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Aboodh Transform Homotopy Perturbation Method For Solving System Of Nonlinear Partial Differential Equations

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Abstract:

In this paper, we apply a new method called Aboodh transform homotopy perturbation method (ATHPM) to solve nonlinear systems of partial differential equations. This method is a combination of the new integral transform “Aboodh transform” and the homotopy perturbation method. This method was found to be more effective and easy to solve linear and nonlinear differential equations.

Key word: Aboodh transform・Homotopy perturbation method・Nonlinear systems of partial differential equations

1.INTRODUCTION

It is well-known that many physical and engineering phenomena such as wave propagation and shallow waterwaves can be modelled by systems of PDEs [1, 2, 3 ,12].Finding accurate and efficient methods for solving non- linear system of PDEs has long been an active researchundertaking. In recent years, many research workers have paid attention to find the solutions of nonlinearPartial differential equations by using various methods. Among these are the Adomian decomposition method [Hashim, Noorani, Ahmed. Bakar. Ismail. and Zakaria, (2006)], the tanh method, thehomotopy perturbation method [Sweilam, Khader (2009), Sharma and GirirajMethi (2011)[4-5], Jafari, Aminataei (2010), (2011)][6], the differential transform method [(2008)], and the variational iteration method. Elzaki transform [Tarig and Salih, (2011), (2012)[7-12]] ,Homotopy Perturbation and ElzakiTransform for Solving system of Nonlinear Partial Differential Equations[Tarig and Eman, (2012)] is totally incapable of handling the nonlinear equations because of the difficulties that are caused by the nonlinear terms. Various ways have been proposed recently to deal with these nonlinearities, one of these combinations of homotopyperturbation method and Aboodh transform which is studies in this paper. The advantage of this method is its capability of combining two powerful methods for obtaining exact solutions for nonlinear partial differential equations.

Aboodh Transform[13-17] is derived from the classical Fourier integral. Based on the mathematical simplicity of the Aboodh Transform and its fundamental properties, AboodhTransform was introduced by Khalid Aboodh in 2013, to facilitate the process of solving ordinary and partial differential equations in the time domain. This transformation has deeper connection with the Laplace and Elzaki Transform. Homotopy perturbation method (HPM) was established in 1999 by He [18-20]. Themethod is a powerful and efficient technique to find the solutions of non-linearequations .

In this paper we present some basic definitions of Aboodh transform andhomotopy perturbation, also we present a reliable combination of homotopy perturbationmethod and Aboodh transform to obtain the solution of system of nonlinear partial differential equations .

2.AboodhTransform:

Definition :

A new transform called the Aboodh transform defined for function of exponential order we consider functions in the set A, defined by:

$$\mathrm{A} = \{\mathrm{f(t)}: \exists\ \mathrm{M}, \mathrm{k}_1, \mathrm{k}_2 > 0\ , |\mathrm{f(t)}| < M\mathrm{e}^{-\mathrm{vt}}$$

For a given function in the set M must be finite number,k_1, k_2 may be finite or infinite. Aboodh transform which is defined by the integral equation

$$A[f(t)] = K(v) = \frac{1}{v}\int_0^\infty f(t)e^{-vt}\,dt \qquad , t \geq 0\ , \mathrm{k}_1 \leq v \leq \mathrm{k}_2 \tag{1}$$

Aboodhtransform of partial derivative :

In this paper, we combined Aboodh transform and homotopy perturbation to solve nonlinear system of partial differential equationsTo obtain Aboodh transform of partial derivative we use integration by parts, and then we have:

$$A\left(\frac{\partial u(x,t)}{\partial t}\right) = vK(x,v) - \frac{u(x,0)}{v} \quad ,$$

$$A\left(\frac{\partial^2 u(x,t)}{\partial t^2}\right) = v^2K(x,v) - \frac{1}{v}\frac{\partial u(x,0)}{\partial t} - u(x,0)$$

Proof: To obtain transforms of partial derivatives we use integration by parts as follows:

$$A\left(\frac{\partial u(x,t)}{\partial t}\right) = \frac{1}{v}\int_0^\infty \frac{\partial u(x,t)}{\partial t} e^{-vt}\, dt = \lim_{p\to\infty}\frac{1}{v}\int_0^p \frac{\partial u(x,t)}{\partial t} e^{-vt}\, dt$$

$$= \lim_{p\to\infty}\left\{\left[\tfrac{1}{v}u(x,t)e^{-vt} + \right]_0^p \tfrac{1}{v}\int_0^p u(x,t)e^{-vt}\, dt\right\} = vK(x,v) - \frac{u(x,0)}{v} \tag{2}$$

We assume that f is piecewise continuous and it is of exponential order.

let $\frac{\partial u(x,t)}{\partial t} = g$ then , by using Eq.(1) we have

$$A\left(\frac{\partial^2 u(x,t)}{\partial t^2}\right) = A\left(\frac{\partial g(x,t)}{\partial t}\right) = vA\big(g(x,t)\big) - \frac{g(x,0)}{v} = v^2K(x,v) - \frac{1}{v}\frac{\partial u(x,0)}{\partial t} - u(x,0) \tag{3}$$

We can easily extend this result to the nth partial derivative by using mathematical induction.

3. Homotopy Perturbation Method:

Let X and Y be the topological spaces. If f and g are continuous maps of the space X into Y, it is said that f is homotopic to g, if there is continuous map $F: X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$, for each $x \in X$,

then the map is called homotopy between f and g.

To explain the homotopy perturbation method, we consider a general equation of the type,

$$L(U) = 0 \tag{4}$$

Where L is any differential operator, we define a convex homotopy $H(U,p)$ by

$$H(U\,,p) = (1-p)F(U) + pL(U) \tag{5}$$

Where $F(U)$ is a functional operator with known solution V_0 which can be obtained easily. It is clear that, for

$$H(U\,,p) = 0 \tag{6}$$

We have: $H(U,0) = F(U), H(U,1) = L(U)$.

In topology this show that $H(U,P)$ continuously traces an implicitly defined carves from a startingpoint $H(V_0\,,0)$ to a solution function $H(f,1)$. The HPM uses the embed ling parameter p as a small parameter and write the solution as a power series

$$U = U_0 + pU_1 + p^2U_2 + p^3U_3 + \ldots \tag{7}$$

If $p \to 1$, then Eq.(7) corresponds to Eq.(5) and becomes the approximate solution of the form,

$$f = \lim_{p\to 1,} U = \sum_{i=0}^\infty U_i \tag{8}$$

We assume that Eq.(8) has a unique solution. The comparisons of like powers of p give solutions of various orders, for more details see [1-4].

4. Applications:

In this section we apply the homotopy perturbation Aboodh transform method for solving system of nonlinear partial differential equations

Example 4.1:

Consider the following system of nonlinear partial differential equations

$$\begin{cases} U_t(x,t) + V(x,t)U_x(x,t) + U(x,t) = 1 \\ V_t(x,t) + U(x,t)V_x(x,t) - U(x,t) = -1 \end{cases} \tag{9}$$

With the initial conditions

$$U(x,0) = e^x, V(x,0) = e^{-x}$$

Taking Aboodh transform of equations Eq. (7) subject to the initial conditions, we have:

$$\begin{cases} A[U(x,t)] = \frac{1}{v^2}e^{x} - \frac{1}{v}A[V(x,t)U_x(x,t) + U(x,t) - 1] \\ A[V(x,t)] = \frac{1}{v^2}e^{-x} - \frac{1}{v}A[U(x,t)V_x(x,t) - V(x,t) + 1] \end{cases} \quad (10)$$

The inverse Aboodh transform implies that:

$$\begin{cases} U(x,t) = e^{x} - A^{-1}\left\{\frac{1}{v}A[V(x,t)U_x(x,t) + U(x,t) - 1]\right\} \\ V(x,t) = e^{-x} - A^{-1}\left\{\frac{1}{v}A[U(x,t)V_x(x,t) - V(x,t) + 1]\right\} \end{cases} \quad (11)$$

Now applying the homotopy perturbation method, we get:

$$\begin{cases} \sum_{n=0}^{\infty} p^n U_n(x,t) = e^{x} - p\left\{A^{-1}\left[\frac{1}{v}A[\sum_{n=0}^{\infty} p^n H_n(U)]\right]\right\} \\ \sum_{n=0}^{\infty} p^n V_n(x,t) = e^{-x} - p\left\{A^{-1}\left[\frac{1}{v}A[\sum_{n=0}^{\infty} p^n H_n(V)]\right]\right\} \end{cases} \quad (12)$$

Where$H_n(U)$,$H_n(V)$are He's polynomials that represents the nonlinear terms .

Or

$$H_n(U): p[V(x,t)U_x(x,t) + U(x,t) - 1] = 0$$
$$H_n(V): p[U(x,t)V_x(x,t) - V(x,t) + 1] = 0$$

where

$$U = U_0 + pU_1 + p^2U_2 + p^3U_3 + \ldots$$
$$V = V_0 + pV_1 + p^2V_2 + p^3V_3 + \ldots$$

The first few components of He's polynomials, are given by

$$H_0(U) = V_0U_{0_x} + U_0 - 1$$
$$H_0(V) = U_0V_{0_x} + V_0 - 1$$
$$H_1(U) = V_0U_{1_x} + V_1U_{0_x} + U_1$$
$$H_1(V) = U_0V_{1_x} + U_1V_{0_x} + V_1$$
$$\vdots$$

Comparing the coefficients of the same powers of p, we get:

$$p^0: U_0(x,t) = e^{x}, V_0(x,t) = e^{-x}, H_0(U) = e^{x}, H_0(V) = -e^{-x}$$

$$p^1: U_1(x,t) = -A^{-1}\left[\frac{1}{v}A[H_0(U)]\right] = -te^{x}, H_1(U) = -te^{x}$$

$$p^1: V_1(x,t) = -A^{-1}\left[\frac{1}{v}A[H_0(V)]\right] = te^{-x}, H_1(V) = -te^{x}$$

$$p^2: U_2(x,t) = -A^{-1}\left[\frac{1}{v}A[H_1(U)]\right] = \frac{t^2}{2}e^{x}$$

$$p^2: V_2(x,t) = -A^{-1}\left[\frac{1}{v}A[H_1(V)]\right] = \frac{t^2}{2}e^{-x}$$

$$p^3: U_3(x,t) = -A^{-1}\left[\frac{1}{v}A[H_2(U)]\right] = -\frac{t^3}{3!}e^{x}$$

$$p^2: V_3(x,t) = -A^{-1}\left[\frac{1}{v}A[H_2(V)]\right] = \frac{t^3}{3!}e^{-x}$$

$$\vdots \qquad \vdots \qquad \vdots$$

Therefore the solutions $U(x,t), V(x,t)$are given by:

$$U(x,t) = U_0(x,t) + U_1(x,t) + U_2(x,t) + \ldots = e^{x-t}$$
$$V(x,t) = V_0(x,t) + V_1(x,t) + V_2(x,t) + \ldots = e^{t-x}$$

*Example 4.2:*Consider the following system of nonlinear partial differential equations
Consider the following system of nonlinear partial differential equations

$$\begin{cases} U_t + V_xW_y - V_yW_x = -U \\ V_t + W_xU_y + W_yU_x = V \\ W_t + U_xV_y + U_yV_x = W \end{cases} \tag{13}$$

With the initial conditions

$$U(x,y,0) = e^{x+y}, V(x,y,0) = e^{x-y}, W(x,y,0) = e^{-x+y}$$

Taking Aboodh transform of equations Eq. (13) subject to the initial conditions, we have:

$$\begin{cases} A[U(x,y,t)] = \frac{1}{v^2}e^{x+y} + \frac{1}{v}A\left[V_yW_x - V_xW_y - U\right] \\ A[V(x,y,t)] = \frac{1}{v^2}e^{x-y} + \frac{1}{v}A\left[V - W_xU_y - W_yU_x\right] \\ A[W(x,y,t)] = \frac{1}{v^2}e^{-x+y} + \frac{1}{v}A\left[W - U_xV_y - U_yV_x\right] \end{cases} \tag{14}$$

The inverse Aboodh transform implies that:

$$\begin{cases} U(x,y,t) = e^{x+y} + A^{-1}\left[\frac{1}{v}A\left[V_yW_x - V_xW_y - U\right]\right] \\ V(x,y,t) = e^{x-y} + A^{-1}\left[\frac{1}{v}A\left[V - W_xU_y - W_yU_x\right]\right] \\ W(x,y,t) = e^{-x+y} + A^{-1}\left[\frac{1}{v}A\left[W - U_xV_y - U_yV_x\right]\right] \end{cases} \tag{15}$$

Now applying the homotopy perturbation method, we get:

$$\begin{cases} \sum_{n=0}^{\infty} p^n U(x,y,t) = e^{x+y} + p\left\{A^{-1}\left[\frac{1}{v}A[\sum_{n=0}^{\infty} p^n H_n(U)]\right]\right\} \\ \sum_{n=0}^{\infty} p^n V(x,y,t) = e^{x-y} + p\left\{A^{-1}\left[\frac{1}{v}A[\sum_{n=0}^{\infty} p^n H_n(V)]\right]\right\} \\ \sum_{n=0}^{\infty} p^n W(x,y,t) = e^{-x+y} + p\left\{A^{-1}\left[\frac{1}{v}A[\sum_{n=0}^{\infty} p^n H_n(W)]\right]\right\} \end{cases} \tag{16}$$

Where$H_n(U)$,$H_n(V)$ and$H_n(V)$are He's polynomials that represents the nonlinear terms .

Or
$$H_n(U): p\left[V_yW_x - V_xW_y - U\right] = 0$$
$$H_n(V): p\left[V - W_xU_y - W_yU_x\right] = 0$$
$$H_n(w): p\left[W - U_xV_y - U_yV_x\right] = 0$$

where
$$U = U_0 + pU_1 + p^2U_2 + p^3U_3 + \dots$$
$$V = V_0 + pV_1 + p^2V_2 + p^3V_3 + \dots$$
$$W = W_0 + pW_1 + p^2W_2 + p^3W_3 + \dots$$

Comparing the coefficients of the same powers of p, we get:

$$p^0: U_0(x,y,t) = e^{x+y}, V_0(x,y,t) = e^{x-y}, W_0(x,y,t) = e^{-x+y}$$
$$H_0(U) = -e^{x+y}, H_0(V) = e^{x-y}, H_0(W) = e^{-x+y}$$
$$p^1: U_1(x,y,t) = A^{-1}\left[\frac{1}{v}A[H_0(U)]\right] = -te^{x+y}, H_1(U) = te^{x+y}$$
$$p^1: V_1(x,y,t) = A^{-1}\left[\frac{1}{v}A[H_0(V)]\right] = te^{x-y}, H_1(V) = te^{x-y}$$
$$p^1: W_1(x,y,t) = A^{-1}\left[\frac{1}{v}A[H_0(W)]\right] = te^{-x+y}, H_1(V) = te^{-x+y}$$
$$p^2: U_2(x,y,t) = A^{-1}\left[\frac{1}{v}A[H_1(U)]\right] = \frac{t^2}{2}e^{x+y} p^2: V_2(x,y,t) = A^{-1}\left[\frac{1}{v}A[H_1(V)]\right] = \frac{t^2}{2}e^{x-y}$$
$$p^2: W_3(x,y,t) = A^{-1}\left[\frac{1}{v}A[H_1(W)]\right] = \frac{t^2}{2}e^{x}$$

$$\vdots \qquad \vdots \qquad \vdots$$

Therefore the solutions $U(x,y,t), V(x,y,t)$and $W_3(x,y,t)$ are given by:

$$U(x,y,t) = U_0(x,t) + U_1(x,t) + U_2(x,t) + \ldots = e^{x+y-t}$$
$$V(x,y,t) = V_0(x,t) + V_1(x,t) + V_2(x,t) + \ldots = e^{x-y+t}$$
$$W(x,y,t) = W_0(x,t) + W_1(x,t) + W_2(x,t) + \ldots = e^{-x-y+t}$$

*Example 4.3:*Consider the following Coupled Burger's system

$$\begin{cases} U_t - U_{xx} - 2UU_x + (UV)_x = 0 \\ V_t - V_{xx} - 2VV_x + (UV)_x = 0 \end{cases} \tag{17}$$

With the initial conditions

$$U(x,0) = \sin x, V(x,0) = \sin x$$

Using the same method in above example to obtain :

$$\begin{cases} U(x,t) = \sin x + A^{-1}\left\{\frac{1}{v}A[U_{xx} + 2UU_x - (UV)_x]\right\} \\ V(x,t) = \sin x + A^{-1}\left\{\frac{1}{v}A[V_{xx} + 2VV_x - (UV)_x]\right\} \end{cases} \tag{18}$$

Now applying the homotopy perturbation method, we get:

$$\begin{cases} \sum_{n=0}^{\infty} p^n U_n(x,t) = \sin x + p\left\{A^{-1}\left[\frac{1}{v}A[\sum_{n=0}^{\infty} p^n H_n(U)]\right]\right\} \\ \sum_{n=0}^{\infty} p^n V_n(x,t) = \sin x + p\left\{A^{-1}\left[\frac{1}{v}A[\sum_{n=0}^{\infty} p^n H_n(V)]\right]\right\} \end{cases} \tag{19}$$

Where$H_n(U)$,$H_n(V)$ are He's polynomials that represents the nonlinear terms .

Or

$$H_n(U): p[U_{xx} + 2UU_x - UV_x - U_xV] = 0$$
$$H_n(V): p[V_{xx} + 2VV_x - UV_x - U_xV] = 0$$

where

$$U = U_0 + pU_1 + p^2U_2 + p^3U_3 + \ldots$$
$$V = V_0 + pV_1 + p^2V_2 + p^3V_3 + \ldots$$

The first few components of He's polynomials, are given by

$H_0(U) = U_{0_{xx}} + 2U_0U_{0_x} - U_0V_{0_x} - U_{0_x}V_0$
$H_0(V) = V_{0_{xx}} + 2V_0V_{0_x} - U_0V_{0_x} - U_{0_x}V_0$
$H_1(U) = U_{1_{xx}} + 2U_0U_{1_x} + 2U_1U_{0_x} - U_0V_{1_x} - U_1V_{0_x} - U_{0_x}V_1 - U_{1_x}V_0$
$H_1(V) = V_{1_{xx}} + 2V_0V_{1_x} + 2V_1V_{0_x} - U_0V_{1_x} - U_1V_{0_x} - U_{0_x}V_1 - U_{1_x}V_0$

$$\vdots$$

Comparing the coefficients of the same powers of p, we get:

$$p^0: U_0(x,t) = \sin x \ , V_0(x,t) = \sin x \, , H_0(U) = -\sin x \ , H_0(V) = -\sin x$$
$$p^1: U_1(x,t) = -A^{-1}\left[\frac{1}{v}A[H_0(U)]\right] = -t\sin x\, ,\ H_1(U) = t\sin x$$
$$p^1: V_1(x,t) = -A^{-1}\left[\frac{1}{v}A[H_0(V)]\right] = -t\sin x\, , H_1(V) = t\sin x$$
$$p^2: U_2(x,t) = -A^{-1}\left[\frac{1}{v}A[H_1(U)]\right] = \frac{t^2}{2}\sin x$$
$$p^2: V_2(x,t) = -A^{-1}\left[\frac{1}{v}A[H_1(V)]\right] = \frac{t^2}{2}\sin x$$
$$p^3: U_3(x,t) = -A^{-1}\left[\frac{1}{v}A[H_2(U)]\right] = -\frac{t^3}{3!}\sin x$$
$$p^2: V_3(x,t) = -A^{-1}\left[\frac{1}{v}A[H_2(V)]\right] = -\frac{t^3}{3!}\sin x$$
$$\vdots \qquad \vdots \qquad \vdots$$

Therefore the solutions $U(x,t), V(x,t)$are given by:

$$U(x,t) = U_0(x,t) + U_1(x,t) + U_2(x,t) + \ldots = e^{-t}\sin x$$
$$V(x,t) = V_0(x,t) + V_1(x,t) + V_2(x,t) + \ldots = e^{-t}\sin x$$

5. CONCLUSION

The main goal of this paper is to show the applicability of the mixture of new integral transform "Aboodh transform" with the homotopy perturbation method to solving system of nonlinear partial differential equations. This combination of two methods successfully implemented by using the initialconditions only. Finally, we conclude thatAboodhtransform homotopy perturbation method considered as a nice refinement in existing numerical techniques.

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