{\alpha}_{3(ML)}}{\hat{\alpha}_{2(ML)}\hat{\alpha}_{3(ML)}\hat{\alpha}_{(ML)}+\hat{\alpha}_{1(ML)}\hat{\alpha}_{3(ML)}\hat{\alpha}_{(ML)}+\hat{\alpha}_{1(ML)}\hat{\alpha}_{2(ML)}\hat{\alpha}_{(ML)}+\hat{\alpha}_{1(ML)}\hat{\alpha}_{2(ML)}\hat{\alpha}_{3(ML)}}$$

4.2 Moment method (MOM)

The method of moments is used for estimating the parameter distribution from a sample. The method is further developed and studied by Chuprov (1874–1926), Thiele, Thorvald Nicolai, Fisher, Ronald Aylmer, and Pearson, Karl, among others (Alwan 2015) since the strength X is weibull random variables with $(\theta, \alpha)$and the stresses y_1, y_2, y_3 are weibull variables with $(\theta,\alpha_1), (\theta,\alpha_2)$, (θ,α_3) respectively then their population means are given by :

$$E(x) = \alpha^{\frac{1}{\theta}}\,\Gamma_{\left(1+\frac{1}{\theta}\right)}$$

$E(y_1) = \alpha_1^{\frac{1}{\theta}}\,\Gamma_{\left(1+\frac{1}{\theta}\right)}$

$E(y_2) = \alpha_2^{\frac{1}{\theta}}\,\Gamma_{\left(1+\frac{1}{\theta}\right)}$

$E(y_3) = \alpha_3^{\frac{1}{\theta}}\,\Gamma_{\left(1+\frac{1}{\theta}\right)}$

according to the method of moment ,equating the samples means with the corresponding populations mean ,then the moments estimator of $\alpha, \alpha_1, \alpha_2, \alpha_3$ denoted by $\hat{\alpha}_{(MOM)}, \hat{\alpha}_{1(MOM)}, \hat{\alpha}_{2(MOM)}, \hat{\alpha}_{3(MOM)}$ respectively, are:

$$\hat{\alpha}_{(MOM)} = \left(\frac{\bar{x}}{\Gamma_{\left(1+\frac{1}{\theta}\right)}}\right)^{\theta}$$

$$\hat{\alpha}_{1(MOM)} = \left(\frac{\overline{y_1}}{\Gamma_{\left(1+\frac{1}{\theta}\right)}}\right)^{\theta}$$

$$\hat{\alpha}_{2(MOM)} = \left(\frac{\overline{y_2}}{\Gamma_{\left(1+\frac{1}{\theta}\right)}}\right)^{\theta}$$

$$\hat{\alpha}_{3(MOM)} = \left(\frac{\overline{y_3}}{\Gamma_{\left(1+\frac{1}{\theta}\right)}}\right)^{\theta}$$

the MOM of R_{1W} say $\hat{R}_{1W(MOM)}$ is given by replacing the MOM parameters estimators instead of the parameters in equation(5) as:

$$\hat{\boldsymbol{R}}_{1W(MOM)} = \mathbf{1} - \frac{\hat{\alpha}_{1(MOM)}}{\hat{\alpha}_{1(MOM)}+\hat{\alpha}_{(MOM)}} - \frac{\hat{\alpha}_{2(MOM)}}{\hat{\alpha}_{2(MOM)}+\hat{\alpha}_{(MOM)}} + \frac{\hat{\alpha}_{1(MOM)}\hat{\alpha}_{2(MOM)}}{\hat{\alpha}_{2(MOM)}\hat{\alpha}_{(MOM)}+\hat{\alpha}_{1(MOM)}\hat{\alpha}_{(MOM)}+\hat{\alpha}_{1(MOM)}\hat{\alpha}_{2(MOM)}}$$

the MOM of R_{2W} say$\hat{R}_{2W(MOM)}$ is given by replacing the MOM parameters estimators instead of the parameters in equation(11) as:

$$\hat{\boldsymbol{R}}_{2W} = \mathbf{1} - \frac{\hat{\alpha}_{1(MOM)}}{\hat{\alpha}_{(MOM)}+\hat{\alpha}_{1(MOM)}} - \frac{\hat{\alpha}_{2(MOM)}}{\hat{\alpha}_{(MOM)}+\hat{\alpha}_{2(MOM)}} - \frac{\hat{\alpha}_{3(MOM)}}{\hat{\alpha}_{(MOM)}+\hat{\alpha}_{3(MOM)}}$$
$$+ \frac{\hat{\alpha}_{1(MOM)}\alpha_{2(MOM)}}{\hat{\alpha}_{2(MOM)}\hat{\alpha}_{(MOM)}+\hat{\alpha}_{1(MOM)}\hat{\alpha}_{(MOM)}+\hat{\alpha}_{1(MOM)}\hat{\alpha}_{2(MOM)}}$$
$$+ \frac{\hat{\alpha}_{1(MOM)}\hat{\alpha}_{3(MOM)}}{\hat{\alpha}_{3(MOM)}\hat{\alpha}_{(MOM)}+\hat{\alpha}_{1(MOM)}\hat{\alpha}_{(MOM)}+\hat{\alpha}_{1(MOM)}\hat{\alpha}_{3(MOM)}}$$
$$+ \frac{\hat{\alpha}_{2(MOM)}\hat{\alpha}_{3(MOM)}}{\hat{\alpha}_{3(MOM)}\hat{\alpha}_{(MOM)}+\hat{\alpha}_{2(MOM)}\hat{\alpha}_{(MOM)}+\hat{\alpha}_{2(MOM)}\hat{\alpha}_{3(MOM)}}$$
$$- \frac{\hat{\alpha}_{1(MOM)}\hat{\alpha}_{2(MOM)}\hat{\alpha}_{3(MOM)}}{\hat{\alpha}_{2(MOM)}\hat{\alpha}_{3(MOM)}\hat{\alpha}_{(MOM)}+\hat{\alpha}_{1(MOM)}\hat{\alpha}_{3(MOM)}\hat{\alpha}_{(MOM)}+\hat{\alpha}_{1(MOM)}\hat{\alpha}_{2(MOM)}\hat{\alpha}_{(MOM)}+\hat{\alpha}_{1(MOM)}\hat{\alpha}_{2(MOM)}\hat{\alpha}_{3(MOM)}}$$

4.3Least Square Method (LS)

The least square method estimator is very popular for model fitting, especially in linear and non-linear regression**.** (Hassan &Basheikh 2012)The least square method estimator scan be produce by minimizing the sum of square error between the value and its expected value, (Ali 2013) The least square of the location and scale parameters of Weibull distribution suggested bySwain et al. (1988)are found by minimizing the following equation: (Kantar& Senoglu 2008)

$$S = \sum_{i=1}^{n}\left[F(X_{(i)}) - E\left(F(X_{(i)})\right)\right]^2 \quad (14)$$

Where$E\left(F(X_{(i)})\right)$ equal to$P_i$the plotting position,where $P_i = \frac{i}{n+1}$and$i = 1,2,\dots,n$ (15)

We can use the least square method for the parameters of the WD by minimizing equation (15) with respect to the unknown parameter α of strength random sample $X \sim W(\theta, \alpha)$ with sample size n.

By taking natural logarithm to $(1 - P_i) = e^{-\alpha\, x^{\theta}}$

, we get:

Where$P_i$plotting position (15), then equating to zero, we obtain:-

$$\ln(1 - P_i) + \frac{x^{\theta}}{\alpha} = 0 \quad (16)$$

Substitution (16) in (14), we get:

$$S = \sum_{i=1}^{n}\left[\ln(1 - P_i) + \frac{x^{\theta}}{\alpha}\right]^2 \quad (17)$$

Deriving (17) with respect to the unknown shape parameter α and equating the result to zero, we will get:

$$\frac{\partial S}{\partial \alpha} = \sum_{i=1}^{n} 2\left[\ln(1 - P_i) + \frac{x^{\theta}}{\alpha}\right]\frac{-x^{\theta}}{\alpha^2}$$

$$\sum_{i=1}^{n} 2\left[\ln(1 - P_i) + \frac{x^{\theta}}{\alpha}\right]\frac{-x^{\theta}}{\alpha^2} = 0 \quad (18)$$

By solving the equation (18), we get:

$$\hat{\alpha}_{(LS)}=\frac{-\sum_{i=1}^{n}{x_{(i)}}^{2\theta}}{\sum_{i=1}^{n}{x_{(i)}}^{\theta}\ln(1-p_i)}$$

In the same way, we will estimate the unknown parameter $\alpha_1,\alpha_2,\alpha_3$ for the stresses random variables Y_1,Y_2,Y_3 of WD with sample size n_1,n_2,n_3, we will obtain:

$$\hat{\alpha}_{1(LS)}=\frac{-\sum_{j_1=1}^{n_1}{y_{1(j_1)}}^{2\theta}}{\sum_{j_1=1}^{n_1}{y_{1(j_1)}}^{\theta}\ln(1-P_{j_1})}$$

$$\hat{\alpha}_{2(LS)}=\frac{-\sum_{j_2=1}^{n_2}{y_{2(j_2)}}^{2\theta}}{\sum_{j_2=1}^{n_2}{y_{2(j_2)}}^{\theta}\ln(1-P_{(j_2)})}$$

$$\hat{\alpha}_{3(LS)}=\frac{-\sum_{j_3=1}^{n_3}{y_{3(j_3)}}^{2\theta}}{\sum_{j_3=1}^{n_3}{y_{3(j_3)}}^{\theta}\ln(1-P_{j_3})}$$

$$where\ P_{j_1}=\frac{j_1}{n_1+1}\quad ,j_1=1,2,\dots,n_1 \qquad (19)$$

$$P_{j_2}=\frac{j_2}{n_2+1}\quad ,j_2=1,2,\dots,n_2 \qquad (20)$$

$$and\ P_{j_3}=\frac{j_3}{n_3+1}\ ,j_3=1,2,\dots,n_3 \qquad (21)$$

and the approximated LS of R_{1W} say $\hat{R}_{1W(LS)}$ is given by replacing the LS parameters estimators instead of the parameters in equation(5) as:

$$\hat{\boldsymbol{R}}_{\boldsymbol{1W(LS)}}=\boldsymbol{1}-\frac{\hat{\alpha}_{1(LS)}}{\hat{\alpha}_{1(LS)}+\hat{\alpha}_{(LS)}}-\frac{\hat{\alpha}_{2(LS)}}{\hat{\alpha}_{2(LS)}+\hat{\alpha}_{(LS)}}+\frac{\hat{\alpha}_{1(LS)}\hat{\alpha}_{2(LS)}}{\hat{\alpha}_{2(LS)}\hat{\alpha}_{(LS)}+\hat{\alpha}_{1(LS)}\hat{\alpha}_{(LS)}+\hat{\alpha}_{1(LS)}\hat{\alpha}_{2(LS)}}$$

the LS of R_{2W} say $\hat{R}_{2W(LS)}$ is given by replacing the Ls parameters estimators instead of the parameters in equation(11) as:

$$\hat{\boldsymbol{R}}_{\boldsymbol{2W(LS)}}=\boldsymbol{1}-\frac{\hat{\alpha}_{1(LS)}}{\hat{\alpha}_{(LS)}+\hat{\alpha}_{1(LS)}}-\frac{\hat{\alpha}_{2(LS)}}{\hat{\alpha}_{(LS)}+\hat{\alpha}_{2(LS)}}-\frac{\hat{\alpha}_{3(LS)}}{\hat{\alpha}_{(LS)}+\hat{\alpha}_{3(LS)}}$$
$$+\frac{\hat{\alpha}_{1(LS)}\hat{\alpha}_{2(LS)}}{\hat{\alpha}_{2(LS)}\hat{\alpha}_{(LS)}+\hat{\alpha}_{1(LS)}\hat{\alpha}_{(LS)}+\hat{\alpha}_{1(LS)}\hat{\alpha}_{2(LS)}}$$
$$+\frac{\hat{\alpha}_{1(LS)}\hat{\alpha}_{3(LS)}}{\hat{\alpha}_{3(LS)}\hat{\alpha}_{(LS)}+\hat{\alpha}_{1(LS)}\hat{\alpha}_{(LS)}+\hat{\alpha}_{1(LS)}\hat{\alpha}_{3(LS)}}$$
$$+\frac{\hat{\alpha}_{2(LS)}\hat{\alpha}_{3(LS)}}{\hat{\alpha}_{3(LS)}\hat{\alpha}_{(LS)}+\hat{\alpha}_{2(LS)}\hat{\alpha}_{(LS)}+\hat{\alpha}_{2(LS)}\hat{\alpha}_{3(LS)}}$$
$$-\frac{\hat{\alpha}_{1(LS)}\hat{\alpha}_{2(LS)}\hat{\alpha}_{3(LS)}}{\hat{\alpha}_{2(LS)}\hat{\alpha}_{3(LS)}\hat{\alpha}_{(LS)}+\hat{\alpha}_{1(LS)}\hat{\alpha}_{3(LS)}\hat{\alpha}_{(LS)}+\hat{\alpha}_{1(LS)}\hat{\alpha}_{2(LS)}\hat{\alpha}_{(LS)}+\hat{\alpha}_{1(LS)}\hat{\alpha}_{2(LS)}\hat{\alpha}_{3(LS)}}$$

4.4Weighted Least Square Method (WLS)

The weighted least squares estimators can be obtained by minimizing the following equation. (Karam& Jani 2015)

$$\sum_{i=1}^{n}w_i\left[F\left(x_{(i)}\right)-E\left(F\left(x_{(i)}\right)\right)\right]^2 \qquad (22)$$

$$w_i=\frac{1}{Var[F(x_{(i)})]}=\frac{(n+1)^2(n+2)}{i(n-i+1)},i=1,2,\dots,n \qquad (23)$$

with respect to the unknown parameter α of strength random variable$X\sim W(\theta,\alpha)$ with sample size n.
By substitution (16) in (22), we get:

$$\sum_{i=1}^{n}w_i\left[\ln(1-P_i)+\frac{{x_{(i)}}^{\theta}}{\alpha}\right]^2=0 \qquad (24)$$

By taking the partial derivative to the equation (24) with respect to α, and simplify the result we obtain:

$$\sum_{i=1}^{n}2w_i\left[\ln(1-P_i)+\frac{{x_{(i)}}^{\theta}}{\alpha}\right]\frac{-{x_{(i)}}^{\theta}}{\alpha^2}=0$$

$$\sum_{i=1}^{n}w_i{x_{(i)}}^{\theta}\ln(1-P_i)+\frac{1}{\alpha}\sum_{i=1}^{n}w_i{x_{(i)}}^{2\theta}=0 \qquad (25)$$

Then, by solving the equation (25), we get:

$$\hat{\alpha}_{(LS)}=\frac{-\sum_{i=1}^{n}w_i{x_{(i)}}^{2\theta}}{\sum_{i=1}^{n}w_i{x_{(i)}}^{\theta}\ln(1-p_i)}$$

Where P_i as in (15) and w_i as in (23)

In the same way, we will estimate the unknown parameter, $\alpha_1,\alpha_2,\alpha_3$ for the stresses random variables Y_1,Y_2,Y_3 of WD with sample size n_1,n_2,n_3, we will obtain:

$$\hat{\alpha}_{1(\mathrm{WLS})}=\frac{-\sum_{j_1=1}^{n_1}\mathrm{w}_{j_1}{y_{1(j_1)}}^{2\theta}}{\sum_{j_1=1}^{n_1}\mathrm{w}_{j_1}{y_{1(j_1)}}^{\theta}\ln(1-P_{j_1})}$$

$$\hat{\alpha}_{2(\mathrm{WLS})}=\frac{-\sum_{j_2=1}^{n_2}\mathrm{w}_{j_2}{y_{2(j_2)}}^{2\theta}}{\sum_{j_2=1}^{n_2}\mathrm{w}_{j_2}{y_{2(j_2)}}^{\theta}\ln(1-P_{(j_2)})}$$

$\hat{\alpha}_{3(\text{WLS})} = \frac{-\sum_{j_3=1}^{n_3} w_{j_3}\, y_{3(j_3)}^{\;2\theta}}{\sum_{j_3=1}^{n_3} w_{j_3}\, y_{3(j_3)}^{\;\theta} \ln\left(1-P_{j_3}\right)}$

Where $P_{j_1}, P_{j_2}, P_{j_3}$ as in (19) ,(20),(21) respectively

and $w_{j_1} = \frac{1}{Var[F(y_{1(i)})]} = \frac{(n_1+1)^2(n_1+2)}{j_1(n_1-j_1+1)}, j_1 = 1,2,\dots,n_1$

$w_{j_2} = \frac{1}{Var[F(y_{2(i)})]} = \frac{(n_2+1)^2(n_2+2)}{j_2(n_2-j_2+1)}, j_2 = 1,2,\dots,n_2$

$w_{j_3} = \frac{1}{Var[F(y_{3(i)})]} = \frac{(n_3+1)^2(n_3+2)}{j_3(n_3-j_3+1)}, j_3 = 1,2,\dots,n_3$

and the approximated WLS of R_{1W} say $\hat{R}_{1W(WLS)}$ is given by replacing the WLS parameters estimators instead of the parameters in equation(5) as:

$$\hat{\boldsymbol{R}}_{1W(WLS)} = 1 - \frac{\hat{\alpha}_{1(WLS)}}{\hat{\alpha}_{1(WLS)}+\hat{\alpha}_{(WLS)}} - \frac{\hat{\alpha}_{2(WLS)}}{\hat{\alpha}_{2(WLS)}+\hat{\alpha}_{(WLS)}} + \frac{\hat{\alpha}_{1(WLS)}\hat{\alpha}_{2(WLS)}}{\hat{\alpha}_{2(WLS)}\hat{\alpha}_{(WLS)}+\hat{\alpha}_{1(WLS)}\hat{\alpha}_{(WLS)}+\hat{\alpha}_{1(WLS)}\hat{\alpha}_{2(WLS)}}$$

the WLS of R_{2W} say $\hat{R}_{2W(WLS)}$ is given by replacing the WLS parameters estimators instead of the parameters in equation(11) as:

$$\hat{\boldsymbol{R}}_{2W(\text{WLS})} = 1 - \frac{\hat{\alpha}_{1(WLS)}}{\hat{\alpha}_{(WLS)}+\hat{\alpha}_{1(WLS)}} - \frac{\hat{\alpha}_{2(WLS)}}{\hat{\alpha}_{(WLS)}+\hat{\alpha}_{2(WLS)}} - \frac{\hat{\alpha}_{3(WLS)}}{\hat{\alpha}_{(WLS)}+\hat{\alpha}_{3(WLS)}}$$
$$+\frac{\hat{\alpha}_{1(WLS)}\hat{\alpha}_{2(WLS)}}{\hat{\alpha}_{2(WLS)}\hat{\alpha}_{(WLS)}+\hat{\alpha}_{1(WLS)}\hat{\alpha}_{(WLS)}+\hat{\alpha}_{1(WLS)}\hat{\alpha}_{2(WLS)}}$$
$$+\frac{\hat{\alpha}_{1(WLS)}\hat{\alpha}_{3(WLS)}}{\hat{\alpha}_{3(WLS)}\hat{\alpha}_{(WLS)}+\hat{\alpha}_{1(WLS)}\hat{\alpha}_{(WLS)}+\hat{\alpha}_{1(WLS)}\hat{\alpha}_{3(WLS)}}$$
$$+\frac{\hat{\alpha}_{2(WLS)}\hat{\alpha}_{3(WLS)}}{\hat{\alpha}_{3(WLS)}\hat{\alpha}_{(WLS)}+\hat{\alpha}_{2(WLS)}\hat{\alpha}_{(WLS)}+\hat{\alpha}_{2(WLS)}\hat{\alpha}_{3(WLS)}}$$
$$-\frac{\hat{\alpha}_{1(WLS)}\hat{\alpha}_{2(WLS)}\hat{\alpha}_{3(WLS)}}{\hat{\alpha}_{2(WLS)}\hat{\alpha}_{3(WLS)}\hat{\alpha}_{(WLS)}+\hat{\alpha}_{1(WLS)}\hat{\alpha}_{3(WLS)}\hat{\alpha}_{(WLS)}+\hat{\alpha}_{1(WLS)}\hat{\alpha}_{2(WLS)}\hat{\alpha}_{(WLS)}+\hat{\alpha}_{1(WLS)}\hat{\alpha}_{2(WLS)}\hat{\alpha}_{3(WLS)}}$$

5.Simulation study

In this section ,Monte Carlo simulation is performed to compare the performances of the ML, MOM ,LS and WLS estimators for R_1 and R_2 (based on 10000 replication).

It made by assuming three cases of R_1 , say[(2.2, 1.9, 1.5), (1.4, 1.2, 0.9), (1.7, 2.3, 1.5)] , three cases of R_2, say[(2.2, 1.2,1.3,0.5), (2.3, 1.4,1.7, 0.6), (2.2,1.3,1.4, 0.4)]

for different sample sizes.

in tables(3),(4), (5),(6) (7)and(8)below we have observed that:-

1- From the tables(3),(4)and(5)below, for $R_1 = 0.4071, 0.4158$,0.3044 we get:

- the MSE value decreasing by increasing sample size for MLE, MOM, LS, and WLS estimators. Thebest MSE value is LS estimator, followed by WLS, MOM and MLE.
- the MAPE value decreasing by increasing sample size for MLE, MOM, LS, and WLS estimators. The best MAPE value is LS estimator, followed by WLS, MOM and MLE.

2-From the tables(6),(7)and(8)below, for $R_2 = 0.6853$, $0.6682, 0.6510$ we get:

- the MSE value decreasing by increasing sample size for MLE, MOM, LS, and WLS estimators.Thebest MSE value is WLS estimator, followed by MOM, MLE and LS.
- the MAPE value decreasing by increasing sample size for MLE, MOM, LS, and WLS estimators. The best MAPE value is WLS estimator, followed by MOM, MLE and LS.

6.Conclusion

The performance LS was the best, followed by WLS, MOM and MLE for all sample sizes, as in the table below.

Table (1): The best estimation method of MSE and MAPE of W for R_1.

Method / Sample size	MLE	MOM	LS	WLS	Best
All sample size	4	3	1	2	LS

The performance WLS was the best, followed by MOM, MLE and LS for all sample sizes, as in the table below.

Table (2): The best estimation method of MSE and MAPE of W for R_2.

Method / Sample size	MLE	MOM	LS	WLS	Best
All sample size	3	2	4	1	WLS

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Table (3): Results of Mean, MSE and MAPE values for WD $R_1 = 0.4071$ for $(\theta, \alpha, \alpha_1, \alpha_2)$ = (2.2, 1.9, 1.5, 1.6), (2.2, 1.9, 1.5, 2.5).

θ = 1.6						
(n, n_1, n_2)		**MLE**	**MOM**	**LS**	**WLS**	**Best**
(10,10,10)	Mean	0.4330	0.4330	0.3615	0.3485	**-**
	MSE	0.0282	0.0272	0.0052	0.0067	**LS**
	MAPE	0.3391	0.3332	0.1461	0.1675	**LS**
(20,20,10)	Mean	0.4441	0.4422	0.3343	0.3281	**-**
	MSE	0.0178	0.0169	0.0072	0.0084	**LS**
	MAPE	0.2662	0.2591	0.1844	0.1993	**LS**
(35,35,35)	Mean	0.4450	0.4453	0.3626	0.3488	**-**
	MSE	0.0106	0.0101	0.0031	0.0049	**LS**
	MAPE	0.2056	0.2002	0.1177	0.1498	**LS**
(35,35,20)	Mean	0.4107	0.4738	0.3506	0.3364	**-**
	MSE	0.0091	0.0130	0.0044	0.0066	**LS**
	MAPE	0.1889	0.2286	0.1434	0.1777	**LS**
(75,75,35)	Mean	0.4475	0.4470	0.3378	0.3300	**-**
	MSE	0.0068	0.0064	0.0055	0.0070	**LS**
	MAPE	0.1623	0.1578	0.1704	0.1903	**MOM**
(100,100,50)	Mean	0.4486	0.4482	0.3416	0.3357	**-**
	MSE	0.0055	0.0052	0.0048	0.0060	**LS**
	MAPE	0.1479	0.1438	0.1611	0.1765	**MOM**
θ = 2.5						
(10,10,10)	Mean	0.4634	0.4697	0.2988	0.3085	**-**
	MSE	0.0606	0.0552	0.0163	0.0144	**WLS**
	MAPE	0.5095	0.4852	0.2765	0.2567	**WLS**
(20,20,10)	Mean	0.4872	0.4839	0.2563	0.2589	**-**
	MSE	0.0448	0.0382	0.0255	0.0248	**WLS**
	MAPE	0.4298	0.3955	0.3708	0.3644	**WLS**
(35,35,35)	Mean	0.4919	0.4964	0.2974	0.3086	**-**
	MSE	0.0318	0.0270	0.0139	0.0119	**WLS**
	MAPE	0.3597	0.3328	0.2699	0.2438	**WLS**
(35,35,20)	Mean	0.4938	0.4935	0.2703	0.2758	**-**
	MSE	0.0333	0.0277	0.0206	0.0194	**WLS**
	MAPE	0.3684	0.3370	0.3361	0.3228	**WLS**
(75,75,35)	Mean	0.5034	0.5032	0.2669	0.2707	**-**
	MSE	0.0237	0.0197	0.0208	0.0201	**MOM**
	MAPE	0.3102	0.2854	0.3444	0.3350	**MOM**
(100,100,50)	Mean	0.5046	0.5037	0.2718	0.2743	**-**
	MSE	0.0208	0.0173	0.0192	0.0190	**MOM**
	MAPE	0.2917	0.2700	0.3322	0.3261	**MOM**

Table (4): Results of Mean, MSE and MAPE values for WD R_1 =0.4158 for $(\theta, \alpha, \alpha_1, \alpha_2)$ = (1.4, 1.2, 0.9, 1.6), (1.4, 1.2, 0.9, 2.5).

$\theta = 1.6$						
(n, n_1, n_2)		**MLE**	**MOM**	**LS**	**WLS**	**Best**
(10,10,10)	Mean	0.4446	0.4459	0.3650	0.3519	**-**
	MSE	0.0285	0.0275	0.0058	0.0073	**LS**
	MAPE	0.3338	0.3281	0.1506	0.1732	**LS**
(20,20,10)	Mean	0.4560	0.4542	0.3380	0.3317	**-**
	MSE	0.0180	0.0171	0.0080	0.0093	**LS**
	MAPE	0.2618	0.2549	0.1913	0.2064	**LS**
(35,35,35)	Mean	0.4576	0.4579	0.3662	0.3522	**-**
	MSE	0.0109	0.0104	0.0036	0.0056	**LS**
	MAPE	0.2043	0.1993	0.1255	0.1578	**LS**
(35,35,20)	Mean	0.4249	0.4848	0.3543	0.3399	**-**
	MSE	0.0092	0.0133	0.0050	0.0073	**LS**
	MAPE	0.1865	0.2268	0.1514	0.1855	**LS**
(75,75,35)	Mean	0.4624	0.4618	0.3422	0.3339	**-**
	MSE	0.0072	0.0068	0.0061	0.0078	**LS**
	MAPE	0.1650	0.1603	0.1773	0.1975	**MOM**
(100,100,50)	Mean	0.4619	0.4614	0.3456	0.3393	**-**
	MSE	0.0059	0.0056	0.0055	0.0068	**LS**
	MAPE	0.1509	0.1471	0.1690	0.1846	**MOM**
$\theta = 2.5$						
(10,10,10)	Mean	0.4779	0.4846	0.2944	0.3039	**-**
	MSE	0.0614	0.0561	0.0193	0.0172	**WLS**
	MAPE	0.5029	0.4795	0.2990	0.2789	**WLS**
(20,20,10)	Mean	0.5020	0.4996	0.2518	0.2544	**-**
	MSE	0.0454	0.0389	0.0296	0.0289	**WLS**
	MAPE	0.4243	0.3918	0.3945	0.3885	**WLS**
(35,35,35)	Mean	0.5085	0.5133	0.2929	0.3050	**-**
	MSE	0.0331	0.0285	0.0170	0.0145	**WLS**
	MAPE	0.3611	0.3355	0.2959	0.2672	**WLS**
(35,35,20)	Mean	0.3691	0.4717	0.2933	0.3035	**-**
	MSE	0.0215	0.0210	0.0165	0.0144	**WLS**
	MAPE	0.2894	0.2860	0.2946	0.2703	**WLS**
(75,75,35)	Mean	0.5161	0.5170	0.2631	0.2664	**-**
	MSE	0.0242	0.0205	0.0245	0.0238	**MOM**
	MAPE	0.3092	0.2878	0.3672	0.3593	**MOM**
(100,100,50)	Mean	0.5213	0.5214	0.2674	0.2700	**-**
	MSE	0.0222	0.0190	0.0229	0.0226	**MOM**
	MAPE	0.2980	0.2805	0.3569	0.3507	**MOM**

Table (5): Results of Mean, MSE and MAPE values for WD $R_1 = 0.3044$ for $(\theta, \alpha, \alpha_1, \alpha_2)$ = (1.7, 2.3, 1.5, 1.6), (1.7, 2.3, 1.5, 2.5).

$\theta = 1.6$						
(n, n_1, n_2)		MLE	MOM	LS	WLS	Best
(10,10,10)	Mean	0.2886	0.2890	0.3220	0.3093	-
	MSE	0.0222	0.0213	0.0033	0.0030	WLS
	MAPE	0.4026	0.3947	0.1482	0.1434	WLS
(20,20,10)	Mean	0.2901	0.2880	0.2965	0.2901	-
	MSE	0.0131	0.0125	0.0018	0.0022	LS
	MAPE	0.3090	0.3011	0.1131	0.1236	LS
(35,35,35)	Mean	0.2786	0.2379	0.3091	0.3029	-
	MSE	0.0016	0.0053	0.0002	0.0004	LS
	MAPE	0.1090	0.2201	0.0416	0.0548	LS
(35,35,20)	Mean	0.2349	0.2872	0.3060	0.2895	-
	MSE	0.0106	0.0069	0.0009	0.0015	LS
	MAPE	0.2868	0.2213	0.0823	0.1016	LS
(75,75,35)	Mean	0.2861	0.2852	0.3001	0.2920	-
	MSE	0.0043	0.0040	0.0006	0.0012	LS
	MAPE	0.1742	0.1685	0.0673	0.0901	LS
(100,100,50)	Mean	0.2856	0.2849	0.3032	0.2972	-
	MSE	0.0034	0.0031	0.0004	0.0009	LS
	MAPE	0.1532	0.1484	0.0573	0.0792	LS
$\theta = 2.5$						
(10,10,10)	Mean	0.2702	0.2690	0.3451	0.3547	-
	MSE	0.0441	0.0392	0.0067	0.0077	LS
	MAPE	0.5846	0.5496	0.2142	0.2301	LS
(20,20,10)	Mean	0.2697	0.2611	0.2987	0.3007	-
	MSE	0.0295	0.0248	0.0032	0.0033	LS
	MAPE	0.4737	0.4351	0.1485	0.1514	LS
(35,35,35)	Mean	0.2593	0.2573	0.3444	0.3554	-
	MSE	0.0196	0.0155	0.0037	0.0050	LS
	MAPE	0.3846	0.3415	0.1627	0.1921	LS
(35,35,20)	Mean	0.2519	0.2844	0.3056	0.3162	-
	MSE	0.0203	0.0148	0.0021	0.0026	LS
	MAPE	0.3919	0.3280	0.1193	0.1326	LS
(75,75,35)	Mean	0.2584	0.2545	0.3110	0.3135	-
	MSE	0.0122	0.0096	0.0014	0.0018	LS
	MAPE	0.3011	0.2680	0.0957	0.1088	LS
(100,100,50)	Mean	0.2560	0.2530	0.3167	0.3185	-
	MSE	0.0102	0.0082	0.0012	0.0017	LS
	MAPE	0.2754	0.2461	0.0906	0.1073	LS

Table (6): Results of Mean, MSE and MAPE values for WD $R_2 = 0.6853$ for $(\theta, \alpha, \alpha_1, \alpha_2, \alpha_3)$ = (2.2, 1.2,1.3,0.5,1.6), (2.2, 1.2,1.3,0.5,2.5).

$\theta = 1.6$						
(n, n_1, n_2, n_3)		**MLE**	**MOM**	**LS**	**WLS**	**Best**
(10,10,10,10)	Mean	0.7789	0.6222	0.6235	0.6274	**-**
	MSE	0.0088	0.0040	0.0038	0.0034	**WLS**
	MAPE	0.1367	0.0921	0.0901	0.0845	**WLS**
(20,20,20,20)	Mean	0.6045	0.6225	0.5954	0.5956	**-**
	MSE	0.0065	0.0039	0.0081	0.0080	**MOM**
	MAPE	0.1179	0.0916	0.1312	0.1308	**MOM**
(20,20,10,10)	Mean	0.6772	0.6202	0.6037	0.6064	**-**
	MSE	0.0001	0.0042	0.0067	0.0062	**MLE**
	MAPE	0.0119	0.0949	0.1191	0.1151	**MLE**
(35,35,20,20)	Mean	0.7762	0.6226	0.5807	0.5848	**-**
	MSE	0.0083	0.0039	0.0109	0.0101	**MOM**
	MAPE	0.1327	0.0915	0.1526	0.1466	**MOM**
(75,50,50,50)	Mean	0.6493	0.6233	0.5630	0.5652	**-**
	MSE	0.0013	0.0038	0.0149	0.0144	**MLE**
	MAPE	0.0525	0.0904	0.1784	0.1752	**MLE**
(100,75,75,75)	Mean	0.6047	0.6227	0.5562	0.5566	**-**
	MSE	0.0065	0.0039	0.0167	0.0166	**MOM**
	MAPE	0.1176	0.0914	0.1883	0.1878	**MOM**
$\theta = 2.5$						
(10,10,10,10)	Mean	0.5284	0.5336	0.7586	0.7584	**-**
	MSE	0.0248	0.0232	0.0054	0.0053	**WLS**
	MAPE	0.2290	0.2214	0.1069	0.1066	**WLS**
(20,20,20,20)	Mean	0.5284	0.5335	0.7348	0.7343	**-**
	MSE	0.0247	0.0231	0.0025	0.0024	**WLS**
	MAPE	0.2289	0.2214	0.0722	0.0715	**WLS**
(20,20,10,10)	Mean	0.4834	0.5308	0.7686	0.7646	**-**
	MSE	0.0408	0.0239	0.0070	0.0063	**WLS**
	MAPE	0.2945	0.2254	0.1216	0.1158	**WLS**
(35,35,20,20)	Mean	0.5361	0.5301	0.7367	0.7325	**-**
	MSE	0.0223	0.0241	0.0026	0.0022	**WLS**
	MAPE	0.2178	0.2265	0.0750	0.0689	**WLS**
(75,50,50,50)	Mean	0.5284	0.5335	0.7106	0.7096	**-**
	MSE	0.0246	0.0231	0.0006	0.0005	**WLS**
	MAPE	0.2290	0.2215	0.0370	0.0355	**WLS**
(100,75,75,75)	Mean	0.5283	0.5335	0.6949	0.6936	**-**
	MSE	0.0247	0.0231	0.0009	0.0006	**WLS**
	MAPE	0.2291	0.2216	0.0140	0.0121	**WLS**

Table (7): Results of Mean, MSE and MAPE values for WD R_2 =0.6682 for $(\theta, \alpha, \alpha_1, \alpha_2, \alpha_3)$ = (2.3, 1.4,1.7, 0.6, 1.6), (2.3, 1.4,1.7, 0.6, 2.5).

$\theta = 1.6$						
(n, n_1, n_2, n_3)		**MLE**	**MOM**	**LS**	**WLS**	**Best**
(10,10,10,10)	Mean MSE MAPE	0.7461 0.0061 0.1165	0.6153 0.0028 0.0791	0.6218 0.0022 0.0695	0.6274 0.0017 0.0611	**-** **WLS** **WLS**
(20,20,20,20)	Mean MSE MAPE	0.7244 0.0032 0.0841	0.6151 0.0028 0.0795	0.5945 0.0054 0.1103	0.5992 0.0048 0.1033	**-** **MOM** **MOM**
(20,20,10,10)	Mean MSE MAPE	0.6058 0.0039 0.0933	0.6243 0.0019 0.0657	0.6044 0.0041 0.0956	0.6046 0.0040 0.0952	**-** **MOM** **MOM**
(35,35,20,20)	Mean MSE MAPE	0.6807 0.0001 0.0187	0.6240 0.0020 0.0661	0.5811 0.0076 0.1304	0.5845 0.0070 0.1253	**-** **MLE** **MLE**
(75,50,50,50)	Mean MSE MAPE	0.6060 0.0039 0.0931	0.6245 0.0019 0.0654	0.5634 0.0110 0.1569	0.5637 0.0109 0.1564	**-** **MOM** **MOM**
(100,75,75,75)	Mean MSE MAPE	0.6043 0.0022 0.0717	0.6221 0.0005 0.0444	0.5564 0.0090 0.1454	0.5568 0.0089 0.1447	 **MOM** **MOM**
$\theta = 2.5$						
(10,10,10,10)	Mean MSE MAPE	0.7709 0.0105 0.1536	0.5782 0.0081 0.1347	0.7715 0.0107 0.1546	0.7787 0.0122 0.1654	**-** **MOM** **MOM**
(20,20,20,20)	Mean MSE MAPE	0.7467 0.0062 0.1175	0.7467 0.0077 0.1313	0.7444 0.0058 0.1141	0.7487 0.0065 0.1205	**-** **LS** **LS**
(20,20,10,10)	Mean MSE MAPE	0.6599 0.0001 0.0125	0.5803 0.0077 0.1316	0.7873 0.0142 0.1783	0.7961 0.0164 0.1914	**-** **MLE** **MLE**
(35,35,20,20)	Mean MSE MAPE	0.6129 0.0031 0.0827	0.5683 0.0100 0.1495	0.7166 0.0023 0.0725	0.7077 0.0016 0.0590	**-** **WLS** **WLS**
(75,50,50,50)	Mean MSE MAPE	0.6404 0.0007 0.0416	0.5770 0.0083 0.1365	0.7177 0.0025 0.0741	0.7259 0.0033 0.0863	**-** **MLE** **MLE**
(100,75,75,75)	Mean MSE MAPE	0.5242 0.0208 0.2155	0.5298 0.0192 0.2071	0.6930 0.0006 0.0372	0.6917 0.0005 0.0352	**-** **WLS** **WLS**

Table (8): Results of Mean, MSE and MAPE values for WD $R_2 = 0.6510$ for $(\theta, \alpha, \alpha_1, \alpha_2, \alpha_3)$ = (2.2,1.3,1.4, 0.4, 1.6), (2.2,1.3,1.4, 0.4, 2.5).

$\theta = 1.6$						
(n, n_1, n_2, n_3)		**MLE**	**MOM**	**LS**	**WLS**	**Best**
(10,10,10,10)	Mean MSE MAPE	0.7734 0.0150 0.1880	0.6224 0.0008 0.0440	0.6240 0.0007 0.0415	0.6282 0.0005 0.0351	- **WLS** **WLS**
(20,20,20,20)	Mean MSE MAPE	0.6851 0.0012 0.0525	0.6207 0.0009 0.0465	0.5952 0.0031 0.0857	0.5991 0.0027 0.0797	- **MOM** **MOM**
(20,20,10,10)	Mean MSE MAPE	0.7187 0.0046 0.1040	0.6229 0.0007 0.0432	0.6037 0.0022 0.0727	0.6078 0.0019 0.0663	- **MOM** **MOM**
(35,35,20,20)	Mean MSE MAPE	0.7768 0.0158 0.1933	0.6219 0.0008 0.0447	0.5809 0.0049 0.1076	0.5851 0.0043 0.1012	- **MOM** **MOM**
(75,50,50,50)	Mean MSE MAPE	0.6905 0.0016 0.0607	0.6227 0.0007 0.0434	0.5630 0.0077 0.1352	0.5653 0.0073 0.1316	- **MOM** **MOM**
(100,75,75,75)	Mean MSE MAPE	0.6043 0.0022 0.0717	0.6221 0.0008 0.0443	0.5564 0.0090 0.1454	0.5568 0.0089 0.1447	- **MOM** **MOM**
$\theta = 2.5$						
(10,10,10,10)	Mean MSE MAPE	0.5708 0.0065 0.1232	0.5268 0.0154 0.1907	0.7581 0.0115 0.1645	0.7591 0.0117 0.1661	- **MLE** **MLE**
(20,20,20,20)	Mean MSE MAPE	0.6055 0.0021 0.0699	0.5293 0.0148 0.1869	0.7343 0.0069 0.1280	0.7295 0.0062 0.1207	- **MLE** **MLE**
(20,20,10,10)	Mean MSE MAPE	0.6101 0.0017 0.0629	0.5288 0.0149 0.1876	0.7690 0.0139 0.1813	0.7628 0.0125 0.1718	- **MLE** **MLE**
(35,35,20,20)	Mean MSE MAPE	0.5689 0.0067 0.1261	0.5313 0.0143 0.1839	0.7369 0.0074 0.1320	0.7310 0.0064 0.1229	- **WLS** **WLS**
(75,50,50,50)	Mean MSE MAPE	0.5082 0.0204 0.2193	0.5327 0.0140 0.1817	0.7109 0.0036 0.0921	0.7088 0.0033 0.0888	- **WLS** **WLS**
(100,75,75,75)	Mean MSE MAPE	0.5273 0.0153 0.1900	0.5325 0.0141 0.1820	0.6949 0.0019 0.0675	0.6936 0.0018 0.0655	- **WLS** **WLS**