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A Two-Dimensional Chebyshev Wavelet Method for Solving Partial Differential Equations

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Abstract

In this paper, we introduce a two-dimensional Chebyshev wavelet method (TCWM) for solving partial differential equations (PDEs) in $L^2(\mathbf{R})$ space. In this method, the spatial variables appearing in the PDE each has its own kernel, as well as wavelet coefficient for approximating the unknown solution of the equation. The approximated solution of the equation is fast and has higher number of vanishing moments as compared to the Chebyshev wavelet method with only one wavelet coefficient for two or more separated kernels for the variables appearing in the PDE.

Mathematics Subject Classification: 44C26; 44C25

Keywords: A two-dimensional Chebyshev wavelet method, separated wavelet coefficient, PDEs, Chebyshev wavelet function

1 Introduction

Most of the physical problems like heat conduction, wave propagation, laser beam models, are modelled as PDEs whose solutions cannot be easily obtained by the classical method. This may be due to either the nonlinearity associated with equation or inappropriate solution space. The Poisson equation with the modulus of x as an inhomogeneous function, in a unit square domain $([0,1] \times [0,1])$, has no classical solution in Hilbert space. Sometimes, a PDE may have no smooth function that describes the problem at hand. For example, Black Scholes equation in finance is of no exception. Solving a PDE in a functional space may not be feasible, the only way is to approximate the solution of the problem. This depends on the kind of the lay-down procedure one adopts. Approximation methods like Runge-Kutta, Adams-Bashford method, provide solution with unsurmountable errors. In a similar development, the local methods such as finite difference method, finite element method, finite volume method, collocation method, provide unstable solution, which when the boundary of the domain of PDE is irregular. Thus, the boundary information is not included in the approximated solution. In addition, they give information only in frequency domain. Numerical approximations of solution of the equation have a number of drawbacks.

A method which is efficient and accurate to obtain the approximated solutions of the PDEs over the past two decades is the wavelet series. This method makes use of both the dilation parameter, for compression of wavelet series solution, and the translation parameter for the location of wavelet series solution in $L^2(\mathbf{R})$ space. The method provides approximated solution to the PDEs in regular domain, as well as irregular domain. In addition, the wavelet function is symmetrical, detects the singularities in the equation, and yields solution which is robust against noise. Again, the wavelet series solution of a PDE converges faster than the traditional methods of approximation on the grounds that the wavelet function has the number of vanishing moments. What is more, in wavelet domain, we are able to obtain both time and frequency information which is not feasible with other methods of approximating PDEs.

Due to the pioneering work by Chen and Hsiao [6] who introduced an integral Haar wavelet method for solving differential equations, number of studies had applied this method in solving differential equations. For example, see research papers by [4, 5, 9, 10, 11, 15, 17, 18, 19, 25]. The approximated Haar wavelet solutions of the Klein-Gordon and Sine-Gordon equations were obtained by [8]. In [21], solution of fractional-order differential equation was obtained by the use of new operational matrix of derivatives. However, the Haar wavelet function, a square-wave, is discontinuous at zero in the domain $[0, 1)$. Thus, the Haar wavelet function lacks vanishing moments on the grounds that it is not $C^2([0,1))$. This shortcoming of the Haar integral wavelet method limits its applicabilities. Some wavelet series methods like Legendre method of different kinds have been introduced for solving the PDEs. For example, see research papers by [1, 2, 20, 22, 27]. In [14], the author used Daubechies wavelet function, which is compactly supported, to obtain the approximated solution of the PDE.

Recently, the Chebyshev wavelet function, analytic in its derivation, and with higher number of vanishing moments has been used in approximating PDEs. In [12], the authors applied Chebyshev wavelet method to obtain the approximated solution of the fractional integrodifferential equation in one spatial variable. In [13, 24], the authors developed the shifted second kind Chebyshev matrix of derivatives for solving ordinary differential equations. The authors in [23] proved the convergence of the shifted fourth kind Chebyshev wavelet solution of the PDE in one spatial variable. A lot of studies have been done on attainment of wavelet solution of PDE in one coordinate variable.

However, only a few works have looked at the approximating PDEs with solutions in two coordinate variables. In [3], the authors obtained approximated solution of the first kind of Fredholm integral equation. The kernel of the integral equation was approximated as an inner product of two different wavelet basis functions as

$$K(x,y) = \sum_{i=1}^{2^{k-1}M} \sum_{j=1}^{2^{k-1}M} K_{ij}\psi_i(x)\psi_j(y),$$

where $i = M(n-1)+m+1$ and $j = M(n'-1)+m'+1$. Again, in [16] the author applied inner product of two wavelet basis kernel functions for solving stochastic Volterra-Fredholm integral equation. Splitting the kernel of the partial differential operator into wavelet kernels that match with the number of spatial variables appearing in the equation cannot be overemphasized. Thus, this method gives the approximated solution to a PDE in the number of dimensions equal to the number of spatial variables appearing in the equation.

In this paper, we make use of Chebyshev wavelet basis which is analytic and compactly supported on the interval $[0,1]$. We introduce a TCWM to obtain approximated solutions of PDEs. In TCWM, each spatial variable occurring in the PDE has its own wavelet coefficients. Thus, we split the unknown solution of the PDE into Chebyshev wavelet functions with each spatial variable having its own kernel, as well as wavelet coefficient, which is then used to obtain the approximated solution of the PDE.

The Chebyshev wavelet function has compact support on $[0,1]$, orthogonal with respect to the weight

$$w(x) = (1-x^2)^{\frac{-1}{2}}.$$

The TCWM solves PDEs by converting the equation into a system of algebraic equations from which the wavelet coefficients are obtained. The wavelet coefficients together with Chebyshev wavelet are used to obtain the approximated solution for the PDE. By and large, this method of approximation is easy and converges faster for a few number of Chebyshev wavelet coefficients. Thus, the number of vanishing moments is higher as compared to [3] and others.

2 Some Properties of Wavelets and Chebyshev wavelets

In this section, we provide some definitions and properties of the wavelets, and Chebyshev wavelets that will enable us to derive our result in subsequent sections.

Definition 2.1 (Wavelet) *A family of functions constructed from the translation and dilation of a single function $\psi(x)$, is called the (mother) wavelet. The dilation parameter j and translation parameter k are varied continuously which gives*

$$\psi_{j,k}(x) = \frac{1}{\sqrt{|j|}}\psi(\frac{x-k}{j}), j, k \in \mathbf{R}, \quad j \neq 0.$$

The dilation parameter which measures the degree of compression or scale, and k is the translation parameter which determines the time location of the wavelet, see [7].

Definition 2.2 (Multiresolution Analysis) *Let $\{\phi_{ok}\}$ be an orthonormal system in $L^2(\mathbf{R})$. The sequence of spaces $\{V_j,\ j \in \mathbf{Z}\}$, generated by $\phi(x)$ is called a multiresolution analysis MRA of $L^2(\mathbf{R})$ if it satisfies the following properties:*

1. $V_j \subset V_{j+1}, \quad j \in \mathbf{Z}$
2. $\bigcup_{j\in Z} V_j = L^2(\mathbf{R})$
3. $\cap_{j\in Z} V_j = \{0\}$
4. $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}$
5. $f(x) \in V_j \Leftrightarrow f(x-k) \in V_j, \forall\, k \in Z$

See [26].

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Theorem 2.3 (The Riesz representation theorem) *Let $f(x)$ be a bounded linear functional on a Hilbert space H. There exists exactly on $x_o \in H$ such that*

$$f(x) = \langle x, x_o \rangle, \quad \forall x \in H.$$

Moreover, we have

$$\|f\| = \|x_o\|.$$

There exists a function ϕ (called scaling function or father wavelet) such that

$$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k), \ \ k \in Z$$

constitute an orthonormal basis for corresponding subspace V_j.

See [26]

Also, we make use of the first kind of the Chebyshev wavelet $\psi_{n,m}(x)$ on the interval $[0,1]$, which is given by

$$\psi_{n,m}(x) = \begin{cases} 2^{\frac{T+1}{2}}\tilde{T_m}(2^{k+1}x - 2n - 1), & \frac{n}{2^k} \le x < \frac{n+1}{2^k} \\ 0, \quad otherwise, & \end{cases}$$

where

$$\tilde{T}(x) = \begin{cases} \frac{1}{\sqrt{\pi}}, & m = 0 \\ \sqrt{\frac{2}{\pi}}T_m(x), & m \ge 1, \end{cases}$$

$m = 0, 1, \ldots, M, \ \ n = 0, 1, \ldots, 2^k - 1$, and k is any positive integer. The $T_m(x)$ are Chebyshev polynomial of the first kind of degree m which are orthogonal with respect to the weight function

$$w(x) = \frac{1}{\sqrt{(1 - x^2)}}$$

and satisfy the following recursive relation

$$T_{m+1}(x) = 2xT_m(x) - T_{m-1}(x), \qquad \forall\, m = 1, 2, \ldots,$$

where

$$T_0(x) = 1$$

and

$$T_1(x) = x$$

see [23].

2.1 A Two-dimensional Chebyshev Wavelet Method

In this section, we introduce TCWM for solving PDEs in both regular and irregular domains. Let the unknown function $w(x, y)$ be defined over $[0, 1] \times [0, 1]$, is expanded in terms of first kind Chebyshev wavelets as

$$w(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} B_{n,m}\psi_{n,m}(x) \times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_{n,m}\psi_{n,m}(y), \tag{1}$$

where

$$\begin{aligned} B_{n,m} &= \langle w(x, .), \psi_{n,m}(x)\rangle \\ B_{n,m} &= \int_0^1 \frac{1}{\sqrt{(1-t)}} w(x, .)\psi_{n,m}(x)dx, \end{aligned}$$

and

$$\begin{aligned} C_{n,m} &= \langle w(., y), \psi_{n,m}(y)\rangle \\ C_{n,m} &= \int_0^1 \frac{1}{\sqrt{(1-t)}} w(., y)\psi_{n,m}(y)dy. \end{aligned}$$

The infinite series is truncated, then it is written as

$$\begin{aligned} w(x, y) &\approx \sum_{n=0}^{2^k-1} \sum_{m=0}^{M} B_{n,m}\psi_{n,m}(x) \times \sum_{n=0}^{2^k-1} \sum_{m=0}^{M} C_{n,m}\psi_{n,m}(y) \\ w(x, y) &= B^T\psi(x)C^T\psi(y), \end{aligned} \tag{4.4}$$

where $C,\ B,\ \psi(x)$ and $\psi(y)$ are $2^k(M + 1) \times 1$ matrices given by

$$B = [\ B_{0,0}, B_{0,1}, \ldots, B_{0,M}, B_{1,0}, B_{1,1}, \ldots, B_{1,M}, B_{2^k-1,0}, B_{2^k-1,1}, \ldots, B_{2^k-1,M}]\ ^T \tag{2}$$

$$C = [\ C_{0,0}, C_{0,1}, \ldots, C_{0,M}, C_{1,0}, C_{1,1}, \ldots, C_{1,M}, C_{2^k-1,0}, C_{2^k-1,1}, \ldots, C_{2^k-1,M}]\ ^T \tag{3}$$

$$\begin{aligned} \psi(x) = [\ &\psi_{0,0}(x), \psi_{0,1}(x), \ldots, \psi_{0,M}(x), \psi_{1,0}(x), \psi_{1,1}(x), \ldots, \psi_{1,M}(x), \psi_{2^k-1,0}(x), \\ &\psi_{2^k-1,1}(x), \ldots, \psi_{2^k-1,M}(x)]\ ^T \end{aligned} \tag{4}$$

$$\begin{aligned} \psi(y) = [\ &\psi_{0,0}(y), \psi_{0,1}(y), \ldots, \psi_{0,M}(y), \psi_{1,0}(y), \psi_{1,1}(y), \ldots, \psi_{1,M}(y), \psi_{2^k-1,0}(y), \\ &\psi_{2^k-1,1}(y), \ldots, \psi_{2^k-1,M}(y)]\ ^T \end{aligned} \tag{5}$$

2.2 The Operational Matrix of Derivatives of the TCWM

In this section, we derive a new operational matrix of the second partial derivatives for approximating the partial coefficients of the PDEs by the following theorem.

Theorem 2.4 *Let $\psi(x)$ and $\psi(y)$ be the first kind Chebyshev wavelets vector defined in* (4) *and* (5) *respectively, then the first derivative of the vectors $\psi(x)$ and $\psi(y)$ are given as*

$$\frac{\partial^2 \psi(x)}{dx^2} = D^2 \psi(x)$$

and

$$\frac{\partial^2 \psi(y)}{dy^2} = D^2 \psi(y)$$

where D^2 is $2^k(M+1)$ square operational matrix of second-order derivatives and given by

$$D^2 = \begin{bmatrix} F & 0 & \cdots & 0 \\ 0 & F & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & F \end{bmatrix} \tag{6}$$

where 0 is square matrix and $(M+1)$ zeros matrix, F is an $(M+1)$ square matrix and its (r,s)th element is defined as follows.

$$F = \begin{cases} 2, & \\ 0, & otherwise \end{cases}$$

Corollary 2.5 *Let $\psi(x)$ be the Chebyshev wavelet vector defined in equation* (4), *then the operational matrix for the n^{th} derivative is*

$$\frac{d^n \psi(x)}{dx^n} = D^n \psi(x),$$

where D^n is the n^{th} power of matrix D.

2.3 Convergence of TCWM

In this section, we show that a two-dimensional Chebyshev wavelet method converges in $L^2(\mathbf{R})$ space.

Theorem 2.6 *Let*

$$w(x,y) = \sum_{n=0}^{2^k-1} \sum_{m=0}^{M} B_{n,m} \psi_{n,m}(x) \times \sum_{n=0}^{2^k-1} \sum_{m=0}^{M} C_{n,m} \psi_{n,m}(y)$$

be the wavelet series solution to the PDE using Chebyshev wavelet then the solution converges to $w(x,y)$ in $L^2(\mathbf{R})$ space.

Proof: Let $L^2(\mathbf{R})$ be a space of square integrable functions with an inner product defined on it, and $\psi_{n,m}(x)$ and $\psi_{n,m}(y)$ defined in equations (4) and (5) form orthonormal basis. Also, let

$$w(x,y) = \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} B_{n,m}\psi_{n,m}(x) \times \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} C_{n,m}\psi_{n,m}(y)$$

be the solution of PDE , where

$$\begin{aligned} B_{n,m} &= \langle w(x,.), \psi_{n,m}(x)\rangle \\ C_{n,m} &= \langle w(.,y), \psi_{n,m}(y)\rangle, \quad for\ a\ fixed\ n. \end{aligned}$$

We define

$$\begin{aligned} \psi_{n,i}(x) &= \Psi(x) \\ \psi_{n,i}(y) &= \Psi(y) \end{aligned}$$

and

$$\begin{aligned} \alpha_j &= \langle w(x,.), \Psi(x)\rangle \\ \gamma_i &= \langle w(.,y), \Psi(y)\rangle. \end{aligned}$$

Let S_n be the partial sum of

$$\langle \alpha_j\Psi(x_j), \gamma_i\Psi(y_i)\rangle.$$

$$\begin{aligned} S_n &= \sum_{j=0}^{n}\sum_{i=0}^{m}\alpha_j\Psi(x_j)\gamma_i\Psi(y_i) \\ \Rightarrow \langle w(x,y), S_n\rangle &= \langle w(x,y), \sum_{j=0}^{n}\sum_{i=0}^{m}\alpha_j\Psi(x_j)\gamma_i\Psi(y_i)\rangle \\ \langle w(x,y), S_n\rangle &= \langle \sum_{j=0}^{n}\sum_{i=0}^{m}\alpha_j\gamma_i\Psi(x_j)\Psi(y_i), \alpha_j\Psi(x_j)\gamma_i\Psi(y_i)\rangle \\ \langle w(x,y), S_n\rangle &= \sum_{j=1}^{n}\sum_{i=1}^{m}\overline{\alpha_j\gamma_i}\alpha_j\gamma_i \\ \langle w(x,y), S_n\rangle &= \sum_{j=1}^{n}\sum_{i=1}^{m}|\alpha_j\gamma_i|^2 \end{aligned}$$

By contradiction, we let S_n and S_m be the partial sum with $m, n > s$.

$$\|S_n - S_m\|^2 = \|\sum_{j=s+1}^{n}\sum_{i=s+1}^{m}\alpha_j\Psi(x_j)\gamma_i\Psi(y_i)\|^2$$

$$\begin{aligned}
\|S_n - S_m\|^2 &= \langle \sum_{j=s+1}^{n} \sum_{i=s+1}^{m} \alpha_j \Psi(x_j)\gamma_i \Psi(y_i), \sum_{j=s+1}^{n} \sum_{i=s+1}^{m} \alpha_j \Psi(x_j)\gamma_i \Psi(y_i)\rangle \\
\|S_n - S_m\|^2 &= \sum_{j=s+1}^{n} \sum_{i=s+1}^{m} \overline{\alpha_j \gamma_i} \alpha_j \gamma_i \langle \Psi(x_j)\Psi(y_i), \Psi(x_j)\Psi(y_i)\rangle \\
\|S_n - S_m\|^2 &= \sum_{j=s+1}^{n} \sum_{i=s+1}^{m} |\alpha_j \gamma_i|^2
\end{aligned}$$

As $m, n \to \infty$. From Bessel's inequality, we see that

$$\sum_{j=s+1}^{n} \sum_{i=s+1}^{m} |\alpha_j \gamma_i|^2$$

is convergent. Thus,

$$\| \sum_{j=0}^{n} \sum_{i=0}^{m} \alpha_j \Psi(x_j)\gamma_i \Psi(y_i)\|^2 \to 0 \;\; as \;\; m, n \to \infty$$

The $\{S_n\}$ converges uniformly. Thus,

$$\lim_{m,n\to\infty} w(x, y) = 0.$$

We see that:

$$\begin{aligned}
\langle L - w(x, y), \Psi(x_j)\Psi(y_i)\rangle &= \langle L, \Psi(x_j)\Psi(y_i)\rangle - \langle w(x, y), \Psi(x_j)\Psi(y_i)\rangle \\
&= \langle \lim_{m,n\to\infty} S_n, \Psi(x_j)\Psi(y_i)\rangle - \alpha_j \gamma_i \langle \Psi(x_j)\Psi(y_i), \Psi(x_j)\Psi(y_i)\rangle \\
&= \alpha_j \gamma_i - \alpha_j \gamma_i \\
&= 0.
\end{aligned}$$

We give the structure of operational matrix of second-order derivatives based on coefficients for each spatial variable appearing in the PDE. We choose $k = 1$ and $M = 3$, we obtain operational matrix of derivatives with respect to x as follows:

$$\begin{aligned}
\psi_1(x) &= \psi_{0,0}(x) = \begin{cases} \frac{2}{\sqrt{\pi}}, & 0 \le x < \frac{1}{2} \\ 0, & otherwise \end{cases} \\
\psi_2(x) &= \psi_{0,1}(x) = \begin{cases} \frac{2\sqrt{2}}{\sqrt{\pi}}(4x - 1), & 0 \le x < \frac{1}{2} \\ 0, & otherwise \end{cases} \\
\psi_3(x) &= \psi_{0,2}(x) = \begin{cases} \frac{2\sqrt{2}}{\sqrt{\pi}}\Big(2(4x - 1)^2 - 1\Big), & 0 \le x < \frac{1}{2} \\ 0, & otherwise \end{cases} \\
\psi_4(x) &= \psi_{0,3}(x) = \begin{cases} \frac{2\sqrt{2}}{\sqrt{\pi}}(256x^3 - 192x^2 + 36x^2 - 1), & 0 \le x < \frac{1}{2} \\ 0, & otherwise \end{cases}
\end{aligned}$$

$$\begin{aligned}
\psi_5(x) &= \psi_{1,0}(x) = \begin{cases} \frac{2}{\sqrt{\pi}}, & \frac{1}{2} \leq x < 1 \\ 0, & otherwise \end{cases} \\
\psi_6(x) &= \psi_{1,1}(x) = \begin{cases} \frac{2\sqrt{2}}{\sqrt{\pi}}(4x-3), & \frac{1}{2} \leq x < 1 \\ 0, & otherwise \end{cases} \\
\psi_7(x) &= \psi_{1,2}(x) = \begin{cases} \frac{2\sqrt{2}}{\sqrt{\pi}}\Big(2(4x-3)^2-1\Big), & \frac{1}{2} \leq x < 1 \\ 0, & otherwise \end{cases} \\
\psi_8(x) &= \psi_{1,3}(x) = \begin{cases} \frac{2\sqrt{2}}{\sqrt{\pi}}(256x^3-576x^2+420x-99), & \frac{1}{2} \leq x < 1 \\ 0, & otherwise \end{cases}
\end{aligned}$$

Also, we derive 8×8 operational matrix of second derivatives. The second-order derivatives of $\psi(x)$, $i = 1, \ldots, 8$ with respect to x is given below

$$\begin{aligned}
\frac{d^2\psi_1(x)}{dx^2} &= 0 \\
\frac{d^2\psi_2(x)}{dx^2} &= 0 \\
\frac{d^2\psi_3(x)}{dx^2} &= \frac{128\sqrt{2}}{\sqrt{\pi}} \\
\frac{d^2\psi_3(x)}{dx^2} &= 64\sqrt{2}\psi_1(x) \\
\frac{d^2\psi_4(x)}{dx^2} &= \frac{2\sqrt{2}}{\sqrt{\pi}}(1536x-384) \\
\frac{d^2\psi_4(x)}{dx^2} &= 384\psi_2(x) \\
\frac{d^2\psi_5(x)}{dx^2} &= \frac{d^2\psi_6(x)}{dx^2} = 0 \\
\frac{d^2\psi_7(x)}{dx^2} &= 64\sqrt{2}\psi_5(x) \\
\frac{d^2\psi_8(x)}{dx^2} &= 384\psi_6(x)
\end{aligned}$$

and the matrix

$$D = \begin{bmatrix} F & 0 \\ 0 & F \end{bmatrix}$$

where,

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 64\sqrt{2} & 0 & 0 & 0 \\ 0 & 384 & 0 & 0 \end{bmatrix} \tag{8}$$

On the other hand, the operational matrix of derivatives with respect to y is similar except that we replace x for y into equation (8)

2.4 The Product Operational Matrix POM of the TCWM

In this section, we derive the product of two Chebyshev wavelet function vectors to obtain the approximated solutions of the PDEs

$$\psi(x)\psi^T(x)C \approx \tilde{C}\psi(x) \qquad (9)$$

We obtain the matrix $\psi(x)\psi^T(x) = \frac{2}{\sqrt{\pi}}\times$

$$\begin{bmatrix} \psi_{0,0} & \psi_{0,1} & \psi_{0,2} & \psi_{0,3} & 0 & 0 & 0 & 0 \\ \psi_{0,1} & \psi_{0,0}+\frac{1}{\sqrt{2}}\psi_{0,2} & \frac{1}{\sqrt{2}}(\psi_{0,1}+\psi_{0,3}) & \frac{1}{\sqrt{2}}\psi_{0,2} & 0 & 0 & 0 & 0 \\ \psi_{0,2} & \frac{1}{\sqrt{2}}(\psi_{0,1}+\psi_{0,3}) & \psi_{0,0} & \frac{1}{\sqrt{2}}\psi_{0,1} & 0 & 0 & 0 & 0 \\ \psi_{0,3} & \frac{1}{\sqrt{2}}\psi_{0,2} & \frac{1}{\sqrt{2}}\psi_{0,1} & \psi_{0,0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \psi_{1,0} & \psi_{1,1} & \psi_{1,2} & \psi_{1,3} \\ 0 & 0 & 0 & 0 & \psi_{1,1} & \psi_{1,0}+\frac{1}{\sqrt{2}}\psi_{1,2} & \frac{1}{\sqrt{2}}(\psi_{1,1}+\psi_{1,3}) & \frac{1}{\sqrt{2}}\psi_{1,2} \\ 0 & 0 & 0 & 0 & \psi_{1,2} & \frac{1}{\sqrt{2}}(\psi_{1,1}+\psi_{1,3}) & \psi_{1,0} & \frac{1}{\sqrt{2}}\psi_{1,1} \\ 0 & 0 & 0 & 0 & \psi_{1,3} & \frac{1}{\sqrt{2}}\psi_{1,2} & \frac{1}{\sqrt{2}}\psi_{1,1} & \psi_{1,0} \end{bmatrix}$$

by the set of rules given below as

$$\begin{aligned} \psi_{n,m}\psi_{l,k}(x) &= 0, \qquad n \neq l \\ \psi_{n,0}\psi_{n,k}(x) &= \frac{2}{\sqrt{\pi}}\psi_{n,k}, \qquad \forall k = 0,1,2 \\ \psi_{n,m}\psi_{n,k}(x) &= \frac{2}{\sqrt{\pi}}\psi_{n,0} + \frac{\sqrt{2}}{\sqrt{\pi}}\psi_{n,m+k}, \qquad for\ m = k \neq 0 \\ \psi_{n,m}\psi_{n,k}(x) &= \frac{\sqrt{2}}{\sqrt{\pi}}\left(\psi_{n,|m-k|} + \psi_{n,m+k}\right), \quad for\ m \neq k,\ m,k \neq 0,\ for\ m+k \leq 3\ and\ n = 0,1 \\ \psi_{n,m}\psi_{n,k}(x) &= \frac{\sqrt{2}}{\sqrt{\pi}}\psi_{n,|m-k|}, \qquad for\ m+k > 3 \end{aligned}$$

Then the 8×8 matrix $\tilde{C}$ in equation (4.10) is written as

$$\tilde{C} = \begin{bmatrix} c_0 & 0 \\ 0 & c_1 \end{bmatrix}$$

where, $c_i,\ i = 0,1$ are 4×4 matrix given below.

$$c_i = \frac{2}{\sqrt{\pi}}\begin{bmatrix} c_{i,0} & c_{i,1} & c_{i,2} & c_{i,3} \\ c_{i,1} & c_{i,0}+\frac{1}{\sqrt{2}}c_{i,2} & \frac{1}{\sqrt{2}}(c_{i,1}+c_{i,3}) & \frac{1}{\sqrt{2}}c_{i,2} \\ c_{i,2} & \frac{1}{\sqrt{2}}(c_{i,1}+c_{i,3}) & c_{i,0} & \frac{1}{\sqrt{2}}c_{i,1} \\ c_{i,3} & \frac{1}{\sqrt{2}}c_{i,2} & \frac{1}{\sqrt{2}}c_{i,1} & \frac{1}{\sqrt{2}}c_{i,0} \end{bmatrix}$$

We increase accuracy of the approximated solutions to PDEs by setting $M = 4$ and $k = 2$. This reduces equation (8) to 20×20 operational matrix of second-order derivatives D^2

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 256\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 1536 & 0 & 0 & 0 \\ 2048\sqrt{2} & 0 & 3072 & 0 & 0 \end{bmatrix}$$

We obtain similar results for $\psi(y)\psi^T(y)$.

3 Conclusion

In TCWM, each spatial variable of the PDE has its own wavelet coefficient, as well as Chebyshev kernel for approximating PDE which produces solution which matches with the spatial variables appearing in the equation. This approach facilitates easy and fast computation of the solution of the PDE on the grounds that, it makes use of a few number of vanishing moments. In addition, we observed that the approximated solution of the equation is fast and has higher number of vanishing moments as compared to [3] and others. Again, the TCWM solves the PDEs in regular, as well as irregular domains. Unlike the local methods such as finite difference method, finite element method, collocation method, which cannot give information on the boundary of the domain to the approximated solution of the PDE, our method provides the boundary information in approximating the unknown solution of the PDE. In addition, the TCWM gives information on both frequency and time domains.

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