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Persistence and Global Dynamics of an Extended Rosenzweig-MacArthur Model

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Abstract
This paper investigates persistence and global dynamics of a tritrophic food chain model consisting of prey, predator, and super-predator. We establish dissipativeness, ultimate boundedness of an invariant region in the state space of this model via the notion of omega-limit sets, absorbing region and global attractor. We explore Freedman-Waltman theorem, and Bendixson-Dulac theorem to guarantee persistence conditions of the model. Lyapunov's functionals and Lyapunov-LaSalle invariance principle ensure the existence of global asymptotic stability of the system. Numerical responses, phase-portrait and phase-flows were used to illustrate propositions and lemmas.
Key words: *Global asymptotic stability; Lyapunov functional; Persistence.*

1. Introduction

Mathematical modelling of multiple interacting species in an ecological system explores robust biological phenomenon such as persistence of the interacting species. This phenomenon occurs naturally in a real ecological setting, which implies the long-term survival of each component populations of the interacting species. (Takeuchi, 1996; Kuang, 2001). Mathematically, persistence of a dynamical system means that the boundary of non-negative octant cone of the phase space is repelling, and no limit cycles (Hofbauer, & Schreiber, 2004; Nindjin, & Aziz-Alaoui, 2007). In vector field analysis of such a non-linear dynamical system, the long-term persistence of this ecological vector field corresponds to a global attractor that is bounded away from extinction; all species are present and none of them follows extinction (Dubey, & Upadhyay, 2004).

On the other hand, dynamical behaviors of multiple interacting species such as global stability analysis ensures the co-existence of the interacting species, while persistence guarantees non-extinction of such biological systems. These phenomena are widely studied in mathematical modelling of multiple interacting species, and models that exhibit these rich dynamics are Holling's type and Kolmogorov's type models. In Butler and Waltman (1985) Mathematical theory which responded to abstraction of persistence in dynamical systems was studied. Freedman and Waltman (1983) investigated the theoretical approaches of persistence and extinction of some class of Kolmogorov's type models. Xu, Shao and Li (2012) provides some sufficient conditions for the uniformly strong persistence of an asymptotically periodic predator-prey delay system. The conditions of coexistence and extinction in two predators-one-prey model with non-periodic solution was investigated by (Alebraheem and Abu-Hasan, 2012).

Kar, and Batabyal (2010) offered some mathematical analysis of the dynamics of a two prey one predator system in the presence of time delay due to gestation.

They derived criteria which guarantee the persistence of the three species and global dynamics. Kesh, Sarkar, Kand, and Ray (2000) proposed and analyzed a mathematical model of two competing prey and one predator species, where the prey species follows Lotka-Volterra dynamics and predator uptake functions are ratio-dependent. They derived conditions for the existence of different boundary equilibria, global stability, and strong persistence of the model.

These models depict and predict a much more realistic ecological system, while incorporating non-linearity assumptions; ecological phenomenal parameters, growth rates, death rates, environmental carrying capacities, stage structures, allee effect, patch-diffusions, predation effect, cascade migrations, spatiotemporal-patterns, discrete delay-in-time, and so on. Therefore, the best we can explore is to formulate and study analyzable models that could describe possible realities in an ecological system. These are made possible using mathematical tools such as maple software and theory of differential equations. One may see for details (Lynch, 2010; Brauer, & Chavez-Castillo, 2012; Nagle, Saff, & Snider, 2012; Banerjee, 2014; Shavin, 2015).

In this paper, we consider an extended Rosenzweig-MarArthur tritrophic food web model, studied by (Feng, Freeze, Xu, & Rocco, 2014). We obtained a topologically equivalence dynamical system using non-dimensionalization of the state variables with their respective ecological dimensionless parameters, which preserves the orientation of the phase space trajectories (Joshua, Akpan, Madubueze & 2016).
The new dimensionless model is as follows:

$$\frac{dx}{d\tau} = \alpha x\left(1-\frac{x}{\kappa}\right) - \eta\frac{x}{1+x}y - \frac{x}{1+x}z$$
$$\frac{dy}{d\tau} = \varepsilon\frac{x}{1+x}y - \xi y - \sigma\frac{y}{1+y}z$$
$$\frac{dz}{d\tau} = \beta\frac{y}{1+y}z - \mu z + \beta\frac{x}{1+x}z \qquad 1.1$$

subject to initial conditions; $x(0) = x_0, y(0) = y_0, z(0) = z_0$, where $x(\tau)$, $y(\tau)$, $z(\tau)$ are the population densities of the interacting species of the model, α is growth rate of the prey, κ is the environmental carrying capacity of the prey, η is the maximum predation rate of prey by predator, ε is the maximum biomass conversion efficiency constant of predator, ξ is the death rate of the predator, μ is the super-predator death rate, β is a free parameter; maximum super-predator biomass conversion efficiency of both prey and predator to biomass of the super-predator.

2.0 Boundedness and Dissipativeness of the Model

The density functions of system (1.1) are continuously differentiable in the non-negative cone of the state space $\Re^3_+ = \{(x(\tau), y(\tau), z(\tau)) \in \Re^3 : x(\tau) \geq 0, y(\tau) \geq 0, z(\tau) \geq 0\}\ \forall\, \tau \geq 0$. We denote the positive octant cone of the solution space as $Int\Re^3_+ = \{(x(\tau), y(\tau), z(\tau)) \in \Re^3 : x(\tau) > 0, y(\tau) > 0, z(\tau) > 0)\}, \forall\, \tau \geq 0$. The phase flows $\Phi_t(t_0; x, y, z)$ of system (1.1) are said to be ultimately bounded with respect to the state space $\Re^3$, if there exists a positively invariant compact region $\mathcal{A} \in \Re^3$ and a finite time $T(T = T(\tau_0; x, y, z))$ such that, for any $(\tau_0; x, y, z) \in \Re \times \Re^3$, $\Phi_\tau(\tau_0; x, y, z) \in \mathcal{A}, \forall\ \tau > T$ (Aziz-Aalaoui, Okiye, 2000). By phase flows of system (1.1), we mean its solution trajectories, and for properties and geometry of flows one may see (Wiggins, 2003; Murza, 2009). If the compact region is absorbing, then the phase flows $\Phi_\tau(\tau_0; x, y, z) \in Int(\mathcal{A})$ for some $\tau > T$.

This means that all orbits generated by the phase flows $\Phi_\tau(\tau_0; x, y, z)$ are eventually absorbed by the interior of $\mathcal{A}$. Moreover, suppose system (1.1) is dissipative, then one can make strong statement about the phase portrait of the system.
By dissipativenes, we mean that all population functions initiating in the nonnegative octant of the state space $\Re^3_+$, are uniformly limited in time by their environmental carrying capacities (Freedman and Hongshun, 1988).

2.1 Lemma 1: (Hpfbauer & Shreiber, 2004; Birnir, 2008).
If system (1.1) has an absorbing region $\mathfrak{D}$, then the $\omega -$limit set of $\mathcal{A}$ defined as; $\omega(\mathfrak{D}) = \bigcap_{t_0>0}\{\overline{\bigcup_{t\geq t_0} \Phi_t(t_0; x, y, z)}: (x, y, z) \in \mathfrak{D}\}$ is a unique global attractor. Equivalently, the $\omega -$limit point of the phase flow is a fixed point $(x^*, y^*, z^*) \in \Re^3_+ \ni, \exists$ a sequence $t_n \to \infty$ as $n \to \infty$ and $lim_{t\to\infty}\Phi_t(t_0; x, y, z) \to (x^*, y^*, z^*)$

2.2 Proposition 1:
Given the phase flows $\Phi_t(t_0; x, y, z)$ of system (1.1) with positive initial conditions. Let $\mathcal{A}$ be a closed and bounded subset of the state space $\Re^3_+$. Then,

(i) $\mathcal{A}$ is a positively invariant subset of $\Re^3_+$.
(ii) phase flows initiating in $\Re^3_+$ are ultimately bounded on $\Re^3_+$ and eventually enters the attracting set in $\mathcal{A}$, if $\frac{\varepsilon\kappa}{1+\kappa} > \xi$ and $\frac{2\beta\kappa+\beta}{1+\kappa} > \mu$.
(iii) system (1.1) is dissipative.
(iv) phase flows of system (1.1) have a unique global attractor in $\mathcal{A}$.

Proof:
(i) Let $(x(\tau), y(\tau), z(\tau))$ be any solution of system (1.1). Define the compact subset $\mathcal{A}$ of $\Re^3_+$ as; $\mathcal{A} = \{(x, y, z) \in \Re^3_+: 0 \leq x(\tau) \leq L_1, 0 \leq y(\tau) \leq L_2, 0 \leq z(\tau) \leq L_3)\}\ \forall\, \tau \geq 0$. Observe that with positive initial conditions, $x(0) = x_0 > 0, y(0) = y_0 > 0, z(0) = z_0 > 0$ $\forall\, \tau \geq 0$. This implies that by choosing $(L_1, L_2, L_3) \equiv (x_0, y_0, z_0)$ then $(x(0), y(0), z(0)) \in \mathcal{A}$. Equivalently, the phase flows at initial conditions $\Phi_\tau(0; x_0, y_0, z_0) \in \mathcal{A}\ \ \forall\, \tau_0 = 0$. Also, it suffices to show that $\Phi_\tau(\tau_0; x, y, z) \in \mathcal{A}\ \ \forall\, \tau > 0$. Since $\Phi_\tau(0; x_0, y_0, z_0) \in \mathcal{A}$, then $\Phi_\tau(\tau_0; x, y, z)$ remains nonnegative $\forall\, \tau \geq 0$, by the properties of phase flows (Wiggins, 2003). By definition of $\mathcal{A}$, there exist positive constants (L_1, L_2, L_3) such that $\overline{lim_{\tau\to+\infty}}x(\tau) \leq L_1$, $\overline{lim_{\tau\to+\infty}}y(\tau) \leq L_2$, $and\ \overline{lim_{\tau\to+\infty}}z(\tau) \leq L_3$. This implies that $\big(x(\tau), y(\tau), z(\tau)\big) \to \mathcal{A}, as\ \tau \to +\infty$ and equivalently the phase flows $\Phi_\tau(\tau_0; x, y, z) \to \mathcal{A}$ as $\tau \to +\infty$. Hence by LaSalle invariance principle (LaSalle, 1968), $\mathcal{A}$ is a well-defined positive invariant subset of the state space $\Re^3_+$, and trapped every flow of system (1.1).
(ii) It suffices to show the conditions for ultimate boundedness of system (1.1). Using the theory of differential inequalities, we have that the first equation of system (1.1) satisfies the logistic assumption,

$$\frac{dx}{d\tau} \leq \alpha x\left(1 - \frac{x}{\kappa}\right)\ \ and\ x(\tau) \leq \frac{x_0\kappa}{x_0 + (x_0 - \kappa)e^{-\kappa\tau}}$$

By standard comparison theorem, we have that $\overline{lim_{\tau\to+\infty}}x(\tau) \leq \kappa = L_1$

Also, the second equation of system (1.1) satisfies,

$$\frac{dy}{d\tau} \leq \epsilon\frac{\kappa}{1+\kappa}y - \xi y, \qquad if\ \ \overline{lim_{\tau\to+\infty}}x(\tau) \leq \kappa = L_1$$
$$\overline{lim_{\tau\to+\infty}}y(\tau) \leq y_0 e^{\left(\frac{\varepsilon\kappa}{1+\kappa} - \xi\right)\tau} = L_2$$

Assuming non-negativity of Holling's type II functional response say,

$$0 \leq \frac{y}{1+y} \leq 1 \; and \; \overline{lim_{\tau \to +\infty}} x(\tau) \leq \kappa = L_1,$$

then we have that the third equation of system (1.1) yields,

$$\frac{dz}{d\tau} \leq \left(\frac{\beta + 2\beta\kappa}{1+\kappa} - \mu\right) z \; and \; \overline{lim_{\tau \to +\infty}} z(\tau) \leq y_0 e^{\left(\frac{\beta+2\beta\kappa}{1+\kappa} - \mu\right)\tau} = L_3$$

Hence, the phase flows of system (1.1) remain positive and are trapped in the attracting set $\mathcal{A} \subset \Re_+^3 \; if \; \frac{\beta+2\beta\kappa}{1+\kappa} > \mu \; and \; \frac{\varepsilon\kappa}{1+\kappa} > \xi \; as \; \tau \to +\infty$.

(iii) Let $(x(\tau), y(\tau), z(\tau))$ be any solution of system (1.1) with positive initial conditions. Define an absolutely continuous function $(\tau) = \frac{\eta_1}{\eta_2} x(\tau) + y(\tau) + \frac{\theta_1}{\theta_2} z(\tau)$. Calculating its time derivative along trajectories of the phase flows of system (1.1) yields,

$$W'(\tau) = \frac{\eta_1}{\eta_2} x'(\tau) + y'(\tau) + \frac{\theta_1}{\theta_2} z'(\tau)$$

$$W'(\tau) + \psi W(\tau) \leq \frac{\eta_1}{\eta_2}(\alpha + \psi) + (\psi - \xi) y + \left(\frac{\theta_1}{\theta_2} - \frac{\mu\theta_1}{\theta_2}\right) z$$

choosing $\psi = \min(\xi, \mu)$, we have that

$$W'(\tau) + \psi W(\tau) \leq \frac{\eta_1}{\eta_2}(\alpha + \psi)$$

By comparison lemma (Aziz-Alaoui, Okiye, 2000), we have that $\forall \, \tau \geq \widetilde{T} \geq 0$

$$W(\tau) \leq \frac{\eta_1}{\eta_2}(\alpha + \psi) - \left(\frac{\eta_1}{\eta_2}(\alpha + \psi) - W(\widetilde{T})\right) e^{-\psi(\tau - \widetilde{T})} \quad \forall \, \tau \geq 0$$

$$W(\tau) \leq \frac{\eta_1}{\eta_2}(\alpha + \psi) - \left(\frac{\eta_1}{\eta_2}(\alpha + \psi) - W(0)\right) e^{-\psi\tau}, \; \widetilde{T} = 0 \qquad 1.2$$

Thus, all solutions of system (1.1) eventually enters the trapped region $\mathfrak{D} \supset \mathcal{A}$ defined as, $\mathfrak{D} = \left\{(x, y, z) \in \Re_+^3 : 0 \leq W(\tau) \leq \frac{\eta_1}{\eta_2}(\alpha + \psi)\right\}$ and

$$\overline{lim_{\tau \to +\infty}} W(\tau) \leq \frac{\eta_1}{\eta_2}(\alpha + \psi) + \frac{\epsilon}{2}, \quad \text{for any } \epsilon > 0 \qquad 1.3$$

and all species are uniformly bounded for all initial conditions in the state space $\Re_+^3$. Since (i), (ii) and (1.3) hold then there exist ω −limit sets, $\omega(\mathcal{A}) \subset \mathcal{A} = \{(x, y, z) \in \Re_+^3 : 0 \leq x(\tau) \leq L_1, 0 \leq y(\tau) \leq L_2, 0 \leq z(\tau) \leq L_3)\} \; \forall \, \tau \geq 0$ (Upadhyay, 2011;Sahoo, 2012). Hence, the proof is completed.

(iv) Since (i), (ii), and (iii) hold, and the trapped region $\mathfrak{D} \supset \mathcal{A}$ is an invariant set, we claim that $\mathfrak{D}$ is an absorbing region of phase flows of system (1.1) (Birnir, 2008), and then $\omega(\mathcal{A}) = \omega(\mathfrak{D}) \subset \mathfrak{D}$ for some $\tau \geq \widetilde{T} \geq 0$. To establish this claim, we seek the positive time $\tau \geq \widetilde{T} \geq 0$ as follows using (1.2) and (1.3);

$$\frac{\eta_1}{\eta_2}(\alpha + \psi) - \left(\frac{\eta_1}{\eta_2}(\alpha + \psi) - W(0)\right) e^{-\psi\tau} \leq \frac{\eta_1}{\eta_2}(\alpha + \psi) + \frac{\epsilon}{2}$$

$$\tau \geq \log_e \left(\frac{2W(0) - \frac{2\eta_1\kappa}{\eta_2\psi}(\alpha + \psi)}{\epsilon}\right)^{\frac{1}{\psi}} \qquad 1.4$$

then applying lemma I, means we have shown that the ω −limit sets of phase flows $\Phi_\tau(\tau_0; x, y, z)$ of system (1.1) are eventually trapped in the absorbing region $\mathfrak{D}$ for positive time τ, and for positive initial conditions as seen in (1.4). Thus, there exists a unique fixed point; a global attractor of the phase flows $\Phi_\tau(\tau_0; x, y, z)$ of system (1.1). Hence, the proof is complete.

3.0 Existence of Positive Equilibrium point.

We obtain the critical point of system (1.1), by solving the planar sub-system during it steady-state; independent of time, and deduce the positivity conditions of each critical point. The model exhibited the following trivial, and semi-trivial equilibria:

$$E_0(x^* = 0, y^* = 0, z^* = 0),\ E_1(x^* = K, y^* = 0, z^* = 0),$$

$$E_2\left(x^* = \frac{\xi}{\epsilon - \xi}, y^* = \frac{\alpha\epsilon(\kappa\epsilon - \kappa\xi - \xi)}{\eta\kappa(\epsilon - \xi)^2}, z^* = 0\right); if\ \epsilon > \xi, \kappa > \frac{\xi}{\epsilon - \xi}$$

$$E_3\left(\check{x} = \frac{\mu}{\beta - \mu}, \check{y} = 0, \check{z} = \frac{\alpha\beta(\kappa\beta - \kappa\mu - \mu)}{\kappa(\beta - \mu)^2}\right);if\ \beta > \mu, \quad \kappa > \frac{\mu}{\beta - \mu}$$

3.1 Lemma 2: Existence of Positive Coexistence Equilibrium Point.

The system (1.1) has a positive coexistence equilibrium point, say; $E_4(X = x^*, Y = y^*, Z = z^*)$ if;

$$(i)\ y^* = \frac{\mu(1 + x^*) - \beta x^*}{(\beta - \mu)(1 + x^*) + \beta x^*}, z^* = \frac{\beta(\epsilon x^* - \xi(1 + x^*))}{\sigma((\beta - \mu)(1 + x^*) + \beta x^*)},$$

$$P(X) = X^3 + P_0X^2 + P_1X + P_2 = 0, P_0 < 0, P_1 < 0, P_2 < 0, \frac{\xi}{\epsilon - \xi} < X < \frac{\mu}{\beta - \mu}$$

$$P_0 = \frac{2\alpha\sigma\kappa\beta - \alpha\sigma\kappa\mu - 3\alpha\sigma\beta + 2\alpha\sigma\mu}{\alpha\sigma\mu - 2\alpha\sigma\beta}$$

$$(iii)\ P_1 = \frac{3\alpha\sigma\kappa\beta - 2\alpha\sigma\kappa\mu + \kappa\sigma\eta\beta - \kappa\sigma\eta\mu - \alpha\sigma\beta + \alpha\sigma\mu + \kappa\beta\xi - \beta\kappa\epsilon}{\alpha\sigma\mu - 2\alpha\sigma\beta}$$

$$P_2 = \frac{\alpha\beta\sigma\kappa - \alpha\sigma\kappa\mu - \eta\kappa\mu\sigma + \kappa\beta\xi}{\alpha\sigma\mu - 2\alpha\sigma\beta}$$

4.0 Persistence of the Model.

The term persistence is given to dynamical systems in which strictly positive solutions do not approach the boundary of the non-negative cone as $\tau \to +\infty$. One requires that the fixed points on the boundaries are repelling with respect to the orthogonal plane that contains them (Butler and Waltman, 1986). Additionally, if there are no limit cycles on the faces of the boundary, then the system persists.

4.1 Lemma 3. (Freedman, & Waltman, 1983).

Consider the Kolmogorov's type dynamical system of model (1.1) as;

$$\frac{dx}{d\tau} = xF(x, y, z)$$
$$\frac{dy}{d\tau} = yG(x, y, z)$$
$$\frac{dz}{d\tau} = zH(x, y, z) \qquad 1.5$$

satisfying positive initial conditions, $x(0) = x_0, y(0) = y_0, z(0) = z_0$.

A1. $x(\tau)$ is prey population function, $y(\tau)$ is predator population function predating exclusively on prey $x(\tau)$, and $z(\tau)$ is super-predator population function predating exclusively on both prey $x(\tau)$, and predator $y(\tau)$, which implies that, $\frac{\partial F}{\partial y} < 0, \frac{\partial F}{\partial z} \leq 0, \frac{\partial G}{\partial x} > 0, \frac{\partial H}{\partial x} \geq 0, \frac{\partial H}{\partial y} > 0, G(0, y, z) < 0, H(0,0, z) < 0$.

A2. The prey population function grows to carrying capacity in the absence of predation effect, which implies that, $F(0,0,0) > 0, \frac{\partial F}{\partial x}(x, y, z) < 0, \exists\ K > 0\ \ni F(K, 0,0) = 0$

A3. There are no equilibrium points on the y or z coordinates and no equilibrium point on $y - z$ plane.
A4. Each predator can survive on the prey, this implies that there exist equilibrium points, $E_2(x^*, y^*, 0)$ and $E_3(x^*, 0, z^*) \ni F(x^*, y^*, 0) = G(x^*, y^*, 0) = 0$ and $F(\check{x}, 0, \check{z}) = H(\check{x}, 0, \check{z}) = 0$ and $x^*, y^*, z^*, \check{x}, \check{z} > 0$; $x^* < K, \check{x} < K$
We proceed to state the theorem that guarantees the non-existence of limit cycles on the boundaries of the invariant region $\mathcal{A}$ of system (1.1). One may see (Brauer, Castillo-Chavez, 2012)

4.2 Lemma 4: (Bendixson-Dulac Theorem,1934).
Consider a smooth differential equation for a planar subsystem,

$$x'(\tau) = g(x, y)$$
$$y'(\tau) = h(x, y) \qquad 1.6$$

If there exists a smooth Dulac function $B(x, y)$ defined on a simply connected region, say $\mathcal{A} \in \Re_+^3$ such that the quantity,

$$div(B(x,y)\bar{F}(x,y)) = \frac{\partial}{\partial x}\{B(x,y).g(x,y)\} + \frac{\partial}{\partial y}\{B(x,y).h(x,y)\}$$

is either strictly positive or strictly negative in $\mathcal{A}$, then system (1.6) has no periodic orbits in $\mathcal{A}$, where $\bar{F}(x, y) = g(x, y)i + h(x, y)j$.

4.3 Proposition 2. (Freedman-Waltman Theorem, 1983).
If conditions of lemma 2 and 3 are satisfied, and growth functions

$$G(\check{x}, 0, \check{z}) > 0, \quad H(x^*, y^*, 0) > 0 \qquad 1.7$$

then system (1.1) persists.

Proof:
Consider the Kolmogorov's growth functions of system (1.1) as;

$$F(x, y, z) = \alpha\left(1 - \frac{x}{\kappa}\right) - \eta\frac{y}{1+x} - \frac{z}{1+x}$$
$$G(x, y, z) = \epsilon\frac{x}{1+x} - \xi - \sigma\frac{z}{1+y}$$
$$H(x, y, z) = \beta\frac{y}{1+y} - \mu + \beta\frac{x}{1+x} \qquad 1.8$$

Observe that $\frac{\partial F}{\partial y} = -\eta\frac{y}{1+x} < 0$; $\frac{\partial F}{\partial z} = -\frac{z}{1+x} < 0$; $\frac{\partial G}{\partial x} = \frac{\varepsilon}{(1+x)^2} > 0$; $\frac{\partial H}{\partial x} = \frac{\beta}{(1+x)^2} > 0$; $\frac{\partial H}{\partial y} = \frac{\beta}{(1+x)^2} > 0$; $G(0, y, z) = -\xi - \sigma\frac{z}{1+y} < 0, H(0,0,z) = -\mu < 0$ So, condition A1 of lemma 3 is satisfied.
Also, we have that $F(0,0,0) = \alpha > 0, \frac{\partial F}{\partial x}(x, y, z) = \frac{\eta y + z}{(1+x)^2} - \frac{\alpha}{\kappa} < 0, \exists\, K > 0 \ni F(K, 0,0) = 0$ for $K = \kappa$ and condition A2 of lemma 3 is satisfied.
There are no equilibrium points on the y or z coordinate axes and no equilibrium on $y - z$ plane, because there are no interspecific competition amongst the predator population functions. Condition A3 of lemma 3 is satisfied.

We obtain the semi-trivial equilibrium points by solving the equations; $F(x^*, y^*, 0) = G(x^*, y^*, 0) = 0$ and $F(\breve{x}, 0, \breve{z}) = H(\breve{x}, 0, \breve{z}) = 0$ say,

$$E_2\left(x^* = \frac{\xi}{\epsilon - \xi}, y^* = \frac{\alpha\epsilon(\kappa\epsilon - \kappa\xi - \xi)}{\eta\kappa(\epsilon - \xi)^2}, z^* = 0\right); if\ \epsilon > \xi, \kappa > \frac{\xi}{\epsilon - \xi}$$

$$E_3\left(\breve{x} = \frac{\mu}{\beta - \mu}, \breve{y} = 0, \breve{z} = \frac{\alpha\beta(\kappa\beta - \kappa\mu - \mu)}{\kappa(\beta - \mu)^2}\right); if\ \beta > \mu, \quad \kappa > \frac{\mu}{\beta - \mu}$$

Thus, condition A4 of lemma 3 is satisfied. Analogously, lemma 3 is verified.

Next, we establish conditions satisfying lemma 4. Consider the $x - y$ planar sub-system of system (1.1),

$$\frac{dx}{d\tau} = \alpha x\left(1 - \frac{x}{\kappa}\right) - \eta\frac{x}{1+x}y = g(x,y)$$
$$\frac{dy}{d\tau} = \epsilon\frac{x}{1+x}y - \xi y = \qquad h(x,y) \qquad 1.9$$

Let $B(x,y) = \frac{1}{xy}$ be a Dulac function. Clearly, $B(x,y) > 0$ is a smooth analytic function in the interior of positive quadrant $x - y$ plane orthogonal to the $z-$ direction. Then, the quantity

$$\frac{\partial}{\partial x}\{B(x,y).g(x,y)\} + \frac{\partial}{\partial y}\{B(x,y).h(x,y)\} = \frac{\eta y}{(1+x)^2} - \frac{\alpha}{y\kappa} \neq 0$$

and is strictly positive if $\frac{\eta y}{(1+x)^2} > \frac{\alpha}{y\kappa}$ or strictly negative if $\frac{\eta y}{(1+x)^2} < \frac{\alpha}{y\kappa}$. Thus, there are no periodic orbits in the positive quadrant of $x - y$ plane. Analogously, using the $x - z$ planar sub-system of system (1.1). Let $B(x,z) = \frac{1}{xz}$ be a Dulac function. Clearly, $B(x,z) > 0$ is a smooth analytic function in the interior of positive quadrant $x - z$ plane, orthogonal to the $y-$ direction. Then, the quantity

$$\frac{\partial}{\partial x}\{B(x,y).g(x,y)\} + \frac{\partial}{\partial y}\{B(x,y).h(x,y)\} = \frac{1}{(1+x)^2} - \frac{\alpha}{z\kappa} \neq 0$$

and is strictly positive if $\frac{1}{(1+x)^2} > \frac{\alpha}{z\kappa}$ or strictly negative if $\frac{1}{(1+x)^2} < \frac{\alpha}{z\kappa}$. Thus, there are no periodic orbits in the positive quadrant of $x - z$ plane.

Furthermore, conditions (1.7) are needed to ensure that the two semi-trivial equilibria in the interior of the coordinate planes are unstable with respect to their orthogonal directions. These conditions are satisfied if there exist at least one positive eigenvalue associated with each equilibrium point. The local stability analysis of system (1.1) at the semi-trivial equilibria guarantee the persistence conditions as follows;

If

$$\frac{\kappa(\varepsilon\mu - \xi\beta)(\beta - \mu)^2 - \alpha\beta^2\sigma(\kappa\beta - \kappa\mu - \mu)}{\kappa(\beta - \mu)^2} > 0, \qquad then\ G(\breve{x}, 0, \breve{z}) > 0$$

$$\frac{(\beta\xi - \mu\varepsilon)[\eta\kappa(\xi - \varepsilon)^2 - \alpha\varepsilon(\kappa\xi - \kappa\varepsilon + \xi)] - \alpha\beta\varepsilon^2(\kappa\xi - \kappa\varepsilon + \xi)}{\varepsilon[\eta\kappa(\xi - \varepsilon)^2 - \alpha\varepsilon(\kappa\xi - \kappa\varepsilon + \xi)]} > 0$$

$then\ H(x^*, y^*, 0) > 0$. Hence, the proof is complete.

4.3 Corollary 1: Nontrivial equilibrium points of system (1.1) are locally asymptotically stable.

5.0 Global stability of the model.

In this section we prove global asymptotic stability of the system (1.1) using strictly positive Lyapunov's functions $V(x,y,z) \in \mathcal{A}$. The essence of $\dot{V}(x,y,z)$ is to show how the system trajectories are evolving w.r.t the contours of $V(x,y,z)$. If $\dot{V}(x,y,z) < 0$ then the trajectories of the system are moving closer to the equilibrium point. If $\dot{V}(x,y,z) > 0$ then the trajectories of the system are moving away from the equilibrium point. If a strong Lyapunov's function exists for system (1.1), then that equilibrium point is globally asymptotically stable. If a strong Lyapunov's function is positive definite, but its time-derivative is negative semi-definite: that is, $\dot{V}(x^*,y^*,z^*) = 0$; $\dot{V}(x,y,z) \leq 0$, $\forall\ (x,y,z) \neq (x^*,y^*,z^*)$, then additional requirement to prove global stability of the equilibrium point is the Lyapunov-LaSalle invariance principle.

5.1 Lemma 4: (Lyapunov-LaSalle, 1968).
If the time-derivative of a strongly positive Lyapunov's function is negative semi-definite, then every solution of system (1.1) approaches the largest invariant subset of the set of points in the state space, $\Re^3_+$ for which $\dot{V}(x,y,z) = 0$ as $\tau \to +\infty$.

5.2 Lemma 4: (Sylvester's criterion).
An arbitrary 3×3 symmetric $M -$ matrix is positive definite iff all the upper-left leading principal minors of A are positive.

5.3 Proposition 3.
The Prey equilibrium point $E_1(x^* = K, y^* = 0, z^* = 0)$ is globally asymptotically stable if $\frac{\varepsilon\kappa}{1+\kappa} < \xi$ and $\frac{\beta+2\beta\kappa}{1+\kappa} < \mu$ in $\mathcal{A}$
Proof:
Consider a positive definite Lyapunov's function $V(x,y,z) = \frac{1}{2}(x-\kappa)^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2$. $V(x,y,z)$ is zero at equilibrium point and is positive for all other positive values of x, y, z. Thus,
$E_1(x^* = K, y^* = 0, z^* = 0)$ is the global minimum of $V(x,y,z)$. The time derivative of $V(x,y,z)$ along the trajectories of system (1.1) yields;

$$\dot{V}(x,y,z) = (x-\kappa)\frac{dx}{d\tau} + y\frac{dy}{d\tau} + z\frac{dz}{d\tau}$$

$$\dot{V}(x,y,z) \leq -\alpha x(x-\kappa)^2 + \left(\frac{\varepsilon\kappa}{1+\kappa} - \xi\right)y^2 + \left(\frac{\beta+2\beta\kappa}{1+\kappa} - \mu\right)z^2 \qquad 2.0$$

From (2.0) $\dot{V}(x,y,z) \leq 0$ if $\frac{\varepsilon\kappa}{1+\kappa} < \xi$ and $\frac{\beta+2\beta\kappa}{1+\kappa} < \mu$. It is easy to see that $\dot{V}(x,y,z) = 0$ when $(x,y,z) = (x^* = \kappa, y^* = 0, z^* = 0)$. Hence by lemma 5.1, $E_1(x^* = K, y^* = 0, z^* = 0)$ is globally asymptotically stable, satisfying the attractivity condition, $lim_{\tau\to+\infty}(x(\tau), y(\tau), z(\tau)) = (\kappa, 0,0)$. Hence, the proof is complete.

5.3 Proposition 4:
The prey-predator equilibrium point $E_2\left(x^* = \frac{\xi}{\varepsilon-\xi}, y^* = \frac{\alpha\varepsilon(\kappa\varepsilon-\kappa\xi-\xi)}{\eta\kappa(\varepsilon-\xi)^2}, z^* = 0\right)$ is globally asymptotically stable in $\mathcal{A}$ if $\frac{\alpha}{\kappa} + \frac{\eta}{2} - \frac{2\eta y^*+\varepsilon}{2(1+x^*)} > 0, \frac{\varepsilon-\eta(1+x^*)}{2(1+x^*)} > 0$.
Proof:
Consider the Lyapunov's function

$$V(x,y,z) = \left(x - x^* - x^* In\frac{x}{x^*}\right) + \left(y - y^* - y^* In\frac{y}{y^*}\right)$$

defined in $\mathcal{A}$. Taking the time derivative of the Lyapunov's function along the trajectories of system (1.1) and a little algebraic manipulation yields;

$$\dot{V}(x,y,z) = \left(\frac{x-x^*}{x}\right)\frac{dx}{d\tau} + \left(\frac{y-y^*}{y}\right)\frac{dy}{d\tau}$$

$$\dot{V}(x,y,z) = \left(\frac{\eta y^*}{1+x^*} - \frac{\alpha}{\kappa}\right)(x-x^*)^2 + \left(\frac{\varepsilon-\eta(1+x^*)}{(1+x^*)(1+x)}\right)(x-x^*)(y-y^*)$$

$$\dot{V}(x,y,z) \le \left(\frac{\eta y^*}{1+x^*} - \frac{\alpha}{\kappa}\right)(x-x^*)^2 + \left(\frac{\varepsilon-\eta(1+x^*)}{(1+x^*)}\right)(x-x^*)(y-y^*)$$

$$\dot{V}(x,y,z) \le -\left[\left(\frac{\alpha}{\kappa} + \frac{\eta(1+x^*)}{2(1+x^*)} - \frac{2\eta y^*+\varepsilon}{2(1+x^*)}\right)(x-x^*)^2 + \left(\frac{\varepsilon-\eta(1+x^*)}{2(1+x^*)}\right)(y-y^*)^2\right]$$

Now $\dot{V}(x,y,z) \le 0$ if $\frac{\alpha}{\kappa} + \frac{\eta}{2} - \frac{2\eta y^*+\varepsilon}{2(1+x^*)} > 0, \frac{\varepsilon-\eta(1+x^*)}{2(1+x^*)} > 0$ and $\dot{V}(x,y,z) = 0$ only when $x = x^*, y = y^*$.Then, the largest invariant subset of the set $E = \{(x,y,z): x = x^*, y = y^*, z \ge 0\}$ is the singleton $E_2\left(x^* = \frac{\xi}{\epsilon-\xi}, y^* = \frac{\alpha\epsilon(\kappa\epsilon-\kappa\xi-\xi)}{\eta\kappa(\epsilon-\xi)^2}, z^* = 0\right)$. Hence, by using lemma 4 the proof is complete.

5.4 Proposition 5:

The prey-super predator equilibrium point $E_3\left(\breve{x} = \frac{\mu}{\beta-\mu}, \breve{y} = 0, \breve{z} = \frac{\alpha\beta(\kappa\beta-\kappa\mu-\mu)}{\kappa(\beta-\mu)^2}\right)$ is globally asymptotically stable in $\mathcal{A}$ if $\frac{1}{2} + \frac{\alpha}{\kappa} - \frac{2\breve{z}+\beta}{(1+\breve{x})} > 0$, $\frac{\beta-(1+\breve{x})}{2(1+\breve{x})} > 0$.

Proof:

Consider the Lyapunov's function

$$V(x,y,z) = \left(x - \breve{x} - \breve{x}In\frac{x}{\breve{x}}\right) + \left(z - \breve{z} - \breve{z}In\frac{z}{\breve{z}}\right)$$

defined in $\mathcal{A}$. Taking the time derivative of the Lyapunov's function along the trajectories of system (1.1) and a little algebraic manipulation yields;

$$\dot{V}(x,y,z) = \left(\frac{x-\breve{x}}{x}\right)\frac{dx}{d\tau} + \left(\frac{z-\breve{z}}{z}\right)\frac{dz}{d\tau}$$

$$\dot{V}(x,y,z) = \left(\frac{\breve{z}}{(1+\breve{x})(1+x)} - \frac{\alpha}{\kappa}\right)(x-\breve{x})^2 + \left(\frac{\beta-(1+\breve{x})}{(1+\breve{x})(1+x)}\right)(x-\breve{x})(z-\breve{z})$$

$$\dot{V}(x,y,z) \le \left(\frac{\breve{z}}{(1+\breve{x})} - \frac{\alpha}{\kappa}\right)(x-\breve{x})^2 + \left(\frac{\beta-(1+\breve{x})}{(1+\breve{x})}\right)(x-x^*)(z-\breve{z})$$

$$\dot{V}(x,y,z) \le -\left[\left(\frac{1}{2} + \frac{\alpha}{\kappa} - \frac{2\breve{z}+\beta}{(1+\breve{x})}\right)(x-\breve{x})^2 + \left(\frac{\beta-(1+\breve{x})}{2(1+\breve{x})}\right)(z-\breve{z})^2\right]$$

Now $\dot{V}(x,y,z) \le 0$ if $\frac{1}{2} + \frac{\alpha}{\kappa} - \frac{2\breve{z}+\beta}{(1+\breve{x})} > 0$, $\frac{\beta-(1+\breve{x})}{2(1+\breve{x})} > 0$ and $\dot{V}(x,y,z) = 0$ only when $x = \breve{x}, z = \breve{z}$. Then, the largest invariant subset of the set $E = \{(x,y,z): x = x^*, y \ge 0, z = \breve{z}\}$ is the point $E_3\left(\breve{x} = \frac{\mu}{\beta-\mu}, \breve{y} = 0, \breve{z} = \frac{\alpha\beta(\kappa\beta-\kappa\mu-\mu)}{\kappa(\beta-\mu)^2}\right)$. Hence, by using lemma 4 the prove is complete.

5.50 Proposition 6:

Suppose lemma 2 holds, then the coexisting equilibrium point of system (1.1) is globally asymptotically stable if $\left(\frac{\alpha}{\kappa} - \frac{\eta y^*+z^*}{(1+x^*)}\right) + \frac{\eta(1+x^*)-\varepsilon}{2(1+x^*)} + \frac{(1+x^*)-\beta}{2(1+x^*)} > 0, \frac{\eta(1+x^*)-\varepsilon}{2(1+x^*)} - \frac{\sigma y^*}{(1+y^*)} + \frac{\sigma(1+y^*)-\beta}{2(1+y^*)} > 0, \frac{(1+x^*)-\beta}{2(1+x^*)} + \frac{\sigma(1+y^*)-\beta}{2(1+y^*)} > 0.$

Proof:

Consider the Lyapunov's function

$$V(x,y,z) = \left(x - x^* - x^* In\frac{x}{x^*}\right) + \left(y - y^* - y^* In\frac{y}{y^*}\right) + \left(z - z^* - z^* In\frac{z}{z^*}\right)$$

defined in $\mathcal{A}$. Taking the time derivative of the Lyapunov's function along the trajectories of system (1.1) and a little algebraic manipulation yields;

$$\dot{V}(x,y,z) = \left(\frac{x - x^*}{x}\right)\frac{dx}{d\tau} + \left(\frac{y - y^*}{y}\right)\frac{dy}{d\tau} + \left(\frac{z - z^*}{z}\right)\frac{dz}{d\tau}$$

$$\dot{V}(x,y,z) = -\left\{\left\{\left(\frac{\alpha}{\kappa} - \frac{\eta y^* + z^*}{(1+x^*)(1+x)}\right)(x - x^*)^2 + \frac{\eta(1+x^*) - \varepsilon}{(1+x^*)(1+x)}(x - x^*)(y - y^*)\right\}\right\}$$
$$-\left\{\left\{\frac{(1+x^*) - \beta}{(1+x^*)(1+x)}(x - x^*)(z - z^*) - \frac{\sigma y^*}{(1+y^*)(1+y)}(y - y^*)^2 + \frac{\sigma(1+y^*) - \beta}{(1+y^*)(1+y)}(y - y^*)(z - z^*)\right\}\right\}$$

$$\leq -\left\{\left\{\left(\frac{\alpha}{\kappa} - \frac{\eta y^* + z^*}{(1+x^*)}\right)(x - x^*)^2 + \frac{\eta(1+x^*) - \varepsilon}{(1+x^*)}(x - x^*)(y - y^*)\right\}\right\} -$$
$$\left\{\frac{(1+x^*) - \beta}{(1+x^*)}(x - x^*)(z - z^*) - \frac{\sigma y^*}{(1+y^*)}(y - y^*)^2 + \frac{\sigma(1+y^*) - \beta}{(1+y^*)}(y - y^*)(z - z^*)\right\}$$

$$\leq -\left\{\left(\left(\frac{\alpha}{\kappa} - \frac{\eta y^* + z^*}{(1+x^*)}\right) + \frac{\eta(1+x^*) - \varepsilon}{2(1+x^*)} + \frac{(1+x^*) - \beta}{2(1+x^*)}\right)(x - x^*)^2\right\}$$
$$-\left\{\left(\frac{\eta(1+x^*) - \varepsilon}{2(1+x^*)} - \frac{\sigma y^*}{(1+y^*)} + \frac{\sigma(1+y^*) - \beta}{2(1+y^*)}\right)(y - y^*)^2 + \left(\frac{(1+x^*) - \beta}{2(1+x^*)} + \frac{\sigma(1+y^*) - \beta}{2(1+y^*)}\right)(z - z^*)^2\right\}$$

The bilinear quadratic form of $\dot{V}(x,y,z)$ yields, $\dot{V}(x,y,z) \leq -X^T MX$

where $= \begin{pmatrix} x - x^* \\ y - y^* \\ z - z^* \end{pmatrix}$, $M = \begin{pmatrix} M_{11} & 0 & 0 \\ 0 & M_{22} & 0 \\ 0 & 0 & M_{22} \end{pmatrix}$ and

$$M_{11} = \left(\frac{\alpha}{\kappa} - \frac{\eta y^* + z^*}{(1+x^*)}\right) + \frac{\eta(1+x^*) - \varepsilon}{2(1+x^*)} + \frac{(1+x^*) - \beta}{2(1+x^*)}$$
$$M_{22} = \frac{\eta(1+x^*) - \varepsilon}{2(1+x^*)} - \frac{\sigma y^*}{(1+y^*)} + \frac{\sigma(1+y^*) - \beta}{2(1+y^*)}$$
$$M_{33} = \frac{(1+x^*) - \beta}{2(1+x^*)} + \frac{\sigma(1+y^*) - \beta}{2(1+y^*)}$$

Using lemma 4 & 5, the 3×3 symmetric M-matrix is positive definite if $M_{11} > 0, M_{11}M_{22} > 0$, $M_{11}M_{22}M_{33} > 0$. $\dot{V}(x,y,z)$ is negative semi-definite. Thus, $V(x,y,z)$ is a strictly positive Lyapunov's function and ensures the global asymptotic stability of coexisting equilibrium point of system (1.1) in the interior of $\mathcal{A}$. Hence, the proof is complete.

6.0 Numerical Responses.

6.1 Numerical Response of Prey Equilibrium point:

Consider the ecological parameters; $\alpha = 0.3421, \kappa = 2.9231, \eta = 3.4462, \varepsilon = 0.98, \xi = 0.76, \sigma = 0.8125, \mu = 0.2222$. System (1.1) has a prey equilibrium point in the absence of predator, and super-predator at $E_1(x^* = 2.9231, y^* = 0, z^* = 0)$ subject to the initial conditions; $x(0) = 3.0769, y(0) = 1.2500, z(0) = 0.9231$.

It is easy to check that the ecological parameters satisfy the conditions of proposition 3. Fig. 1.1 illustrates the global asymptotic stability of prey equilibrium point, while others follow extinction. The population of the prey then converges to the environmental carrying capacity as $\tau \to +\infty$.

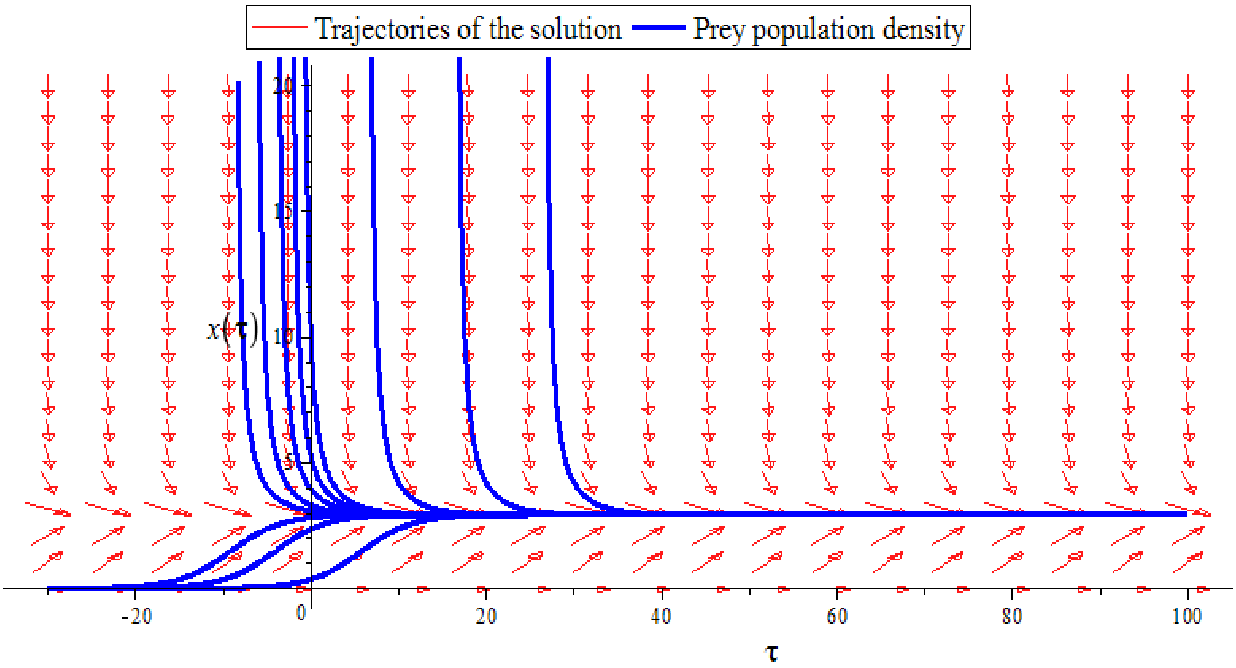

Fig. 1.1 Phase Portrait of Prey Equilibrium point

6.2 Numerical Response of Prey-predator Equilibrium point:

Consider the ecological parameters; $\alpha = 2.60, \kappa = 2.9231, \eta = 2.40, \varepsilon = 0.96, \xi = 0.60, \sigma = 1.0, \mu = 1.60, \beta = 0.15$. System (1.1) has prey-predator equilibrium point in the absence super-predator at $E_2(x^* = 1.6667, y^* = 1.2417, z^* = 0.)$ with eigenvalues $-1.4232, -0.320 \pm 0.3127i$ subject to the initial conditions; $x(0) = 0.3077, y(0) = 1.8462, z(0) = 1.5385$. It is easy to check that the ecological parameters satisfy the conditions of proposition 4. Fig. 1.2 illustrates global asymptotic stability of prey-predator equilibrium point, while super-predator goes to extinction. The population of the prey-predator species spiral towards its equilibrium point as, $\tau \to +\infty$.

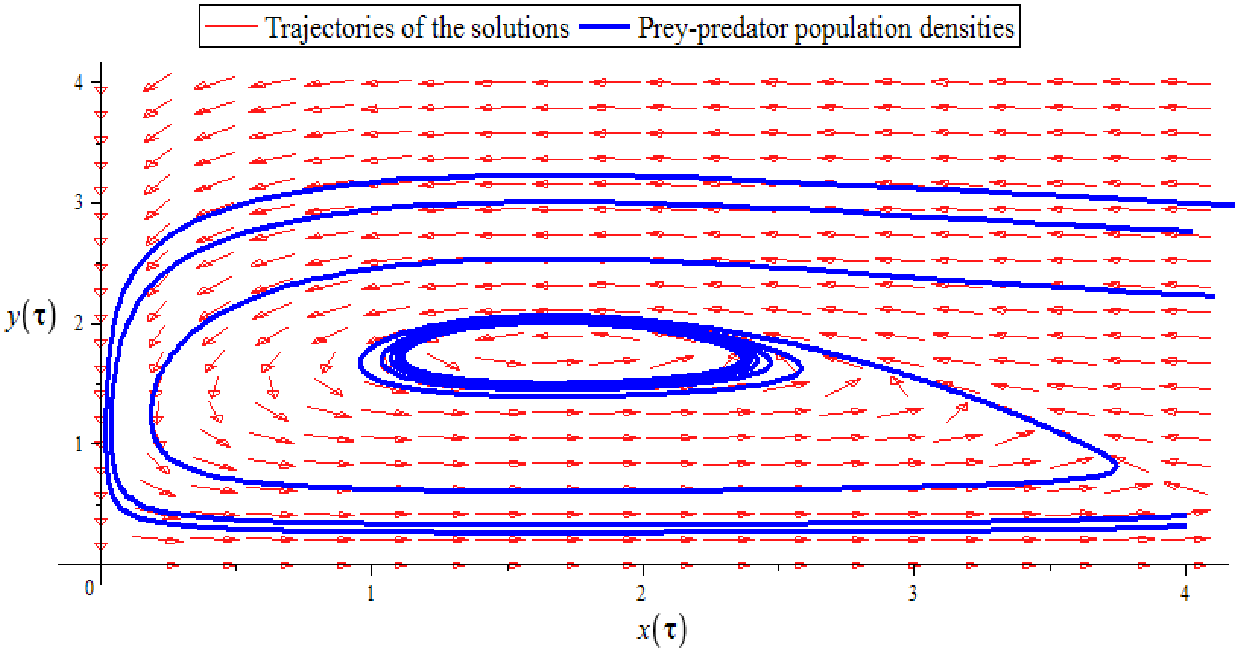

Fig. 1.2 Phase Portrait of Prey-predator Equilibrium point

6.3 Numerical Response of Prey-super-predator Equilibrium point:

Consider the ecological parameters; $\alpha = 0.9091, \kappa = 4.75, \eta = 0.4545, \varepsilon = 0.1136, \xi = 0.9091, \sigma = 0.5333, \mu = 0.2727, \beta = 0.4110$. System (1.1) has a prey-super predator equilibrium point in the absence of predator at $E_3(x^* = 1.9724, y^* = 0, z^* = 1.7381)$ with eigenvalues $-1.67672, -0.01237 \pm 0.2205i$ subject to the initial conditions; $x(0) = 1.25, y(0) = 2.6667, z(0) = 3.75$. It is easy to check that the ecological parameters satisfy the conditions of proposition 5. Fig. 1.3 illustrates global asymptotic stability of prey-superpredator equilibrium point, while super-predator goes to extinction. The population of the prey and super-predator species spiral towards its equilibrium point as, $\tau \to +\infty$.

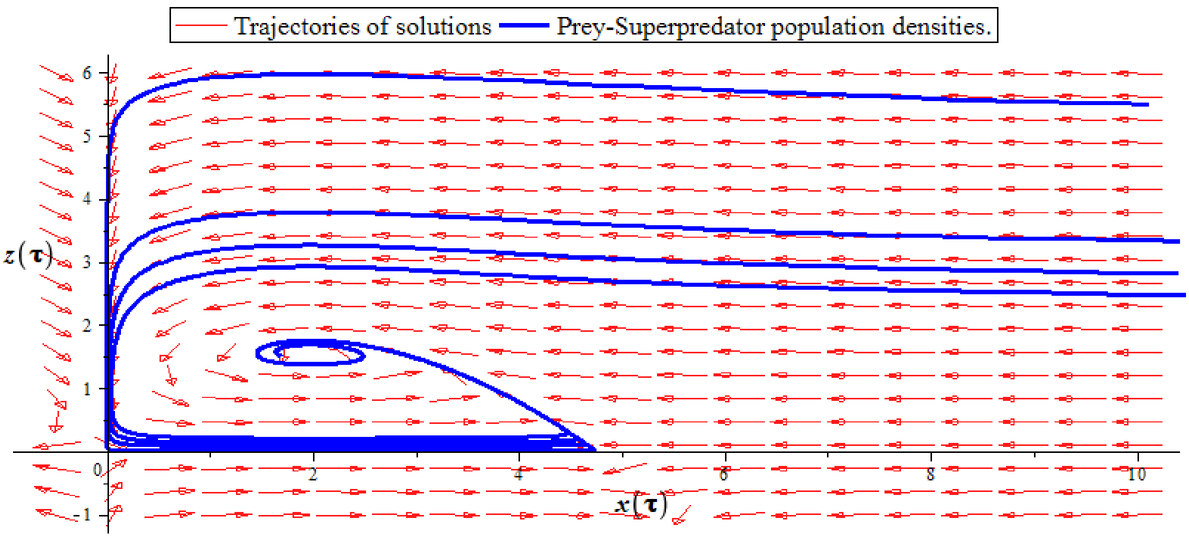

Fig. 1.3 Phase Portrait Prey-Superpredator Equilibrium point

6.4 Numerical Responses of Coexisting Equilibrium point:

It is easy to check that the ecological parameters; $\alpha = 0.7688, \kappa = 2.0064, \eta = 0.1673, \varepsilon = 0.1248, \xi = 0.041, \sigma = 1.0755, \mu = 0.3804, \beta = 0.3655$ satisfy conditions of propositions 1, 2 and 6. These parameters ensure coexistence, and persistence for a globally asymptotically stable equilibrium point point $E_4(2.0390, 2.1138 \times 10^{-10}, 0.6315)$ of system (1.1) with negative eigenvalues $(-0.78981, -0.1377, -0.0251)$ subject to initial conditions $x(0) = 1.06783, y(0) = 1.373, z(0) = 0.6383$. Fig. (1.4-1.6) illustrate the solution profiles of the interacting species, ensuring their long-term survival and globally asymptotically stable oscillations.

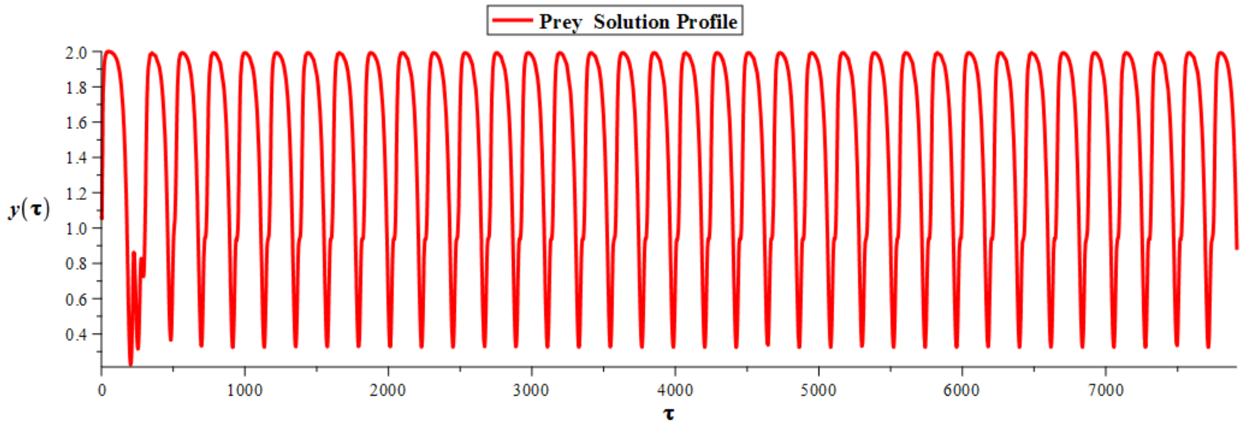

Fig. 1.4 The Dynamics of Prey Component of the Solutions of (x(τ),y(τ),z(τ))

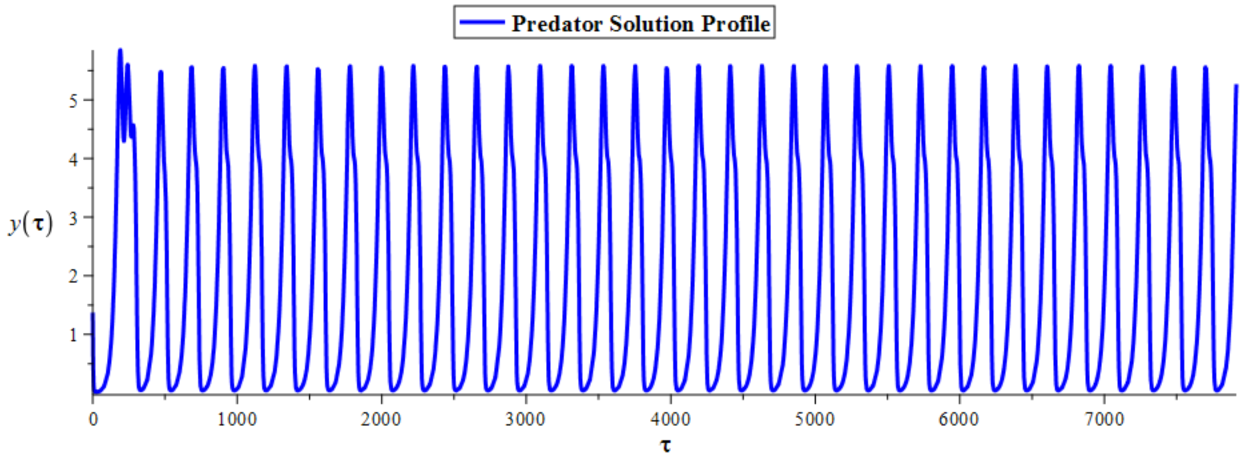

Fig. 1.5 The Dynamics of Predator Component of the Solution of (x(τ),y(τ),z(τ)

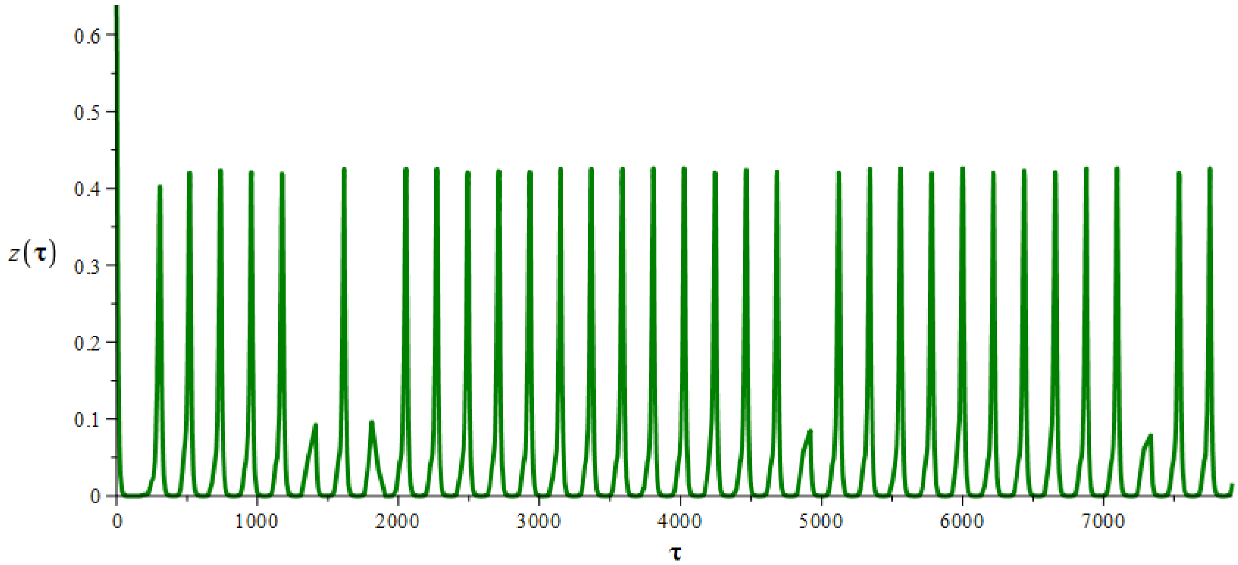

Fig. 1.6 The Dynamics of Super-predator Component of the Solutions of (x(τ),y(τ),z(τ))

7.0 Conclusion

In this paper, we have established ultimate boundedness, dissipativeness and existence of positive equilibrium points of the model. The long-term survival of all interacting species of the system is obtained. We constructed suitable Lyapunov's functionals and used LaSalle extension of Lyapunov's direct method to ensure the existence of global asymptotic behaviors of the system. The claims in propositions and lemmas are illustrated using numerical response, phase-portrait, and phase-flow diagrams.

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