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Numerical Solution to Parabolic PDE Using Implicit Finite Difference Approach

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Abstract

This paper examines an implicit Finite Difference approach for solving the parabolic partial differential equation (PDE) in one dimension. We consider the Crank Nicolson scheme which offers a better truncation error for both time and spatial dimensions as compared with the explicit Finite Difference method. In addition the scheme is consistent and unconditionally stable. One downside of implicit methods is the relatively high computational cost involved in the solution process, however this is compensated by the high level of accuracy of the approximate solution and efficiency of the numerical scheme. A physical problem modelled by the heat equation with Neumann boundary condition is solved using the Crank Nicolson scheme. Comparing the numerical solution with the analytical solution, we observe that the relative error increases sharply at the right boundary, however it diminishes as the spatial step size approaches zero.

Keywords: Partial Differential Equation, Implicit Finite Difference, Crank Nicolson Scheme

1. Introduction

Myriads of physical phenomenon like diffusion, wave propagation, laser beam models, financial models, etc., are modelled by partial differential equations. Most often the equations under consideration are so complicated that finding their solutions analytically (e.g. by Laplace and Fourier transform methods, or in the form of a power series) is either impossible or impracticable (Süli, 2012). This may be due to several reasons including nonlinearity of problems, variable coefficients, difficulty in evaluating integrals analytically, approximation based on truncation of infinite series required, inappropriate solution space, etc. And, even problems that could be solved analytically, we depend on numerical procedure to plot the solution.

Consequently, numerical methods are employed as alternative approach for the approximation of the unknown analytical solution (Bruaset, and Tveito, 2006). The main idea of any numerical method for a differential equation is to discretize the given continuous problem with infinitely many degrees of freedom to obtain a discrete problem of system of equations with only finitely many unknowns that may be solved using numerical algorithms (Claes, 2009). Although there are several different numerical methods available for solving the parabolic equations, we focus on finite difference schemes. This is motivated by the fact that they are very simple to understand and they are easy to generalize to more complex boundary value problems, and also are easily implemented on a computer. The finite difference method comes in two forms namely; the explicit methods and the implicit methods. Explicit methods generally are consistent, however their stability is restricted (LeVeque, 2007). On the other hand the implicit methods are consistent as well as unconditionally stable, however they are computationally costly compared to the explicit methods (Douglas and Kim, 2001). This is compensated by the high level of accuracy of approximate solution and efficiency of the numerical scheme. In this paper we use the Crank Nicolson method which is an implicit finite difference method to solve a physical problem modelled by the parabolic equation. We choose this method over the explicit method because of its attractive properties mentioned.

2. Finite Difference Method

In this section, we discuss the concept of the finite difference method by providing useful terminologies, definitions and theorems. The differencing method solves a PDE numerically by setting up a regular grid in space and time to compute approximate solutions at space or time points of this grid (Strauss, 2008). The method consist of approximating the differential operator by replacing the derivatives in the equation using differential

quotients. This process is commonly termed as discretization. The error between the numerical solution and the exact solution is determine by the error that is committed by going from a differential operator to a difference operator. This error is called the discretization error or truncation error.

Definition 2.1. *Suppose the function u is C^2 continuous in the neighbourhood of x. For any h > 0 we have*

$$u(x+h) = u(x) + hu'(x) + \frac{h^2}{2}u''(x+h_1) \quad (1)$$

where h_1 is a number between 0 and $h: x+h_1 \in]x, x+h[$.
Truncating (1) *at the first order gives*

$$u(x+h) = u(x) + hu'(x) + O(h^2) \quad (2)$$

where the term $O(h^2)$ indicates that error of the approximation is proportional to h^2.

We refer to equation (2) as the *forward difference* approximation of u' . We deduce from equation (2) that there exists a constant $C > 0$ such that, for *h* > 0 sufficiently small, we have

$$\left| \frac{u(x+h) - u(x)}{h} - u'(x) \right| \leq C\, h,$$

where

$$C = \sup_{\xi \in [x, x+h_0]} \frac{|u''(\xi)|}{2}$$

for $h \leq h_0, h_0 > 0$ given. The approximation of u' at the point *x* is said to be *consistent* at the first order. The order of a first derivative can be defined in a more general sense as follows;

Definition 2.2. *The approximation of the derivative u' at point x is of order p (p > 0) if there exists a constant C > 0, independent of h, such that the error between the derivative and its approximation is bounded by $C\,h^p$, i.e., is exactly $O(h^p)$.*

Similarly, the first order *backward difference approximation* of u' at point *x* can be defined as

$$u(x-h) = u(x) - hu'(x) + O(h^2) \quad (3)$$

In order to improve the accuracy of the approximation of u' we define a consistent approximation, called the central difference approximation, also referred to as symmetric quotient, by combining the forward difference and the backward difference (Lapidus, and Pinder,1999). That is, taking the points $x-h$ and $x+h$ into consideration.

Definition 2.3. *Suppose that the function $u \in C^3(\bar{\Omega})$ where $\Omega =]x-h, x+h[$, then we write,*

$$u(x+h) = u(x) + hu'(x) + \frac{h^2}{2}u''(x) + \frac{h^3}{6}u'''(\tau^+) \quad (6)$$

$$u(x-h) = u(x) - hu'(x) + \frac{h^2}{2}u''(x) - \frac{h^3}{6}u'''(\tau^-) \quad (7)$$

where $\tau^+ \in]x, x+h[$ and $\tau^- \in]x-h, x[$.
Subtracting (7) from (6) to we obtain,

$$\frac{u(x+h) - u(x-h)}{2h} = u'(x) + \frac{h^3}{6}u'''(\tau^+) \quad (8)$$

where $\tau \in]x-h, x+h[$.

It follows from the definition that, for every $h \in]0, h_0[$, we have a bound on the approximation error as:

$$\left|\frac{u(x+h)-u(x-h)}{2h}-u'(x)\right|\le Ch^2,\quad C=\sup_{\xi\in[x-h_0,x+h_0]}\frac{|u'''(\xi)|}{6}$$

If u is C^2 continuous, then the approximation is consistent at the order one only. The order of the approximation is related to the regularity of the function u (Alan, 2002).

The approximation of the second derivative and its associated truncation error is given by the following lemma.

Lemma 2.1. *Suppose u is a C^4 continuous function on an interval $[x-h_0, x+h_0]$, $h_0>0$. Then, there exists a constant $C > 0$ such that for every $h\in]0,h_0[$, we have*

$$\left|\frac{u(x+h)-2u(x)+u(x-h)}{h^2}-u''(x)\right|\le C\,h^2. \qquad (9)$$

The differential quotient $\frac{1}{h^2}(u(x+h)-2u(x)+u(x-h))$ is consistent second-order approximation of the second derivative u'' of u at point x.

Proof. We begin by using the Taylor expansion up to the fourth order to obtain the following:

$$u(x+h)=u(x)+hu'(x)+\frac{h^2}{2}u''(x)+\frac{h^3}{6}u'''(x)+\frac{h^4}{24}\mathrm{u}^{(4)}(\tau^+) \qquad (10)$$

$$u(x-h)=u(x)-hu'(x)+\frac{h^2}{2}u''(x)-\frac{h^3}{6}u'''(\tau^-)+\frac{h^4}{24}u^{(4)}(\tau^-) \qquad (11)$$

where $\tau^+\in]x,x+h[$ *and* $\tau^-\in]x-h,x[$. By the intermediate value theorem, we can write:

$$\frac{u(x+h)-2u(x)+u(x-h)}{h^2}=u''(x)+\frac{h^2}{12}u^{(4)}(\tau),$$

for $\tau\in]x-h,x+h[$. Hence, from equation (9) we deduce the constant

$$C=\sup_{\xi\in[x-h_0,x+h_0]}\frac{|u^{(4)}(\xi)|}{12}.$$

□

Remark 2.1. The error estimates (9) depends on the regularity of the function u. If u is C^3 continuous, then the error is of order h only.

3. Parabolic Problem in 1-dimension

Parabolic PDE describes time dependent physical phenomenon and is mostly used in modeling diffusion problems such as heat transfer (Morton and Mayer, 2005). In this section, we consider the one-dimensional homogenous heat equation posed in the bounded domain $\Omega=[0,L]\subset\mathbb{R}$:

$$\begin{cases} \dfrac{\partial u}{\partial t}-\alpha\dfrac{\partial^2 u}{\partial t^2}=0, & \text{for } (x,t)\in\Omega\times\mathbb{R}_+ \\ u(x,0)=u_0(x), & \text{for } x\in\Omega, \\ \mathrm{BC}: u(0,t)=\beta,\quad u(L,t)=\gamma, & \text{for } \beta,\gamma\in\mathbb{R} \end{cases} \qquad (12)$$

where $\alpha\in\mathbb{R}_+$ is a given scalar value and $u_0\in C^0(\Omega,\mathbb{R})$. The boundary condition (BC) specified in (12) is of the Dirichlet boundary condition. The boundary condition can be replaced by Neumann boundary condition given as,

$$\frac{\partial u}{\partial x}(0,t)=\beta,\qquad \frac{\partial u}{\partial x}(L,t)=\gamma,\qquad \beta,\gamma\in\mathbb{R}$$

3.1. Analytic Solution

The general analytic solution of the heat equation (12) with homogeneous Dirichlet boundary condition is as follows (Kersale, 2003):

$$u(x,t)=\sum_{n=1}^{\infty}b_n\sin\left(\frac{\pi nx}{L}\right)e^{-\alpha\left(\frac{\pi n}{L}\right)^2 t}$$

where

$$b_n=\frac{2}{L}\int_0^L u_0(x)\sin\left(\frac{\pi n}{L}\right)dx,\quad n=1,2,3,\ldots$$

This solution can be obtained by using the method of separation of variables and Fourier transform. When the boundary conditions are of the Neumann kind then the solution is written as:

$$u(x,t)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n e^{-\alpha\left(\frac{\pi n}{L}\right)^2 t}\cos\left(\frac{\pi nx}{L}\right)$$

where

$$a_n=\frac{2}{L}\int_0^L u_0(x)\cos\left(\frac{\pi n}{L}\right)dx,\quad n=0,1,2,\ldots$$

3.2. Implicit Finite Difference Scheme

To begin with, we introduce equi-distributed grid points $(x_i)_{0\le i\le N+1}$ given by $x_i=ih$, with $h=L/(N+1)$ to discretize the domain, Ω in space. Furthermore, we consider time step, $k>0$ to discretize the time domain, hence defining a regular grid, $t^j=jk, j\in\mathbb{N}$. We denote by $U(x_i,t^j)$ the value of the numerical solution at point (x_i,t^j) and $u(x,t)$ the exact solution of the equation (12). The initial data u_0 is discretized by:

$$U_i^0=u_0(x_i)\qquad \text{for } i\in\{0,\ldots,n+1\}.$$

The boundary conditions are also handled in similar vein. Dirichlet boundary conditions is represented in a discrete form in a straight forward manner as:

$$U_0^j=\beta,\quad U_{N+1}^j=\gamma\qquad \forall\, j>0.$$

In the case of Neumann boundary conditions, in order to improve the accuracy of the solution we apply the central difference approximation to discretize the derivatives involved. This is given by

$$U_{-1}^j-U_1^j=2h\beta,\quad U_{N+2}^j-U_N^j=2h\gamma,\ \forall\, j>0$$

Using the finite difference approximations for the derivative in the heat equation we obtain the following:

$$\frac{\partial u}{\partial t}(x,t)=\frac{u(x,t+k)-u(x,t)}{k}+O(k) \tag{13}$$

$$\frac{\partial^2 u}{\partial x^2}(x,t)=\frac{u(x+h,t)-2u(x,t)+u(x-h,t)}{h^2}+O(h^2) \tag{14}$$

Considering equations (13) and (14), the finite difference schemes can be written in a more general form as a weighted average of the explicit and implicit schemes, producing the theta scheme, also written as the θ-scheme:

$$\frac{U_i^{j+1}-U_i^j}{k}-\alpha(1-\theta)\frac{U_{i+1}^j-2U_i^j+u_{i-1}^j}{h^2}-\alpha\theta\frac{U_{i+1}^{j+1}-2U_i^{j+1}+U_{i-1}^{j+1}}{h^2}=0 \tag{15}$$

We simplify equation (15) further as:

$$U_i^{j+1}=U_i^j+\zeta\theta\left[U_{i+1}^{j+1}-2U_i^{j+1}+U_{i-1}^{j+1}\right]+\zeta(1-\theta)\left[U_{i+1}^j-2U_i^j+u_{i-1}^j\right] \tag{16}$$

where θ denotes a positive parameter, $\theta\in[0,1]$ and

$$\zeta=\alpha\frac{k}{h^2}$$

The forward and backward Euler methods, and the Crank Nicolson method are considered to be special cases of the θ-scheme defined by (16), with $\theta=0$ for the forward Euler, $\theta=1$ for backward Euler, and $\theta=1/2$ for the Crank Nicolson method. The θ-scheme is an implicit scheme for $\theta\neq 0$.

The difference equation of the θ-scheme (16) can be written in a vector form as:

$$\left(I-\zeta\theta A\right)\vec{U}^{j+1}=\left(I+\zeta(1-\theta)A\right)\vec{U}^{j} \qquad (17)$$

where I is a unit matrix,

$$\vec{U}^{j}=\left[U_1^j,U_2^j,\ldots,U_{N-1}^j\right]^T$$

and

$$A=\begin{pmatrix} -2 & 1 & 0 & . & . & 0 \\ 1 & -2 & 1 & 0 & . & 0 \\ . & . & . & . & . & . \\ 0 & . & 0 & 1 & -2 & 1 \\ 0 & . & . & 0 & 1 & -2 \end{pmatrix}$$

Thus, to find $\vec{U}^{j+1}$, we need to solve a tridiagonal linear system, which can be done by employing Thomas algorithm.

Lemma 3.1. *Let T be an* $N\times N$ *tridiagonal matrix of the form:*

$$T=\begin{pmatrix} b & c & 0 & . & . & 0 \\ a & b & c & 0 & . & 0 \\ . & . & . & . & . & . \\ 0 & . & 0 & a & b & c \\ 0 & . & . & 0 & a & b \end{pmatrix}.$$

The eigenvalues and the corresponding eigenvectors of T are:

$$\lambda_k=b+2\sqrt{ac}\cos\frac{\pi k}{N+1},\quad \vec{v}_k=\begin{pmatrix} \left(\frac{a}{c}\right)^{1/2}\sin\frac{1\cdot\pi k}{N+1} \\ \left(\frac{a}{c}\right)^{2/2}\sin\frac{2\cdot\pi k}{N+1} \\ \vdots \\ \left(\frac{a}{c}\right)^{N/2}\sin\frac{N\cdot\pi k}{N+1} \end{pmatrix},\quad k=1,\ldots,N$$

(Kicaid and Cheney, 2002)

Lemma 3.2. *Consider the* θ-scheme (14), *for* $\theta\in[0,1]$.

(i) If $\frac{1}{2}\le\theta\le 1$, *the* θ-scheme *is unconditionally stable in* L^2 *- norm (and convergent)*

(ii) If $0\le\theta<\frac{1}{2}$, *the* θ-scheme *is stable in* L^2 *- norm (and convergent of this norm) under the condition:*

$$\zeta\le\frac{1}{2(1-2\theta)}.$$

Proof. To establish the stability of the θ-scheme, we represent the scheme (17) in a matrix form as

$$\vec{U}^{j+1}=\Lambda\vec{U}^{j} \qquad (18)$$

and we determine whether the magnitude of any eigenvalue of coefficient matrix Λ exceeds 1. From equation (17), we can write,

$$\vec{U}^{j+1}=\left(I-\zeta\theta A\right)^{-1}\left(I+\zeta(1-\theta)A\right)\vec{U}^{j} \qquad (19)$$

It follows from (19) that,

$$\Lambda=\left(I-\zeta\theta A\right)^{-1}\left(I+\zeta(1-\theta)A\right)$$

Since the eigenvalues of $(I+\zeta(1-\theta)A)$ and $(I-\zeta\theta A)^{-1}$ are the same, the eigenvalues appearing in the matrix Λ, are easily found be

$$\frac{1+\zeta(1-\theta)\lambda_k}{1-\zeta\theta\lambda_k}$$

where λ_k are the eigenvalues of *A*. The values of λ_k can directly be deduced from lemma 3.1. as,

$$\lambda_k = -2+2\cos\frac{\pi k}{N} = -4\sin^2\left(\frac{\pi k}{2N}\right), \qquad k=1,\cdots,N \qquad (20)$$

For stability of the scheme, it is necessary that,

$$\left|\frac{1+\zeta(1-\theta)\lambda_k}{1-\zeta\theta\lambda_k}\right| \le 1, \quad k=1,\ldots,N \qquad (21)$$

We denote

$$\phi_k = \frac{\pi k}{2N} \qquad (22)$$

Substituting equations (20) and (22) into (21) gives,

$$\left|1-4\zeta(1-\theta)\sin^2\phi_k\right| \le \left|1+4\zeta\theta\sin^2\phi_k\right|. \qquad (23)$$

Since $\theta \ge 0$, by assumption and $\zeta > 0$ by definition, then the right hand side of (23) is positive, and therefore the inequality can be written as

$$-(1+4\zeta\theta\sin^2\phi_k) \le 1-4\zeta(1-\theta)\sin^2\phi_k \le 1+4\zeta\theta\sin^2\phi_k \qquad (24)$$

The right part of the inequality is automatically satisfied for all ϕ_k. The left part is satisfied when

$$4\zeta(1-\theta)\sin^2\phi_k < 2 \qquad (25)$$

The strongest restriction on ζ (and hence on the step size in time) occurs when $\sin^2\phi_k$ assumes its largest value, that is 1. In that case, (25) yields

$$(1-2\theta)\zeta \le \frac{1}{2} \qquad (26)$$

Considering (26), we note that the θ-scheme is unconditionally stable if $\frac{1}{2} \le \theta \le 1$, for any arbitrary ζ.

If $\frac{1}{2} \le \theta \le 1$, then the θ-scheme is stable provided that,

$$\zeta \le \frac{1}{2(1-2\theta)}$$

□

3.2.1. Crank Nicolson Scheme

We consider Crank Nicolson method for solving the heat equation in this paper due to the fact that the method has second order accuracy in both time and spatial dimensions, and is unconditionally stable (Akrivis, et. al., 2006). This means that the stability of the scheme is not affected by any step size chosen for the time dimension as seen in lemma 3.2. The method as mentioned earlier can be extracted from the θ-scheme by setting $\theta = 1/2$. We can write the scheme as follows:

$$\frac{U_i^{j+1}-U_i^j}{k} = -\frac{\alpha}{2}\left[\frac{U_{i+1}^j-2U_i^j+u_{i-1}^j}{h^2}+\frac{U_{i+1}^{j+1}-2U_i^{j+1}+U_{i-1}^{j+1}}{h^2}\right] \qquad (27)$$

Rearranging (27) gives,

$$U_i^{j+1}-\frac{\zeta}{2}\left\{U_{i-1}^{j+1}-2U_i^{j+1}+U_{i+1}^{j+1}\right\} = U_i^j+\frac{\zeta}{2}\left\{U_{i-1}^j-2U_i^j+U_{i+1}^j\right\} \qquad (28)$$

where

$$\zeta = \alpha \frac{k}{h^2}$$

Now, before we generate the system of linear equations out of (28), we recall the discrete form of the boundary conditions discussed earlier. Let us consider for instance the discrete homogeneous Neumann boundary condition given by

$$U_{-1}^j = U_1^j, \quad U_{N+2}^j = U_N^j$$

For $i = 1, \ldots, N-1$, we generate a system of linear equations out of equation (28), taking into consideration the boundary conditions. We obtain the following system:

$$\begin{aligned}
(1+\zeta)U_0^{j+1} - \zeta U_1^{j+1} &= U_0^j + \zeta\left\{U_1^j - U_0^j\right\} \\
-\frac{\zeta}{2}U_0^{j+1} + (1+\zeta)U_1^{j+1} - \frac{\zeta}{2}U_2^{j+1} &= U_1^j + \frac{\zeta}{2}\left\{U_2^j - 2U_1^j + U_0^j\right\} \\
-\frac{\zeta}{2}U_1^{j+1} + (1+\zeta)U_2^{j+1} - \frac{\zeta}{2}U_3^{j+1} &= U_2^j + \frac{\zeta}{2}\left\{U_3^j - 2U_2^j + U_1^j\right\} \\
-\frac{\zeta}{2}U_2^{j+1} + (1+\zeta)U_3^{j+1} - \frac{\zeta}{2}U_4^{j+1} &= U_3^j + \frac{\zeta}{2}\left\{U_4^j - 2U_3^j + U_2^j\right\} \\
\ddots \quad \ddots &\quad \vdots \\
-\zeta U_9^{j+1} + (1+\zeta)U_{10}^{j+1} &= U_9^j + \zeta\left\{U_{10}^j - U_9^j\right\}
\end{aligned}$$

The above system can be written in a matrix form as

$$\begin{pmatrix}
(1+\zeta) & -\zeta & & & & & \\
-\zeta/2 & (1+\zeta) & -\zeta/2 & & & 0 & \\
 & -\zeta/2 & (1+\zeta) & -\zeta/2 & & & \\
 & & \ddots & & \ddots & & \\
 & & & \ddots & & \ddots & \\
 & 0 & & & & & \\
 & & & & -\zeta & (1+\zeta)
\end{pmatrix}
\begin{pmatrix} U_0^{j+1} \\ U_1^{j+1} \\ U_2^{j+1} \\ \vdots \\ \vdots \\ \\ U_{10}^{j+1} \end{pmatrix}
=
\begin{pmatrix} U_0^j + \zeta(U_1^j - U_0^j) \\ U_1^j + \frac{\zeta}{2}(U_2^j - 2U_1^j + U_0^j) \\ U_2^j + \frac{\zeta}{2}(U_3^j - 2U_2^j + U_1^j) \\ \vdots \\ \vdots \\ \\ U_9^j + \zeta(U_{10}^j - U_9^j) \end{pmatrix} \qquad (29)$$

Each time step, *j* requires that we solve the tridiagonal system (29). This makes it computationally costly compared to the explicit methods. However, for one dimensional heat problem, there is not much difference between the explicit method and the implicit method in terms of computational cost.

4. Test Case

Consider the equation describing one-dimensional, single-phase, slightly compressible flow in a producing petroleum reservoir which is given, for $0 \le x \le 500$ and $t > 0$, by;

$$\frac{\varphi\mu C}{K}\frac{\partial p(x,t)}{\partial t} = \frac{\partial^2 p(x,t)}{\partial x^2}, \qquad (30)$$

the porous medium and the reservoir are homogeneous, that the liquid is ideal, and that gravitational effects are negligible. The distance is represented by *x*, time by *t*, *p* the pressure, ϕ the dimensionless constant porosity of the medium, μ the viscosity the permeability of the medium and *C* the compressibility (in {pounds per square inch}$^{-1}$). Assuming that $\alpha = \phi\mu C/K = 0.00004$ days/ft^2, and that the following conditions hold:

$$p(x,0) = 2.5\times10^7, \qquad 0 \le x \le 500,$$

$$\frac{\partial p(0,t)}{K\partial x} = \frac{\partial p(500,t)}{\partial x} = 0, \quad t > 0.$$

Determine the pressure p at t = 5, using the Crank Nicolson method with $k = \Delta t = 0.5$ and $h = \Delta x = 50$ (Burden et. al., 1978).

The analytical solution for the heat problem described above is given by

$$p(x_n, t) = 2.5\times10^7 + \sum_{n=1}^{\infty}\left(\frac{5\times10^7}{n\pi}\sin\pi n\right)e^{-0.00004\left(\frac{\pi n}{500}\right)^2 t}\cos\left(\frac{\pi n x_n}{500}\right).$$

The pressure p, at time t = 5 given at the spatial step, $\Delta x = 50$ is provided as follows

x	$p(x,5)$
0	2.499999999999923 x10^7
50	2.499999999999956 x10^7
100	2.499999999999939 x10^7
150	2.499999999999987 x10^7
200	2.499999999999970 x10^7
250	2.499999999999937 x10^7
300	2.499999999999939 x10^7
350	2.499999999999990 x10^7
400	2.499999999999956 x10^7
450	2.499999999999965 x10^7
500	2.499999999980735 x10^7

4.1. Numerical Solution

We solve the heat problem (30) numerically using the Crank-Nicholson method with $k = \Delta t = 0.5$ and $h = \Delta x = 50$. We represent the approximate solution of the heat problem (30) at the next time step by $P(x_i, t^{j+1})$, which for simplicity can be written as P_i^{j+1}. Expressing (30) by the Crank Nicolson scheme (28), we obtain

$$P_i^{j+1} - \frac{\zeta}{2}\left\{P_{i-1}^{j+1} - 2P_i^{j+1} + P_{i+1}^{j+1}\right\} = P_i^j + \frac{\zeta}{2}\left\{P_{i-1}^j - 2P_i^j + P_{i+1}^j\right\} \qquad (31)$$

To incorporate the Neumann boundary condition,

$$\frac{\partial P(0,t)}{K\partial x} = \frac{\partial P(500,t)}{\partial x} = 0,$$

we discretize the boundary data using central difference to obtain,

$$\frac{1}{2h}\left(P_{i+1}^{\ j} - P_{i-1}^{\ j}\right) = 0.$$

The spatial domain is discretized using $x = ih$. At the left boundary i = 0 and so x = 0. The boundary reduces to

$$P_1^j = P_{-1}^j.$$

We handle the right boundary in similar manner by setting i = 10, which implies x = 500. The boundary can then be written as

$$P_{11}^j = P_9^j$$

Now, we generate a system of equations which will enable us solve for the interior nodes.
For i = 0 in equation (31) we have,

$$P_0^{j+1} - \frac{\zeta}{2}\left\{P_{-1}^{j+1} - 2P_0^{j+1} + P_1^{j+1}\right\} = P_0^j + \frac{\zeta}{2}\left\{P_{-1}^j - 2P_0^j + P_1^j\right\}$$

But $P_1^j = P_{-1}^j$ *and* $P_{-1}^{j+1} = P_1^{j+1}$

$$P_0^{j+1} - \frac{\zeta}{2}\left\{2P_1^{j+1} - 2P_0^{j+1}\right\} = P_0^j + \frac{\zeta}{2}\left\{2P_1^j - 2P_0^j\right\}$$

For $i=1$

$$P_1^{j+1}-\frac{\zeta}{2}\left\{P_0^{j+1}-2P_1^{j+1}+P_2^{j+1}\right\}=P_1^{j}+\frac{\zeta}{2}\left\{P_0^{j}-2P_1^{j}+P_2^{j}\right\}$$

For $i = 2$

$$P_2^{j+1}-\frac{\zeta}{2}\left\{P_1^{j+1}-2P_2^{j+1}+P_3^{j+1}\right\}=P_2^{j}+\frac{\zeta}{2}\left\{P_1^{j}-2P_2^{j}+P_3^{j}\right\}$$

For $i = 3, 4\ldots\ldots10$, we get a system of equations which can be written in a matrix form as

$$\begin{pmatrix} (1+\zeta) & -\zeta & & & & & \\ -\zeta/2 & (1+\zeta) & -\lambda/2 & & & 0 & \\ & -\zeta/2 & (1+\zeta) & -\zeta/2 & & & \\ & & \ddots & & \ddots & & \\ & & & \ddots & & \ddots & \\ & 0 & & & & & \\ & & & & -\zeta & (1+\zeta) \end{pmatrix}\begin{pmatrix} P_0^{j+1} \\ P_1^{j+1} \\ P_2^{j+1} \\ \vdots \\ \vdots \\ \\ P_{10}^{j+1} \end{pmatrix}=\begin{pmatrix} P_0^{j}+\zeta(P_1^{j}-P_0^{j}) \\ P_1^{j}+\frac{\zeta}{2}(P_2^{j}-2P_1^{j}+P_0^{j}) \\ P_2^{j}+\frac{\zeta}{2}(P_3^{j}-2P_2^{j}+P_1^{j}) \\ \vdots \\ \vdots \\ \\ P_9^{j}+\zeta(P_{10}^{j}-P_9^{j}) \end{pmatrix} \quad (32)$$

Following from the heat problem (30), we let $\frac{\zeta}{2}=0.25$ and $(1+\zeta)=1.5$.

The initial conditions, $P(x,0)=2.5\times10^7$ can also be written in a discrete form as follows:

$$\begin{aligned} P_i^0 &= 2.5\times10^7 \\ P_0^0 &= 2.5\times10^7 \\ &\vdots \\ P_{10}^0 &= 2.5\times10^7 \end{aligned}$$

Substituting the value of ζ and the initial condition into (32), we obtain a tridiagonal system:

$$\begin{pmatrix} 1.5 & -0.5 & & & & & & & & \\ -0.25 & 1.5 & -0.25 & & & & & & & \\ & -0.25 & 1.5 & -0.25 & & & & & & \\ & & -0.25 & 1.5 & -0.25 & & & & & \\ & & & -0.25 & 1.5 & -0.25 & & & & \\ & & & & -0.25 & 1.5 & -0.25 & & & \\ & & & & & -0.25 & 1.5 & -0.25 & & \\ & & & & & & -0.25 & 1.5 & -0.25 & \\ & & & & & & & -0.25 & 1.5 & -0.25 \\ & & & & & & & & -0.5 & 1.5 \end{pmatrix}\begin{pmatrix} P_0^{j+1} \\ P_1^{j+1} \\ P_2^{j+1} \\ P_3^{j+1} \\ P_4^{j+1} \\ P_5^{j+1} \\ P_6^{j+1} \\ P_7^{j+1} \\ P_8^{j+1} \\ P_9^{j+1} \\ P_{10}^{j+1} \end{pmatrix}=\begin{pmatrix} 2.5\times10^7 \\ 2.5\times10^7 \\ 2.5\times10^7 \\ 2.5\times10^7 \\ 2.5\times10^7 \\ 2.5\times10^7 \\ 2.5\times10^7 \\ 2.5\times10^7 \\ 2.5\times10^7 \\ 2.5\times10^7 \\ 2.5\times10^7 \end{pmatrix} \quad (33)$$

Hence the approximate solution as a result of solving the tridiagonal system (33), for $j = 0, 1, 2,\ldots 9$, is given as follows:

$x_i = ih$	$P(x_i, 5)$
0	25000000
50	25000000
100	25000000
150	25000000
200	25000000
250	25000000
300	25000000
350	25000000
400	25000000
450	25000000
500	25000000

The solution of both the analytical approach and Crank Nicolson scheme for spatial step, h = 50 and h = 20 are compared respectively in figure 1. Also the values of the relative error of both solutions are plotted in figure 2.

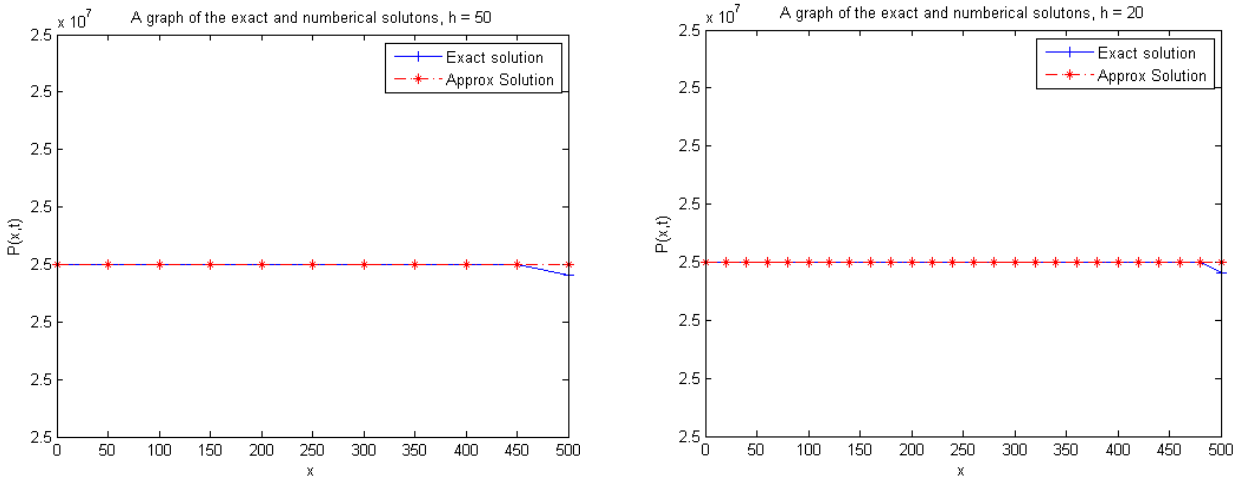

Figure 1: Comparison of exact solution and approximate solution for h = 50 and h = 20.

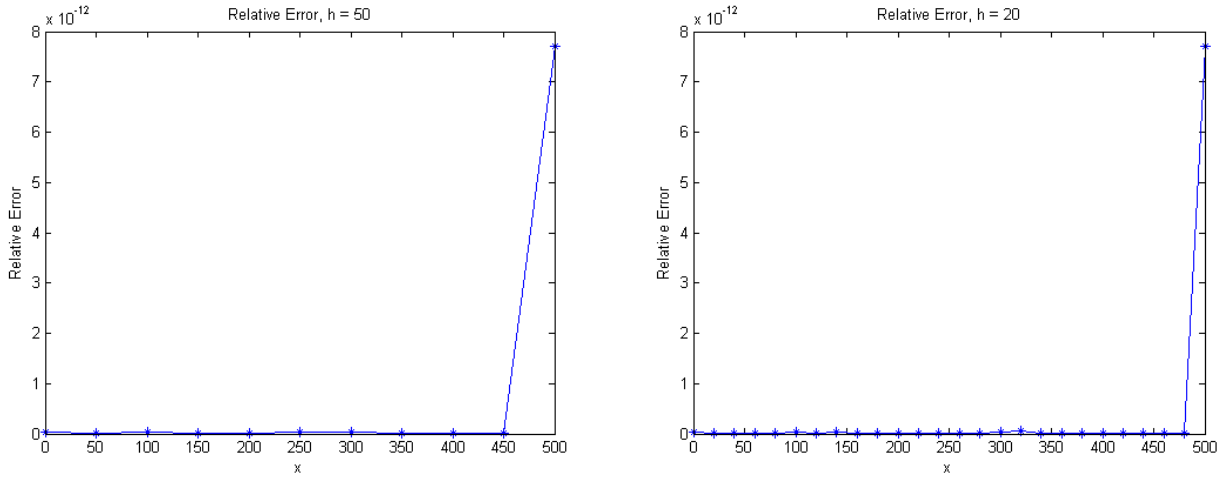

Figure 2: Relative error of the approximate solution for h = 50 and h = 20.

Clearly, we see from figure 1 that, the Crank Nicolson method has good approximate solution. However, the relative error has a sharp increase at the right boundary as shown in figure 2. We observe that the error associated with the numerical scheme diminishes as the spatial step size, h approaches zero.

5. Conclusion

An implicit Finite Difference approach for solving the parabolic partial differential equation in one dimension has been discussed. We considered the Crank Nicolson scheme due its attractive properties such as the second order accuracy in both time and spatial dimensions and stability which is unconditional. These make it preferable to the explicit method which has first order accurate in time dimension and has a restricted stability. We note that, though implicit method are computationally costly due to that fact that at each time step, a whole system of equations is solved, this is compensated by the accuracy and the unrestricted stability. The Test case has shown that the Crank Nicolson method is a good alternative to solving parabolic PDEs, where analytical solution is difficult to obtain or even impracticable to use. To further improve on the accuracy of the numerical solution to the parabolic PDE, we shall explore the avenues of wavelet approaches.

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