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A predator-prey mathematical model with competitive interaction amongst two species

T. G. Kassem[1], J. N. Ndam[2], J. P. Chollom[3] and I.A. Nyam[4]
[1,2,3,4]Department of Mathematics, University of Jos, Nigeria.

Abstract
A mathematical model is constructed to study the effect of predation on two competing species in which one of the competing species is a prey to the predator whilst the other species is not under predation. We assume that all species can move by diffusion and study the spatial structure of the species and obtained conditions for the existence and stability of equilibrium solutions. The results indicate the possibility of a stable coexistence of the three interacting species in form of stable oscillations under the reflecting boundary conditions. Numerical simulations supported our theoretical predictions. By utilizing Liapunov-like functions and differential inequalities we were able to establish that the system is dissipative.

1. Introduction

The dynamic relationship between predators and their preys has long been and will continue to be one of the dominating themes in both ecology and mathematical biology due to its universal existence and importance (Bohner et al, 2006). In predation-mediated coexistence, predation may have a tendency to increase species diversity in competitive communities. Kan-On and Mimura (1998) showed that in a 3-component diffusion system where all the species can move by diffusion, the spatial structure of the competing species may coexistence in the presence of predators.
Similarly, Ndam et al (2012) using a combination of Holling's type III and BD functional responses set out conditions for diffusive instability in a 3-component system. In this paper, we consider a mathematical model in which two species are engaged in a competition for resources and a third other species is predating one of the competing species. We assume that all species can move by diffusion and study the spatial structure of competing species that may coexist in the presence of predation. The aim of this paper is to investigate the effects of predation mediated coexistence and density dependence diffusion and cross diffusion to produce segregation effects and the creation of spatial niches.

2. Model equations

In order to study this situation, we propose here the following 3-component reaction-diffusion system for two competing species N_1 and N_2 in which one of the species N_1 is prey to a predator N_3 :

$$
\begin{aligned}
\frac{dN_1}{dt} &= r_1 N_1\left(1-\frac{N_1}{K_1}-b_{12}\frac{N_2}{K_1}\right)+\nabla^2\left(D_1 N_1 D_4 N_3\right)\\
\frac{dN_2}{dt} &= r_2 N_2\left(1-\frac{N_2}{K_2}-b_{21}\frac{N_1}{K_2}\right)+D_2\nabla^2 N_1 \qquad (1)\\
\frac{dN_3}{dt} &= c_2\frac{N_1^2 N_3}{a^2+N_1^2}-dN_3-qN_3+\nabla^2\left(D_3 N_3+D_5 N_1\right)
\end{aligned}
$$

Where $\nabla^2=\frac{\partial^2}{\partial x^2}$ with $N_1(x,0)>0, N_2(x,0)>0, N_3(x,0)>0$ subject to the reflecting boundary condition

$$D_1 \frac{\partial N_1(0,t)}{\partial x} = D_1 \frac{\partial N_1(L,t)}{\partial x} = 0, D_2 \frac{\partial N_2(0,t)}{\partial x} = D_2 \frac{\partial N_2(L,t)}{\partial x} = 0,$$
$$D_3 \frac{\partial N_3(0,t)}{\partial x} = D_3 \frac{\partial N_3(L,t)}{\partial x} = 0 \tag{2}$$

System (1) can be scaled as follows:

$$X = \frac{N_1}{K_1}, Y = \frac{N_2}{K_2}, Z = \frac{N_3}{K_1}, \tau \equiv r_1 t, \rho = \frac{r_2}{r_1}, \alpha_{12} = \frac{b_{12}K_2}{K_1}, \alpha_{21} = \frac{b_{21}K_1}{K_2}, K_1 = a, \omega = \frac{c_1}{r_1}.$$

$$\frac{\partial X}{\partial \tau} = X(1 - X - \alpha_{12}Y) - \omega \frac{X^2 Z}{1+X^2} + \nabla^2(X + \varphi Z),$$
$$\frac{\partial Y}{\partial \tau} = \rho Y(1 - Y - \alpha_{21}X) + \theta \nabla^2 Y, \tag{3}$$
$$\frac{\partial Z}{\partial \tau} = \sigma \frac{X^2 Z}{1+X^2} - \mu Z - \gamma Z + \nabla^2(\zeta Z + \psi X).$$

$$X(x,0) = X_0, Y(x,0) = Y_0, Z(x,0) = Z_0,$$

$$\frac{\partial X(0,\tau)}{\partial x} = \frac{\partial X(L,\tau)}{\partial x} = 0, \theta \frac{\partial Y(0,\tau)}{\partial x} = \theta \frac{\partial Y(L,\tau)}{\partial x} = 0, \zeta \frac{\partial Z(0,\tau)}{\partial x} = \zeta \frac{\partial Z(L,\tau)}{\partial x} = 0,$$

$$\psi \frac{\partial X(0,\tau)}{\partial x} = \psi \frac{\partial X(L,\tau)}{\partial x} = 0, \varphi \frac{\partial Z(0,\tau)}{\partial x} = \varphi \frac{\partial Z(L,\tau)}{\partial x} = 0,$$

where we have taken $\theta = D_2 / D_1, \zeta = D_3 / D_1, \varphi = D_4 / D_1, \psi = D_5 / D_1$ and α_{12}, α_{21} are the interspecific competitions rates.

Let us begin by reviewing the qualitative behaviours of the diffusionless systems deriving from (3):

$$\frac{dX}{d\tau} = X(1 - X - \alpha_{12}Y) - \omega \frac{X^2 Z}{1+X^2} \equiv f_1(X,Y,Z), \tag{4a}$$
$$\frac{dY}{d\tau} = \rho Y(1 - Y - \alpha_{21}X) \equiv f_2(X,Y,Z), \tag{4b}$$
$$\frac{dZ}{d\tau} = \sigma \frac{X^2 Z}{1+X^2} - \mu Z - \gamma Z \equiv f_3(X,Y,Z). \tag{4c}$$

The steady states, are singularities of the solutions $f_1(X, Y, Z) = f_2(X, Y, Z) = f_3(X, Y, Z) = 0$, which, from (4a), (4b) and (4c), are
$X^* = 0, Y^* = 0, Z^* = 0$ and
$X^* = \pi, Y^* = 1 - \alpha_{12}\pi$

$$Z^* = \frac{\sigma}{\omega\pi(\sigma - \mu - \gamma)}\left(1 - \alpha_{12} + (\alpha_{12}\alpha_{21} - 1)\pi\right),$$

where $\pi = \sqrt{\dfrac{\mu+\gamma}{\sigma - \mu - \gamma}}$ (5)

provided $\sigma > \mu + \gamma$.

The stability of the steady states is again determined by the community matrix which, for equations (4a), (4b) and (4c) is given as:

$$J_{(0,0,0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & -\mu-\gamma \end{pmatrix}. \tag{6}$$

The characteristic polynomial is

$$\lambda^3 - (-\mu - \gamma + 1 + \rho)\lambda^2 - (\mu + \mu\rho + \gamma + \gamma\rho)\lambda + (\mu + \gamma)\rho = 0 \tag{7}$$

and the corresponding eigenvalues are $\lambda := 1, \rho, -\mu-\gamma$. This steady state is unstable. Next we find the community matrix at the endemic state (X^*, Y^*, Z^*) which is given as:

$$J_{(X^*,Y^*,Z^*)} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & 0 & 0 \end{pmatrix} \tag{8}$$

where

$$a_{11} = \frac{1}{\sigma}\left(\sigma\alpha_{12}(1-\alpha_{21}\pi) + 2(\mu+\gamma)\left(1-\alpha_{12}+(\alpha_{12}\alpha_{21}-1)\pi\right) - \sigma\right)$$

$$a_{12} = -\pi\alpha_{12}, a_{13} = -\frac{\omega(\mu+\gamma)}{\sigma}, a_{21} = -\rho(1-\alpha_{21}\pi)\alpha_{21}, a_{22} = -\rho(1-\alpha_{21}\pi),$$

$$a_{31} = \frac{2}{\sigma}(\sigma-\mu-\gamma)\left(1-\alpha_{12}+(\alpha_{12}\alpha_{21}-1)\pi\right).$$

The characteristic equation of the variational matrix $J_{(X^*,Y^*,Z^*)}$ is

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \tag{9}$$

where

$$\begin{aligned} a_1 &= -(a_{11}+a_{22}) \\ &= \frac{1}{\sigma}\left(\sigma\alpha_{12}(\alpha_{21}\pi-1) + 2(\mu+\gamma)\left(\alpha_{12}-1+(1-\alpha_{12}\alpha_{21})\pi\right)+\sigma\right) + \alpha_{12}\pi, \\ a_2 &= -(a_{13}a_{31} + a_{21}a_{12} - a_{22}a_{11}) \\ &= \frac{2\omega(\mu+\gamma)}{\sigma^2}\left((\sigma-\mu-\gamma)\left(1-\alpha_{12}+(\alpha_{12}\alpha_{21}-1)\pi\right)\right) + \sigma\pi\alpha_{12}\alpha_{21}(\alpha_{21}\pi-1) \\ &+ \frac{\alpha_{12}\pi}{\sigma}\left(\sigma\alpha_{12}(\alpha_{21}\pi-1) + 2(\mu+\gamma)\left(\alpha_{12}-1+(1-\alpha_{12}\alpha_{21})\pi\right)+\sigma\right), \\ a_3 &= a_{31}a_{12}a_{22} \\ &= \frac{\rho\alpha_{12}(1-\alpha_{21})\pi}{\sigma}\left(\sigma\alpha_{12}(1-\alpha_{21}\pi) + 2(\mu+\gamma)\left(1-\alpha_{12}+(\alpha_{12}\alpha_{21}-1)\pi\right)-\sigma\right) \end{aligned}$$

According to Routh-Hurwitz criterion, the interior equilibrium E = (X^*, Y^*, Z^*) is locally asymptotically stable if the following conditions hold:

$$a_1 > 0, a_2 > 0, a_1 a_2 > a_3. \tag{10}$$

Following a similar procedure as outlined in Freedman and Ruan(1995), we

next show that the system is dissipative. Let
$X(t) = x(t) + X^*, Y(t) = y(t) + Y^*, Z(t) = z(t) + Z^*$

then the equations (4a), (4b) and (4c) can be transformed into

$$\dot{x}(t) = \left(x(t) + X^*\right)\left(-x(t) - \alpha_{12} y(t)\right) - \frac{\omega\left(x(t) + X^*\right)^2 \left(z(t) + Z^*\right)}{1 + \left(x(t) + X^*\right)^2} \tag{11}$$

$$\dot{y}(t) = \left(y(t) + Y^*\right)\left(-y(t) - \alpha_{21} x(t)\right) \tag{12}$$

$$\dot{z}(t) = \frac{\sigma\left(x(t) + X^*\right)^2 \left(z(t) + Z^*\right)}{1 + \left(x(t) + X^*\right)^2} - (\mu + \gamma)\left(z(t) + Z^*\right). \tag{13}$$

Choose a Liapunov function as follows

$$V = x(t) - X^* \ln\left(1 + \frac{x(t)}{X^*}\right) + y(t) - Y^* \ln\left(1 + \frac{y(t)}{Y^*}\right) + \frac{q}{\sigma}\left[z(t) - Z^* \ln\left(1 + \frac{z(t)}{Z^*}\right)\right] \tag{14}$$

where $-1 \le q \le 1$ is a given constant. If $x(t) = y(t) = z(t) = 0$, then $V = 0$, and V is

positive definite for bounded

$$x(t) > \pi, y(t) > 1 - \alpha_{12}\pi, z(t) > \frac{\sigma}{\omega\pi(\sigma - \mu - \gamma)}\left(1 - \alpha_{12} + (\alpha_{12}\alpha_{21} - 1)\pi\right).$$

We have

$$\dot{V}\left(x(t), y(t), z(t)\right) = \frac{x(t)}{x(t) + X^*}\dot{x}(t) + \frac{y(t)}{y(t) + Y^*}\dot{y}(t) + \frac{q}{\sigma}\frac{z(t)}{z(t) + Z^*}\dot{z}(t) \tag{15}$$

$$\dot{V}\left(x(t), y(t), z(t)\right) = x(t)\left(-x(t) - \alpha_{12} y(t) - \frac{\omega\left(x(t) + X^*\right)\left(z(t) + Z^*\right)}{1 + \left(x(t) + X^*\right)^2}\right)$$

$$+ y(t)\left(-y(t) - \alpha_{21} x(t)\right) + \frac{q\left(x(t) + X^*\right)^2}{1 + \left(x(t) + X^*\right)^2} - \frac{q}{\sigma}(\mu + \gamma) \tag{16}$$

$$\le -x^2(t) - \alpha_{12} x(t) y(t) - \frac{q\left(x^2(t) + x(t) X^*\right)\left(z(t) + Z^*\right)}{1 + \left(x(t) + X^*\right)^2} - y^2(t) - \alpha_{21} x(t) y(t).$$

Hence the diffusionless system is dissipative.

3. Model with Cross and Self-diffusion

To examine the linear stability of the steady state, we recast the system (4a), (4b) and (4c) in the form

$$\frac{\partial X}{\partial \tau}=\frac{\partial^2 X}{\partial x^2}+\varphi\frac{\partial^2 Z}{\partial x^2}+F_1(X,Y,Z,\tau),$$
$$\frac{\partial Y}{\partial \tau}=\theta\frac{\partial^2 Y}{\partial x^2}+F_2(X,Y,Z,\tau), \tag{17}$$
$$\frac{\partial Z}{\partial \tau}=\zeta\frac{\partial^2 Z}{\partial x^2}+\psi\frac{\partial^2 X}{\partial x^2}+F_3(X,Y,Z,\tau),$$

Where

$$F_1(X,Y,Z,\tau)=X(1-X-\alpha_{12}Y)-\omega\frac{X^2Z}{1+X^2},$$
$$F_2(X,Y,Z,\tau)=\rho Y(1-Y-\alpha_{21}X), \quad 19$$
$$F_3(X,Y,Z,\tau)=\sigma\frac{X^2Z}{1+X^2}-\mu Z-\gamma Z.$$

Now, assume a spatially uniform steady state (X^*, Y^*, Z^*) such that $F(X^*, Y^*, Z^*) = 0$, $k = 1, 2, 3$ and perturb the population densities of the species as

$$X(x,t)=X^*+X'(x,\tau),$$
$$Y(x,t)=Y^*+Y'(x,\tau), \tag{18}$$
$$Z(x,t)=Z^*+Z'(x,\tau).$$

Substituting (18) into and linearizing, one obtains

$$\frac{\partial X}{\partial \tau}=\frac{\partial^2 X}{\partial x^2}+\varphi\frac{\partial^2 Z}{\partial x^2}+a_{11}X'+a_{12}Y'+a_{13}Z',$$

$$\frac{\partial Y}{\partial \tau}=\theta\frac{\partial^2 Y}{\partial x^2}+a_{21}X'+a_{22}Y'+a_{23}Z', \tag{19}$$

$$\frac{\partial Z}{\partial \tau}=\zeta\frac{\partial^2 Z}{\partial x^2}+\psi\frac{\partial^2 X}{\partial x^2}+a_{31}X'+a_{32}Y'+a_{33}Z'.$$

where

$$(a_{ij})=\begin{pmatrix}\frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z}\\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z}\\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z}\end{pmatrix}. \tag{20}$$

is the Jacobian of the system evaluated at the equilibrium point (X^*, Y^*, Z^*).

The linear stability of (19) can be examined using the normal modes approach, by assuming solutions of the form $X' = e^{\lambda\tau + ilx}, Y' = e^{\lambda\tau + ilx}, Z' = e^{\lambda\tau + ilx}$ where λ and l are constants. Hence the eigenvalues satisfy the equation

$$\begin{vmatrix} \lambda + (1+\varphi)l^2 - a_{11} & -a_{12} & -a_{13} \\ -a_{21} & \lambda + \theta l - a_{22} & 0 \\ -a_{31} & 0 & \lambda + (\zeta + \psi)l^2 - a_{33} \end{vmatrix} = 0 \tag{21}$$

The characteristic equation is given as

$$\begin{aligned} &\lambda^3 + \lambda^2\left((1+2\varphi+\zeta+\theta)l^2 - a_{11} - a_{22}\right) \\ &+\lambda\{\left((\zeta+\varphi)(1+\varphi+\theta)+\theta(1+\varphi)\right)l^4 + a_{11}(a_{22}+a_{33}) + a_{22}a_{33} + \{(a_{11}\theta - (\zeta+\varphi)(a_{11}+a_{22}) \\ &\quad -\theta a_{33} - (1+\varphi)(a_{22}+a_{33}))\}l^2 - a_{12}a_{21} - a_{13}a_{31}\} \\ &\quad -a_{11}\theta(\zeta+\varphi)l^4 + \{a_{11}a_{33}\theta + (a_{11}a_{22} - a_{12}a_{21})(\zeta+\varphi) - a_{13}a_{31}\theta\}l^2 \\ &\qquad + a_{13}a_{22}a_{31} + a_{12}a_{21}a_{33} - a_{11}a_{22}a_{33} = 0 \end{aligned} \tag{22}$$

where

$$\beta_1 = (1+2\varphi+\zeta+\theta)l^2 - a_{11} - a_{22} \tag{23a}$$

$$\begin{aligned} \beta_2 &= \left((\zeta+\varphi)(1+\varphi+\theta)+\theta(1+\varphi)\right)l^4 + a_{11}(a_{22}+a_{33}) + a_{22}a_{33} \\ &\quad + \left\{\left(a_{11}\theta - (\zeta+\varphi)(a_{11}+a_{22}) - \theta a_{33} - (1+\varphi)(a_{22}+a_{33})\right)\right\}l^2 - a_{12}a_{21} - a_{13}a_{31} \end{aligned} \tag{23b}$$

$$\begin{aligned} \beta_3 &= -a_{11}\theta(\zeta+\varphi)l^4 + \{a_{11}a_{33}\theta + (a_{11}a_{22} - a_{12}a_{21})(\zeta+\varphi) - a_{13}a_{31}\theta\}l^2 \\ &\quad + a_{13}a_{22}a_{31} + a_{12}a_{21}a_{33} - a_{11}a_{22}a_{33}. \end{aligned} \tag{23c}$$

For stability the following conditions should be satisfied

$$\beta_1 > 0, \beta_2 > 0 \text{ and } \beta_1\beta_2 > \beta_3. \tag{24}$$

Diffusive instability sets in if any of conditions (24) is violated.

4. Numerical Simulation

In this section, we give some numerical simulations supporting our theoretical predictions. Figure 1a-d demonstrate that solutions for the 3-component system exist such that X, Y and Z exhibit damped oscillations for the parameters: $\rho = 8/3, a_{12} = 0.8, a_{21} = 1.2, \mu = 0.2, \omega = 0.9, \sigma = 1.2, \gamma = 0.02$. When Z is plotted against X, we observed a stable spiral and when the competing species are viewed against each other, a more complex scenario is observed. First there is a spiraling in and what looks more like a limit cycle. In Figure 2, the values for α_{12} and α_{21} were perturbed to demonstrate the effect of interspecific competition amongst the competing species. We noticed that as the interspecific competition (α_{12}) increases the X species are driven to extinction whereas species Y increases. Similarly as (α_{21}) increases the Y species are driven to extinction whereas the X species increases in the presence of predation. These indicate that the interspecific competitions drive the dynamic behaviour of the diffusionless system. Figure 3 is the numerical simulation of the 3-component problem with self and cross diffusion. The numerical solutions showed that species X, Y and Z stably exist in a sustained oscillation. What we do not know is whether cycles are drawn by the classical predator-prey mechanism or some other factors may be involved in producing the oscillations. In figure 4 competing species experience habitat segregation effects and the creation of spatial niches under predation. Figure

5 shows the effects of cross-diffusion on the spatial density distribution of the interacting species. Here species Y diffuse faster than the others. This scenario is understandable since the specie is not under predation

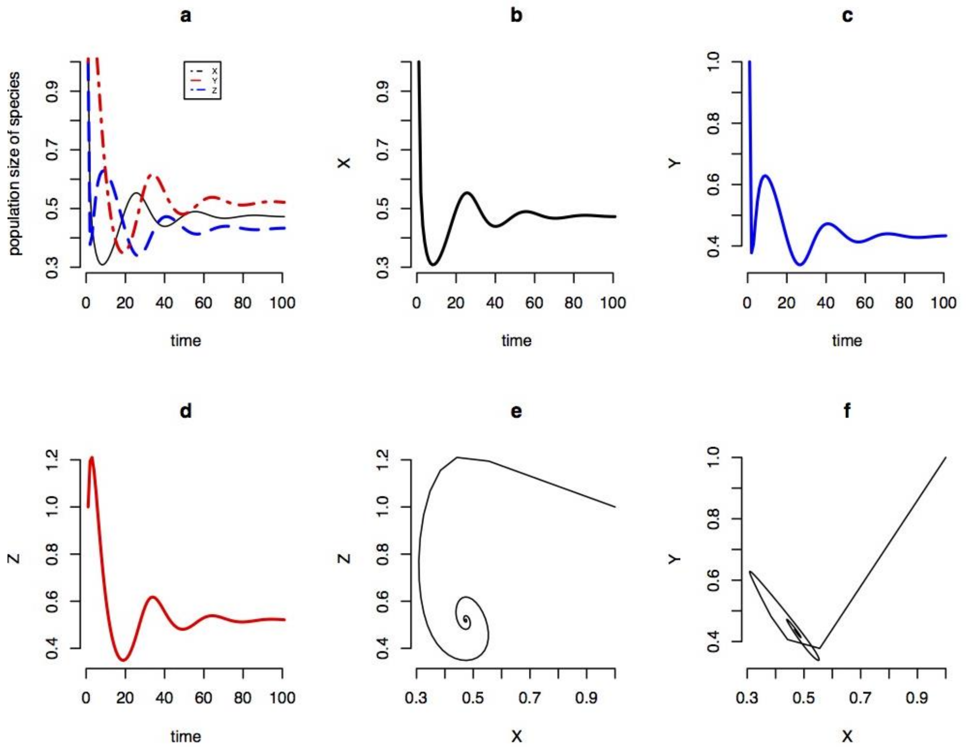

Figure 1. Dynamic behaviour of predator-prey-competition model. Parameters: $\rho = 8/3, a_{12} = 0.8, a_{21} = 1.2, \mu = 0.2, \omega = 0.9, \sigma = 1.2, \gamma = 0.02$

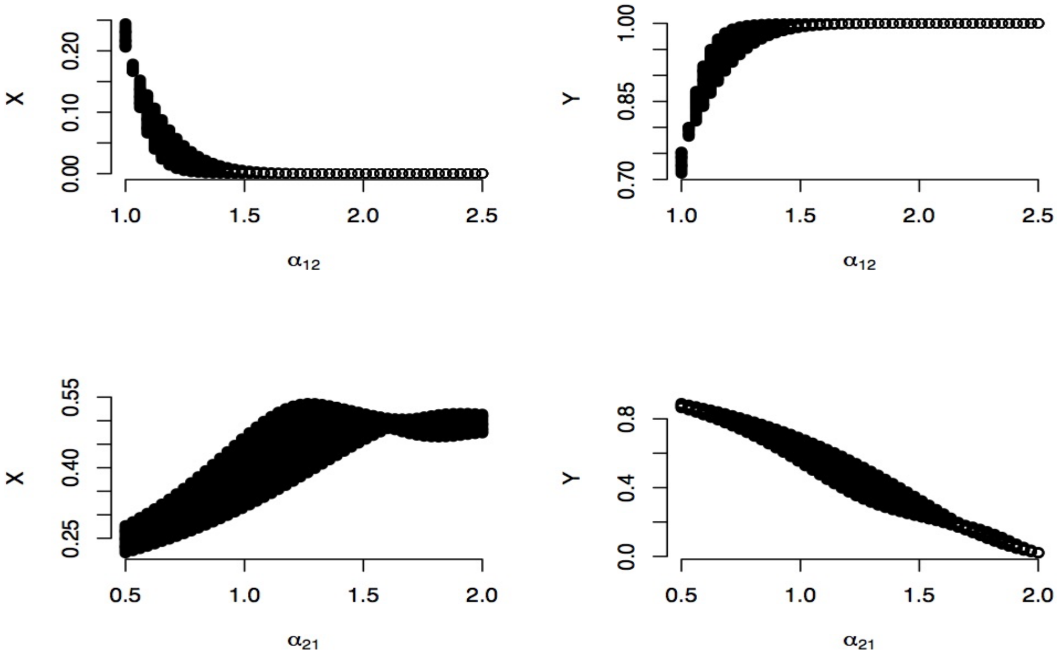

Figure 2. Bifurcation picture as α_{12} & α_{21} are varied. The parameters chosen for the model are: $\rho = 8/3, \alpha_{12} = 1-2.5, \alpha_{21} = 0.5-2, \mu = 0.2, \omega = 1.6, \sigma = 1.6, \gamma = 0.02$

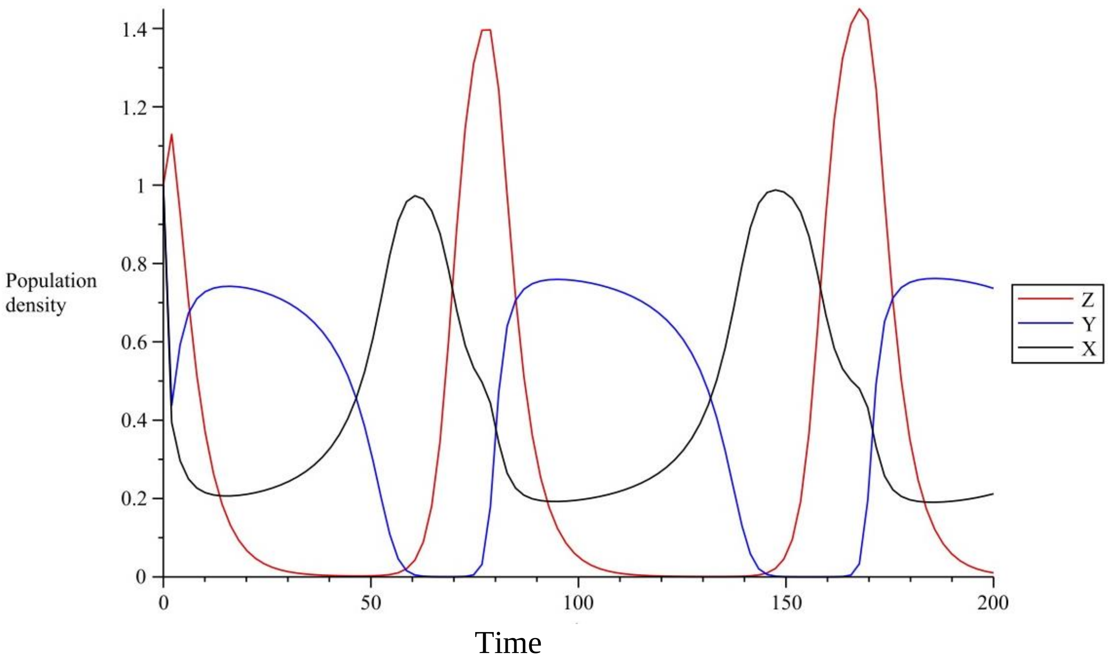

Figure 3. Sustained oscillations. Parameters: $\rho = 8/3, a_{12} = 0.8, a_{21} = 1.2, \mu = 0.02, \omega = 0.9, \sigma = 1.2, \gamma = 0.2, \phi = -0.1, \theta = 0.1, \zeta = 0.1, \psi = 0.1$

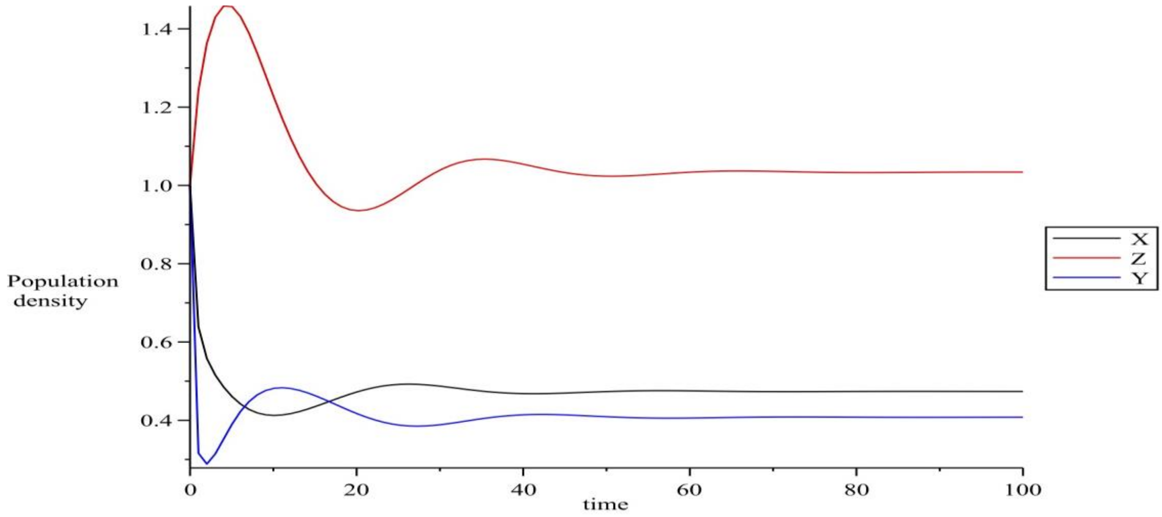

Figure 4. Spatially segregating coexistence of the competing species: $\rho = 8/3, a_{12} = 0.8, a_{21} = 1.2, \mu = 0.02, \omega = 0.5, \sigma = 1.2, \gamma = 0.2, \phi = -0.1, \theta = 0.1, \zeta = 0.1, \psi = 0.1$

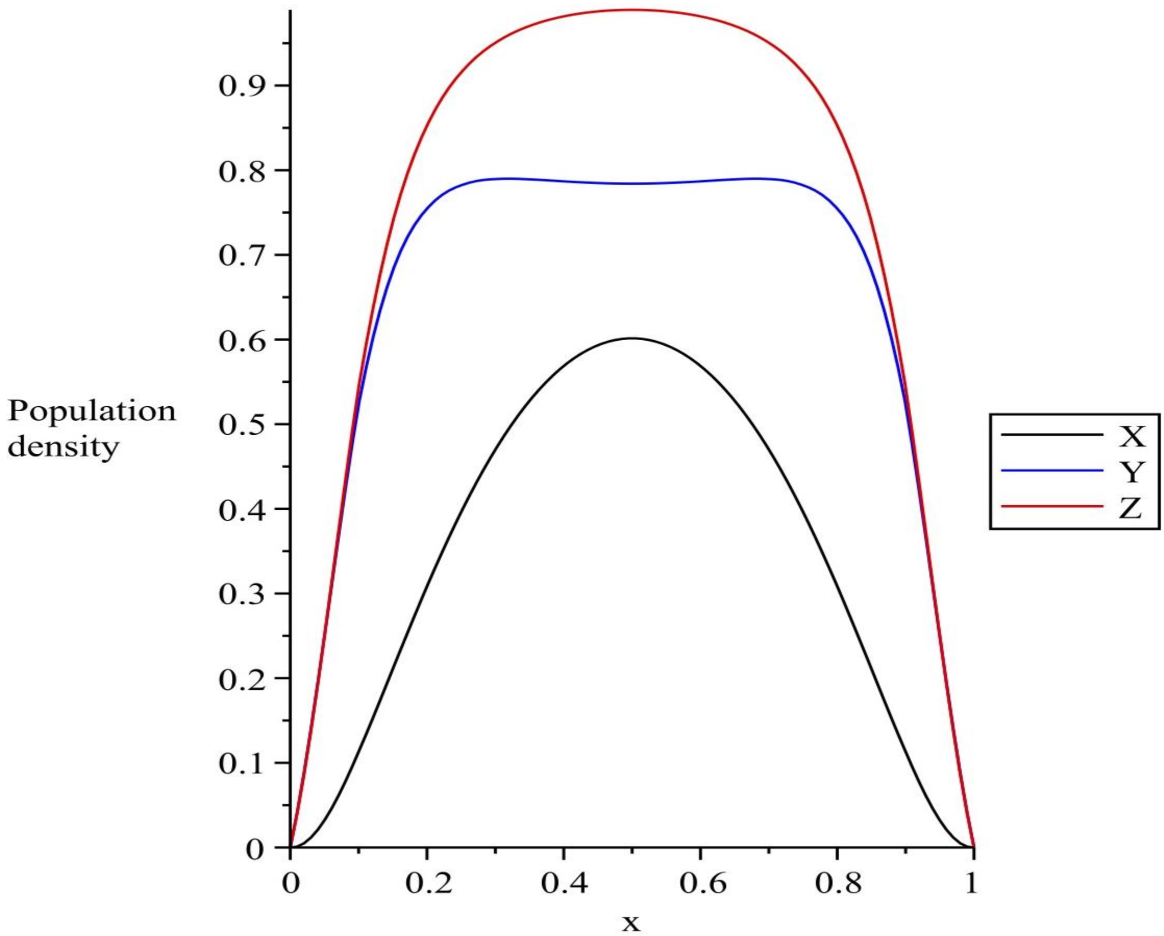

Figure 5. Cross and self diffusion of the interacting species: $\rho = 8/3, a_{12} = 0.8, a_{21} = 1.2, \mu = 0.02, \omega = 0.5, \sigma = 1.2, \gamma = 0.2, \phi = -0.1, \theta =$

$0.1, \zeta = 0.1, \psi = 0.1$

5. Conclusion

In this paper, we have studied the ecological model in which one of two species that are competing is under predation. The conditions of existence of equilibrium points and their stability were obtained. Theoretical analysis of the dissipativity of the system was presented. Numerical simulations were illustrated to demonstrate that the three species' coexistence is dependent on values of interspecific competition factors α_{12} and α_{21} between Y and X rather than predation. However, the extinction of the predator is attained when the measure of efficiency of the searching and the capture of predator is equal to efficiency of converting prey into predator births when the predators and species in competition experienced damped oscillation under that circumstance.

References

[1] M. Bohner, M. Fan and J. Zhang (2006). Existence of periodic solutions in predator-prey and competition dynamic systems. Nonlinear analysis: Real world applications, 7, 1193 - 1204.

[2] Y. Kan-On and M. Mimura (1998). Singular perturbation approach to a 3-component reaction-diffusion system arising in population dynamics. SIAM Journal of Mathematical Analysis, vol. 29, no. 6, pp. 1519-1536.

[3] J. N. Ndam, J. P. Chollom and T. G. Kassem (2012). A mathematical model of three-species interactions in an acquatic habitat, ISRN Applied Mathematics

[4] H. I. Freedman and S. Ruan (1995). Uniform persistence in functional differential equations, Journal of Differential Equations ; 115, 173-192