

A Common Unique Random Fixed Point Theorem in 2 - Hilbert Space

[1]V. K. Agrawal, [2]Ajay soni, [3]Amit Kumar Diwakar, [4]K. K. Wadhwa

[1]Govt. Home Science College, Hoshangabad, India.

[2]Research Scholar, Barkatullah University, Bhopal, India

[3]Research Scholar, Barkatullah University, Bhopal, India.

[4]Govt. Narmada Mahavidhyalay, Hoshangabad, India

Abstract:
The object of this paper is to obtain a common unique fixed point theorem for two continuous random operators defined on a non empty closed subset of a separable 2 - Hilbert space.
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1. Introduction

In recent years, the study of random fixed points have attracted much attention, some of the recent literatures in random fixed point may be noted in [1,2]. In this paper we construct a sequence of measurable functions and consider its convergence to the common unique random fixed point of two continuous random operators defined on a non-empty closed subset of a separable Hilbert space. For the purpose of obtaining the random fixed point of the two continuous random operators. We have used a rational inequality (from [4]) and the parallelogram law. Throughout this paper, (Ω, Σ) denotes a measurable space consisting of a set Ω and sigma algebra Σ of subsets of Ω. H stands for a separable Hilbert space, and C is a nonempty closed subset of H.

2 Preliminary:

Definition 2.1. A function $f : \Omega \to C$ is said to be measurable if $f^{-1}(B \cap C) \in \Sigma$ for every Borel subset B of H.

Definition 2.2. A function $F : \Omega \times C \to C$ is said to be a random operator if $F(., x) : \Omega \to C$ is measurable for every $x \in C$.

Definition 2.3. A measurable function $g : \Omega \to C$ is said to be a random fixed point of the random operator $F : \Omega \times C \to C$ if $F(t, g(t)) = g(t)$ for all $t \in \Omega$.

Definition 2.4. A random operator $F : \Omega \times C \to C$ is said to be continuous if for fixed $t \in \Omega$, $F(t, .) : C \to C$ is continuous.

Condition (A). Two mappings $S, T : C \to C$, where C is a non-empty closed subset of a Hilbert space H, is said to satisfy condition (A) if

$$\| Sx - Ty, t \|^2 \leq a \frac{\| y - Ty,\ t \|^2\ [1 + \| x - Sx,\ t \|^2]}{1 + \| x - y,\ t \|^2} + b[\| x - Sx, t \|^2 \| y - Ty, t \|^2]$$

for each $x, y \in C, a, b$ being positive real numbers such that $0 < a + b < \frac{1}{2}$.

3.Main Result:

Theorem 3.1: Let C be a non-empty closed subset of a separable 2 - Hilbert space H. Let S and T be two continuous random operators defined on C such that for $t \epsilon \Omega$, $s(t,.), T(t,.): C \to C$ satisfy condition (A) Then S and T have a common unique random fixed point in C.

Proof: We define a sequence of functions $\{g_n\}$ as g_0 $\Omega \epsilon$ C is arbitrary measurable function for $t \epsilon \Omega$, and n = 0,1,2,3.......

$$g_{2n+1}(t) = S(t, g_{2n}(t)), g_{2n+2}(t) = T(t, g_{2n+1}(t))$$

If $g_{2n}(t) = g_{2n+1}(t) = g_{2n+2}(t)$ for $t \epsilon \Omega$ for some n then me see that $g_{2n}(t)$ a random fixed point of S and T. So we assume that no two conseutine terms of sequence $\{g_n\}$ are equal.

Now consider for $t \epsilon \Omega$

$$\| g_{2n+1}(t) - g_{2n+2}(t), a \|^2 = \| S(t, g_{2n}(t)) - T(t, g_{2n+1}(t), a \|^2$$

$$\leq \frac{a \| g_{2n+1}(t) - T(t, g_{2n+1}(t), a) \|^2 \ [1 + \| g_{2n}(t) - S(t, g_{2n}(t), a) \|]^2}{1 + \| g_{2n}(t) - g_{2n+1}(t), a \|^2}$$

$$+b[\| g_{2n}(t) - S(t, g_{2n}(t), a) \|]^2 + \| g_{n+1}(t) - T(t, g_{(2n+1)}(t), a) \|^2]$$

$$= \frac{a \| g_{2n+1}(t) - g_{2n+2}(t), a \|^2 \ [1 + \| g_{2n}(t) - S(t, g_{2n}(t), a) \|]^2}{1 + \| g_{2n}(t) - g_{2n+1}(t), a \|^2} + b \| g_{2n}(t) - g_{2n+1}(t), a \|^2$$
$$+ \| g_{2n+1}(t) - g_{2n+2}(t), a \|^2$$

$$(a+b) \ \| g_{2n+1}(t) - g_{2n+2}(t), a \|^2 + b \| g_{2n}(t) - g_{2n+1}(t), a \|^2$$

$$\Rightarrow [1 - (a+b)] \ \| g_{2n+1}(t) - g_{2n+2}(t), a \|^2 \leq b \| g_{2n}(t) - g_{2n+1}(t), a \|^2$$

$$\Rightarrow \| g_{2n+1}(t) - g_{2n+2}(t), a \|^2 \leq \frac{b}{1-(a+b)} \| g_{2n}(t) - g_{2n+1}(t), a \|,$$

Where $K = [\frac{b}{[1-(a+b)]}]^2 \leq \frac{1}{2}$

In general

$$\| g_n(t) - g_{n+1}(t), a \| \leq k \| g_{n-1}(t) - g_n(t), a \|$$

$$\Rightarrow \| g_n(t) - g_{n+1}(t), a \| \leq k^n \| g_0(t) - g_1(t), a \| \text{ for all } t \epsilon \Omega$$

Now, we shall prove that for $t \epsilon \Omega$, $\{g_n(t)\}$ is a Cauchy sequence . for this for every position integer i we have, for $t \epsilon \Omega$.

$$\| g_n(t) - g_{n+p}(t), a \| = \| g_n(t) - g_{n+1}(t) + \cdots + g_{n+p-1}(t) + -g_{n+p}(t), a \|$$

$\leq \| \boldsymbol{g_n(t) - g_{n+1}(t), a} \| + \| \boldsymbol{g_{n+1}(t) - g_{n+2}(t), all} + \dots . \| \boldsymbol{g_{n+p-1}(t) - g_{n+p}(t), a} \|$

$$\leq [k^{\boldsymbol{n}} + k^{\boldsymbol{n+1}} + \cdots . + k^{\boldsymbol{n+p-1}}] \ \| g_0(t) - g_1(t), a \|$$

$=k^n[1 + k + k^{\boldsymbol{2}} + \cdots . + k^{\boldsymbol{p-1}}] \ \| g_0(t) - g_1(t), a \|$

$$\leq \frac{k^n}{1-k} \| g_0(t) - g_1(t), a \| \, for \ all \ t \in \Omega$$

as $n \to \infty$ $\| g_n(t) - g_{n+p}(t), a \to 0$, if follows that for $t \in \Omega$ $\{g_n(t)\}$ is a Cauchy sequence and hence is convergent is 2 - Hilbert space H. for $t \in \Omega$.

Let $\{g_n(t) \to g(t)$ as $n \to \infty$ ....(iii)

Since C is closed g is a function from C to C.

Existence of random fixed point: for $for \ t \in \Omega$,

$\| g(t) - T(t, g(t)), a \|^2 = \| g(t) - g_{2n+1}(t) + g_{2n+1} - T(t, g(t)), a \|^2$

$$\leq 2 \| g(t) - g_{2n+1}(t), a \|^2 + 2 \| g_{2n+1} - T(t, g(t)), a \|^2$$

[By parallelogram law $\| x + y \|^2 \leq \mathbf{2} \| x \|^2 + \mathbf{2} \| y \|^2$]

$=2 \| g(t) - g_{2n+1}(t), a \|^2 + 2 \| S(t, g_{2n+1}(t) - T(t, g(t)), a \|^2$

$$\leq \frac{2 \| g(t) - g_{2n+1}(t), a \|^2 + 2a \| g(t) - T(t, g(t)), a \|^2 \ [1 + \| g_{2n}(t) - S(t, g_{2n}(t)), \mathrm{a} \|^2}{1 + \| g_{2n}(t) - g(t) \|^2}$$

$+2\mathrm{b}[\| g_{2n}(t) - S(t, g_{2n}(t)), \mathrm{a} \|^2 + \| g(t) - T(t, g(t)), a \|^2]$

$=2 \| g(t) - g_{2n+1}(t), a \|^2 + \frac{2a \| g(t) - T(t, g(t)), a \|^2 [1 + g_{2n}(t) - g_{2n+1}(t), a \|^2]}{1 + \| g_{2n}(t) - g(t), a \|^2}$

$+2\mathrm{b}[\| g_{2n}(t) - S(t, g_{2n+1}(t)), \mathrm{a} \|^2 + \| g(t) - T(t, g(t)), a \|^2]$

As $\{g_{2n+1}(t)\}$ and $g_{2n+2}(t)$ are sub sequence of $\{\{g_n(t)\}$, as $n \to \infty$, $\{g_{2n+1}(t)\} \to g(t)$ and $\{g_{2n+2}(t)\} \to g(t)$

Therefore,

$\| g(t) - T(t, g(t)), \mathrm{a} \|^2 \leq 2 \| g(t) - g(t), a \|^2 + \frac{2a \| g(t) - T(t, g(t)), a \|^2 [1 + \| g(t) - g(t), a \|^2}{1 + \| g(t) - g(t), a \|^2}$

$+2\mathrm{b}\| g(t) - g(t), a \|^2 + \| g(t) - T(t, g(t)), a \|^2$

$$\leq 2(a+b) \ \| g(t) - T(t, g(t)), a \|^2$$

$\Rightarrow$[1-2$(a+b)$ $\| g(t) - T(t, g(t)), a \|^2 \leq 0$

$$\Rightarrow \| g(t) - T(t, g(t)), a \|^2 = 0 (as\ 2(a+b) < 1)$$

$\Rightarrow T(t, g(t)) = g(t)$ ¥ $t \in \Omega$

In an exactly similar may me can prove that for all $t \in \Omega$.

$$S(t, g(t)) = g(t) \ldots \ldots \ldots \ldots (V)$$

Again , If A: $\Omega \times C \to C$ is a continuous random operator on a non-empty subset C of a Separable 2 - Hilbert space H, then for any measurable function f: $\Omega \to C$, the function h(t)= $A(t, f(t))$ is also measurable [3].

It follows from the construction of $\{ g_n \} (by\ (i))$ and the above consideration that $\{ g_n \}$ is a sequence of measurable function . This fact along with (4) and (5) shows that g:$\Omega \to C$ is a common random fixed point of S and T.

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