Decomposition Method for Kdv Boussinesq and Coupled Kdv Boussinesq Equations

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Abstract: This paper obtains the solitary wave solutions of two different forms of Boussinesq equations that model the study of shallow water waves in lakes and ocean beaches. The decomposition method using He's polynomials is applied to solve the governing equations. The travelling wave hypothesis is also utilized to solve the generalized case of coupled Boussinesq equations, and, thus, an exact soliton solution is obtained. The results are also supported by numerical simulations.

Keywords: Decomposition Method, He's polynomials, cubic Boussinesq equation, Coupled Boussinesq equations

1. Introduction

In recent years, remarkable developments have taken place in the study of nonlinear evolutionary partial differential equations. It is realized that many such equations possess special solutions in the form of pulses which retain their shapes and velocities after interacting with each other. Such solutions are called solitons. Many equations admitting soliton solutions are as follows: sine Gordon and double sine Gordon equations, Schrodinger equation, and KDV, MKDV, and complex modified KDV equations many research works have been done on these equations. Most of the current research is directed to solve coupled nonlinear systems analytically and numerically. Solitons are of great interest in many physical areas, as, for example, in dislocation theory of crystals, plasma and fluid dynamics, magneto hydrodynamics, laser and fiber optics, and the study of the water waves. Many research works on Boussinesq equation have been developed analytical solution of this equation was studied by many authors [1-32], such as the construction of N-soliton solutions using the bilinear form [4], multiple soliton solutions for the GB equation using a simplified version of Hirota method [27] and decomposition method [28]. Construction of soliton solutions and periodic solution of Boussinesq equation by modified decomposition method are given in [29, 30]. A variational iteration method is developed for GB equation [31]. A solitary wave solution of the Boussinesq equation with power law nonlinearity is derived in [32]. Many numerical methods have been developed for solving the Boussinesq equation, suchas Petrov-Galerkin method [19]. Mohebbi and Asgari [26] also have solved the GB equation using a fourth order time stepping schemes with combination of discrete Fourier transform. Split step Fourier method is also used to solve Boussinesq-type equations.

2. Decomposition Method Using He's Polynomials

To illustrate the basic concept of modified decomposition method, we consider the following general differential equation

$$L(u) + N(u) = g(x) \tag{1}$$

Where L is the liner operator and N is the nonlinear operator and g(x) is the homogeneous term.

According to the ADM we construct the

$$u_{n+1} = u_n - {L_t}^{-1}\{N(u) + g(x)\}, \tag{2}$$

Where${L_t}^{-1} = \int_0^t dt$,. The embedding parameter p ∈(0, 1] can be considered as an Expanding parameters. The homotopy perturbation method uses the homotopy parameter p as an expanding parameter to obtain

$$u = \sum_{n=0}^{\infty} p^n u_n = u_0 + pu_1 + p^2u_2 + p^3u_3 + \cdots, \tag{3}$$

If$p \to 1$, the approximate solution of the form,

$$f = \lim_{p\to 1} u = \sum_{n=0}^{\infty} u_{n,} \tag{4}$$

It is well known that series (4) is convergent for most of the cases and also the rate of convergence is dependent on L (u). We assume that (4) has a unique solution. The comparisons of like powers of p give solutions of various orders. In sum, according to, He's considers the solution $u(x)$, of the homotopy equation in a series of p as follows:

$$u(x) = \sum_{n=0}^{\infty} p^n u_n = u_0 + pu_1 + p^2u_2 + p^3u_3 + \cdots,$$

And the method consider the nonlinear term N(u), as

$$N(u) = \sum_{n=0}^{\infty} p^n H_n = H_0 + pH_1 + p^2H_2 + p^3H_3 + \cdots,$$

Where H_n are so called He's polynomials which can be calculated by using the formula

$$H_n(u_0, u_1, u_2, \dots) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i u_i\right)\right]_{p=0}, \qquad n = 0,1,2,3,\dots, \tag{5}$$

The successive approximation $u_{n+1}, n \geq 0$ of the solution of u will be obtained by selective functionu_0, Consequently the solution is given by $u = \lim_{n\to\infty} u_n$.

3. Numerical Applications

In this section, we apply the decomposition method using He's polynomials for solving Boussinesq and coupled Boussinesq equations.

Example 3.1 Consider the cubic KDV Boussinesq equation

$$u_{tt} - u_{xx} + 2(u^3)_{xx} - u_{xxxx} = 0 \tag{6}$$

With subject to initial condition

$$\mathrm{u}(\mathrm{x}, 0) = \frac{1}{x}, \quad u_t(x,0) = -\frac{1}{x^2} \tag{7}$$

According to the above procedure

$$u(x,t) =$$

$$\frac{1}{x} - \frac{1}{x^2}t + pL_t^{-1}L_t^{-1}\{\sum_{n=0}^{\infty} p^n(u_{nxx}) - 2\sum_{n=0}^{\infty} p^n(u^3)_{nxx} + \sum_{n=0}^{\infty} p^n(u_{nxxxx})\}, \quad (8)$$

$$u(x,t) =$$

$$\frac{1}{x} - \frac{1}{x^2}t + pL_t^{-1}L_t^{-1}\{\sum_{n=0}^{\infty} p^n(u_{nxx}) - 2\sum_{n=0}^{\infty} p^n\big(H_n(u)\big) + \sum_{n=0}^{\infty} p^n(u_{nxxxx})\}, \quad (9)$$

Equating the like power components of p, we get

$$p^0: u_0 = \frac{1}{x} - \frac{1}{x^2}t,$$

$$p^1: u_1 = L_t^{-1}L_t^{-1}\{u_{0xx} - 2H_0 + u_{0xxxx}\},$$

$$= \frac{t^2}{x^3} - \frac{t^3}{x^4} - \frac{180t^4}{12x^7} + \frac{84t^5}{20x^8} = \frac{t^2}{x^3} - \frac{t^3}{x^4} + small\ terms,$$

$$p^2: u_2 = L_t^{-1}L_t^{-1}\{u_{1xx} - 2H_1 + u_{1xxxx}\},$$

$$= \frac{t^4}{x^5} - \frac{t^6}{x^7} + small\ terms,$$

And so on, summing all components of u, we get

$$u(x,t) = u_0 + u_1 + u_2 + \cdots,$$

$$= \frac{1}{x} - \frac{1}{x^2}t + \frac{t^2}{x^3} - \frac{t^3}{x^4} + \frac{t^4}{x^5} - \frac{t^6}{x^7} + small\ terms,$$

$$= \frac{1}{x+t}.$$

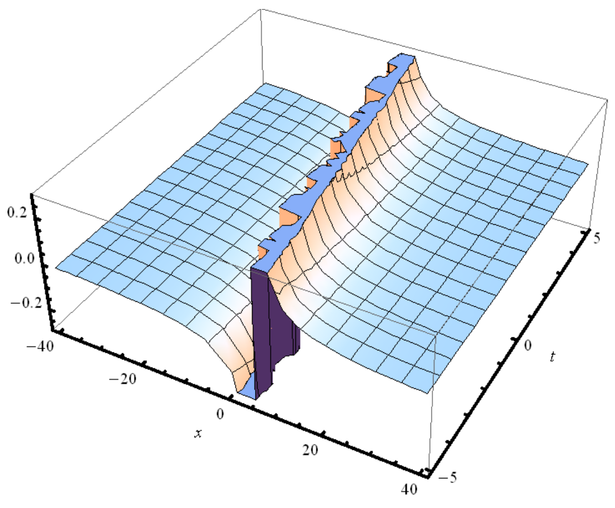

Fig1: behavior of u(x,t) with $-40 \le x \le 40$ and $-5 \le t \le 5$.

Example 3.2 Consider the coupled Boussinesq equations

$$u_t + v_x + uu_x = 0,$$

$$v_t + (vu)_x + u_{xxx} = 0, \qquad (10)$$

With subject to initial conditions

$$u(x,0) = \frac{\lambda}{k} + 2ktanh(kx),$$

$$v(x,0) = 2k^2sech^2(kx), \qquad (11)$$

According to the above procedure

$$u(x,t) = \frac{\lambda}{k} + 2ktanh(kx) - pL_t^{-1}\{\sum_{n=0}^{\infty} p^n\,(v_{nx}) + \sum_{n=0}^{\infty} p^n\,(u_n u_{nx})\}, \qquad (12)$$

$$v(x,t) = 2k^2sech^2(kx) - pL_t^{-1}\{\sum_{n=0}^{\infty} p^n (vu)_{nx} - \sum_{n=0}^{\infty} p^n\,(u_{nxxx})\}, \qquad (13)$$

$$u(x,t) = \frac{\lambda}{k} + 2ktanh(kx) - pL_t^{-1}\{\sum_{n=0}^{\infty} p^n\,(v_{nx}) + \sum_{n=0}^{\infty} p^n\,(H_n)\}, \qquad (14)$$

$$v(x,t) = 2k^2sech^2(kx) - pL_t^{-1}\{\sum_{n=0}^{\infty} p^n (M_n) - \sum_{n=0}^{\infty} p^n\,(u_{nxxx})\}, \qquad (15)$$

Where H_n *and* M_n are nonlinear terms, equating like power components of p, we get

$$p^0 : u_0 = \frac{\lambda}{k} + 2ktanh(kx),$$

$$p^0 : v_0 = 2k^2sech^2(kx),$$

$$p^1 : u_1 = -L_t^{-1}\{v_{0x} + H_0\},$$

$$= 4k^3sech^2(kx)\tanh(kx)t - 2\lambda ksech^2(kx)t - 4k^3sech^2(kx)\tanh(kx)t,$$

$$= -2\lambda ksech^2(kx)t,$$

$$p^1 : v_1 = -L_t^{-1}\{M_0 + u_{0xxx}\},$$

$$= -4k^4sech^4(kx)t + 4\lambda k^2sech^2(kx)\tanh(kx)t + 8k^4sech^2(kx)t$$

$$- 8k^4sech^4(kx)t - 8k^4sech^2(kx)t + 12k^4sech^4(kx)t,$$

$$=4\lambda k^2sech^2(kx)\tanh(kx)t,$$

And so on, summing all components of u and v, we get

$$u(x,t) = \frac{\lambda}{k} + 2ktanh(kx) - 2\lambda ksech^2(kx)t - \cdots,$$

$$u(x,t) = \frac{\lambda}{k} + 2ktanh(kx - \lambda t).$$

$$v(x,t) = 2k^2sech^2(kx) + 4\lambda k^2sech^2(kx)\tanh(kx)t + \cdots,$$

$$v(x,t) = 2k^2sech^2(kx - \lambda t).$$

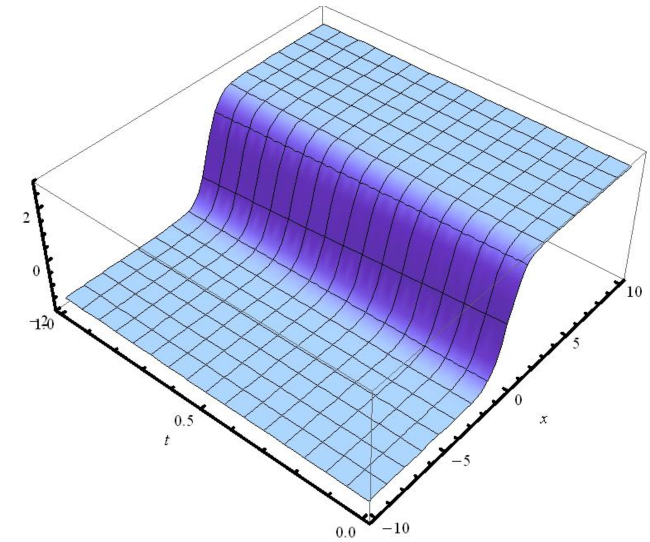

Fig 2: Behavior of u(x,t) with $-10 \le x \le 10$ and $0 \le t \le 1$ and $k = 1, \lambda = 0.5$.

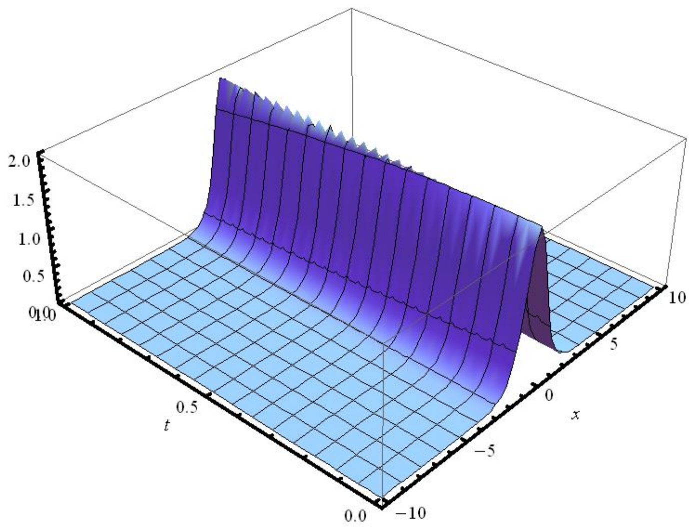

Fig 3: Behavior of v(x, t) with $-10 \le x \le 10$ and $0 \le t \le 1$ and $k = 1, \lambda = 0.5$.

4. Conclusion

There are two main goals that we aimed for this work. The first is to show the power of the modified decomposition method using He's polynomials and its significant features. The second is to employ this method to obtain rational solutions of nonlinear dispersive equations. It is obvious that the method gives rapid convergent successive approximations without any restrictive assumptions or transformation that may change the physical behavior of the problem. Moreover, the decomposition method reduces the size of calculations by not requiring the tedious working. The cubic Boussinesq and coupled Boussinesq equations were examined for rational solutions only and soliton solution are obtained. The desired solutions were obtained rapidly and in a direct way.

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