Bayesian estimation of the scale parameter and survival function of weighted weibull distribution under different loss functions using r software

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Abstract: In this paper, we propose to obtain the Bayesian estimators of the scale parameter of a three parameter weighted weibull distribution, based on non-informative and informative priors using Entropy loss function and Quadratic loss function. The risk functions of these estimators have been studied. A real life example has been used to compare the performance of the estimates under different loss functions.

Keywords: Weighted Weibull distribution, Jeffery's prior and Gamma prior, loss functions.

1 Introduction

The Weibull distribution is a well known common distribution and has been a powerful probability distribution in reliability analysis, while weighted distributions are used to adjust the probabilities of the events as observed and recorded. The Weibull distribution can also be used as an alternative to Gamma and Log–normal distribution in reliability engineering and life testing. Gupta and Kundu (2009) proposed a weighted exponential distribution by using the method of Azzalini (1985). The proposed model can be used as an alternative to Gamma and Weibull distribution. Saman *et al.* (2010) proposed the weighted Weibull model based on an idea of Azzalini (1985). They studied basic properties of the distribution including moments, generating function, hazard rate function and estimation of parameters. Hamdy M. Salem (2013) worked on Inference on Stress-Strength Reliability for Weighted Weibull Distribution. S.Dey et al. (2014) discussed the properties and methods of estimation for the weighted weibull distribution. Farahani and Khorram (2014) considered the Bayesian statistical inference for the weighted exponential distribution.

A new three-parameter distribution, called the new weighted weibull distribution (NWW) has been introduced recently by Suleman Nasiru (2015). The new weighted weibull distribution has the probability density function (pdf)

$$f(x) = (1+\lambda^{\theta})\alpha\theta x^{\theta-1}e^{-(\alpha x^{\theta}+\alpha(\lambda x)^{\theta})}, \quad x>0, \alpha, \theta, \lambda > 0 \tag{1.1}$$

and the cumulative distribution function cdf of the distribution is

$$F(x) = 1-e^{-(\alpha x^{\theta}+\alpha(\lambda x)^{\theta})} \tag{1.2}$$

With one scale parameter α and two shape parameters θ and λ.

The corresponding survival function is given by

$$S(x) = 1-F(x) = e^{-(\alpha x^{\theta}+\alpha(\lambda x)^{\theta})} \tag{1.3}$$

and the hazard function is

$$h(x)=(1+\lambda^{\theta})\alpha\theta x^{\theta-1} \tag{1.4}$$

2. Prior and Loss Functions

The Bayesian inference requires appropriate choice of prior(s) for the parameter(s). From the Bayesian viewpoint, there is no clear cut way from which one can conclude that one prior is better than the other. Nevertheless, very often priors are chosen according to one's subjective knowledge and beliefs. However, if one has adequate information about the parameter(s), it is better to choose informative prior(s); otherwise, it is preferable to use non-informative prior(s). In this paper we consider both types of priors: the Jeffrey' s prior and the natural conjugate prior.

The Jeffrey's prior proposed by Jeffrey, H.(1964), is given as:

$$g_1(\alpha) \propto \frac{1}{\alpha} \quad , \quad \alpha > 0 \tag{2.1}$$

The conjugate prior in this case will be the gamma prior, and the probability density function is taken as

$$g_2(\alpha) = \frac{b^a}{\Gamma a} e^{-b\alpha}\, \alpha^{a-1} \; , \qquad a,b,\alpha > 0 \tag{2.2}$$

With the above priors, we use two different loss functions for the model (1.1).

a. Quadratic Loss Function (QLF)

The use of a quadratic loss function is common, for example when using least squares techniques. It is often more mathematically tractable than other loss functions because of the properties of variances, as well as being symmetric: an error above the target causes the same loss as the same magnitude of error below the target. If the target is t, then a quadratic loss function

$$\lambda(x) = C(t-x)$$

For some constant C, the value of the constant makes no difference to a decision, and can be ignored by setting it equal to 1.

The quadratic loss function can also be defined as

$$l(\hat{\alpha},\alpha)=\left(\frac{\alpha-\hat{\alpha}}{\alpha}\right)^2$$

b. Entropy Loss Function

In many practical situations, it appears to be more realistic to express the loss in terms of the ratio $\hat{\alpha}/\alpha$. In this case, Calabria and Pulcini (1994) point out that a useful asymmetric loss function is the entropy loss function:

$$L(\delta^p)\propto[\delta^p-p\log(\delta)-1]$$

where $\delta=\frac{\hat{\alpha}}{\alpha}$ and p>0, whose minimum occurs at $\hat{\alpha}=\alpha$. Also, the loss function $L(\delta)$ has been used in Dey et al (1987) and Dey and Liu (1992), in the original form having p =1. Thus, $L(\delta)$ can be written as

$$L(\delta)=c[\delta-\log(\delta)-1]\,;c>0.$$

3. Maximum likelihood Estimation of the scale Parameter α

Let us consider a random sample $\underline{x}=(x_1,x_2,...,x_n)$ of size n from the weighted weibull family. Then the log-likelihood function for the given sample observation is

$$L(\underline{x}/\alpha)=(1+\lambda^\theta)^n(\alpha\theta)^n\prod_{i=1}^{n}x_i^{\theta-1}e^{-(\alpha x_i^\theta+\alpha(\lambda x_i)^\theta)} \tag{3.1}$$

The log-likelihood function is

$$\ln L(\underline{x}/\alpha)=n\ln(1+\lambda^\theta)+n\ln\alpha+n\ln\theta+(\theta-1)\sum_{i=1}^{n}\ln x_i-\alpha(1+\lambda^\theta)\sum_{i=1}^{n}x_i^\theta \tag{3.2}$$

As shape parameters θ *and* λ are assumed to be known, the ML estimator of scale parameter α is obtained by solving the

$$\frac{\partial\ln L(\underline{x}/\alpha)}{\partial\alpha}=\frac{n}{\alpha}-(1+\lambda^\theta)\sum_{i=1}^{n}x_i^\theta$$

$$\hat{\alpha}_{MLE}=\frac{n}{(1+\lambda^\theta)\sum_{i=1}^{n}x_i^\theta} \tag{3.3}$$

4. Bayesian Estimation of scale parameter α and *S* under the Assumption of Jeffrey's Prior

4.1 Bayes estimator of α

Combining the prior distribution in (2.1) and the likelihood function, the posterior density of α is derived as follows:

$$p_1(\alpha/\underline{x}) \propto (1+\lambda^{\theta})^n(\alpha\theta)^n \prod_{i=1}^{n} x_i^{\theta-1} e^{-(\alpha x_i^{\theta}+\alpha(\lambda x_i)^{\theta})} \frac{1}{\alpha}$$

$$p_1(\alpha/\underline{x}) = k\alpha^{n-1} e^{-\alpha(1+\lambda^{\theta})\sum_{i=1}^{n} x_i^{\theta}}$$

$$p_1(\alpha/\underline{x}) = k\alpha^{n-1} e^{-\alpha\beta_1} \qquad (4.1)$$

where k is independent of α, $\beta_1 = (1+\lambda^{\theta})\sum_{i=1}^{n} x_i^{\theta}$ and $k^{-1} = \int_0^{\infty} \alpha^{n-1} e^{-\alpha\beta_1}\, d\alpha$

$$\Rightarrow k^{-1} = \frac{\Gamma(n)}{\beta_1^{\,n}}$$

Therefore from (4.1) we have

$$p_{1J}(\alpha/\underline{x}) = \frac{\beta_1^{\,n}}{\Gamma(n)} \alpha^{n-1} e^{-\alpha\beta_1} \qquad \alpha > 0 \qquad (4.2)$$

which is the density kernel of gamma distribution having parameters $\alpha_1 = n$ *and* $\beta_1 = (1+\lambda^{\theta})\sum_{i=1}^{n} x_i^{\theta}$. So the posterior distribution of $(\alpha/\underline{x}) \sim G(\alpha_1, \beta_1)$

4.1.1 Estimation under Quadratic loss function

By using quadratic loss function $l(\hat{\alpha},\alpha) = \left(\frac{\alpha-\hat{\alpha}}{\alpha}\right)^2$ the risk function is given by

$$R(\hat{\alpha},\alpha) = \int_0^{\infty} \left(\frac{\alpha-\hat{\alpha}}{\alpha}\right)^2 \frac{\beta_1^{\,n}}{\Gamma(n)} \alpha^{n-1} e^{-\alpha\beta_1}\, d\alpha$$

$$R(\hat{\alpha},\alpha)=\frac{\beta_1{}^n}{\Gamma(n)}\left[\int_0^\infty \alpha^{n-1}e^{-\alpha\beta_1}d\alpha \ +\hat{\alpha}^2\int_0^\infty \alpha^{n-2-1}e^{-\alpha\beta_1}d\alpha-2\hat{\alpha}\int_0^\infty \alpha^{n-1-1}e^{-\alpha\beta_1}d\alpha\right]$$

$$R(\hat{\alpha},\alpha)=\frac{\beta_1{}^n}{\Gamma(n)}\left[\frac{\Gamma(n)}{\beta_1{}^n}+\hat{\alpha}^2\frac{\Gamma(n-2)}{\beta_1{}^{n-2}}-2\hat{\alpha}\frac{\Gamma(n-1)}{\beta_1{}^{n-1}}\right]$$

Now solving $\frac{\partial R(\hat{\alpha},\alpha)}{\partial\hat{\alpha}}=0$, we obtain the Baye's estimator as

$$\hat{\alpha}_{1Q}=\frac{(n-2)}{\beta_1} \tag{4.3}$$

4.1.2 Estimation under Entropy Loss Function

By using entropy loss function $L(\delta)=c[\delta-\log\delta-1]$ for some constant c the risk function is given by

$$R(\hat{\alpha},\alpha)=\int_0^\infty c(\delta-\log(\delta)-1)\,p_1(\alpha\,|\,\underline{x})\,d\alpha$$

$$R(\hat{\alpha},\alpha)=\int_0^\infty c\left(\frac{\hat{\alpha}}{\alpha}-\log\left(\frac{\hat{\alpha}}{\alpha}\right)-1\right)\frac{\beta_1{}^n}{\Gamma(n)}\alpha^{n-1}e^{-\alpha\beta_1}d\alpha$$

$$R(\hat{\alpha},\alpha)=c\frac{\beta_1{}^n}{\Gamma(n)}\left[\hat{\alpha}\int_0^\infty \alpha^{n-1-1}e^{-\alpha\beta_1}d\alpha-\ln(\hat{\alpha})\int_0^\infty \alpha^{n-1}e^{-\alpha\beta_1}d\alpha+\int_0^\infty \ln(\alpha)\alpha^{n-1}e^{-\alpha\beta_1}d\alpha-\int_0^\infty \alpha^{n-1}e^{-\alpha\beta_1}d\alpha\right]$$

$$R(\hat{\alpha},\alpha)=c\frac{\beta_1{}^n}{\Gamma(n)}\left[\hat{\alpha}\frac{\Gamma(n-1)}{\beta_1{}^{n-1}}-\ln(\hat{\alpha})\frac{\Gamma(n)}{\beta_1{}^n}+\frac{\Gamma'(n)}{\beta_1{}^n}-\frac{\Gamma(n)}{\beta_1{}^n}\right]$$

Now solving $\frac{\partial R(\hat{\alpha},\alpha)}{\partial\hat{\alpha}}=0$, we obtain the Baye's estimator as

$$\hat{\alpha}_{1E}=\frac{(n-1)}{\beta_1} \tag{4.4}$$

4.2 Bayes estimator of $S(x)$

By using posterior distribution function (4.2), we can found the survival function such that

$$\hat{S}_{1J}(x)=\int_0^\infty \left\{e^{-(\alpha x^\theta+\alpha(\lambda x)^\theta)}\right\} p_1(\alpha/\underline{x})d\alpha$$

$$\hat{S}_{1J}(x)=\int_0^\infty \left\{e^{-\alpha(1+\lambda^\theta)x^\theta}\right\} \frac{\beta_1^{\ n}}{\Gamma(n)}\alpha^{n-1}e^{-\alpha\beta_1}d\alpha$$

$$\hat{S}_{1J}(x)=\frac{\beta_1^{\ n}}{\Gamma(n)}\int_0^\infty \alpha^{n-1}e^{-\alpha(\beta_1+(1+\lambda^\theta)x^\theta)}d\alpha$$

$$\hat{S}_{1J}(x)=\left(\frac{\beta_1}{\beta_1+(1+\lambda^\theta)x^\theta}\right)^n \tag{4.5}$$

Where β_1 has been defined above

5. Bayesian Estimation of α and *S* under the Assumption of Gamma Prior

5.1 Bayes estimator of α

Combining the prior distribution in (2.2) and the likelihood function, the posterior density of α is derived as follows:

$$p_2(\alpha/\underline{x}) \propto (1+\lambda^\theta)^n(\alpha\theta)^n\prod_{i=1}^{n} x_i^{\ \theta-1}e^{-(\alpha x_i^\theta+\alpha(\lambda x_i)^\theta)}\frac{b^a}{\Gamma a}e^{-b\alpha}\alpha^{a-1}$$

$$p_2(\alpha/\underline{x})=k\alpha^{n+a-1}e^{-\alpha\left((1+\lambda^\theta)\sum_{i=1}^{n}x_i^\theta+b\right)}$$

$$p_2(\alpha/\underline{x})=k\alpha^{n+a-1}e^{-\alpha\beta_2} \tag{5.1}$$

where k is independent of α, $\beta_2=\left((1+\lambda^\theta)\sum_{i=1}^{n}x_i^{\ \theta}+b\right)$ and $k^{-1}=\int_0^\infty \alpha^{n+a-1}e^{-\alpha\beta_2}\,d\alpha$

$$\Rightarrow k^{-1}=\frac{\Gamma(n+a)}{\beta_2^{\ n+a}}$$

Therefore from (5.1) we have

$$p_{2G}(\alpha/\underline{x}) = \frac{\beta_2^{\,n+a}}{\Gamma(n+a)}\alpha^{n+a-1}e^{-\alpha\beta_2} \qquad \alpha > 0 \tag{5.2}$$

which is the density kernel of gamma distribution having parameters $\alpha_2 = (n+a)$ and $\beta_2 = \left((1+\lambda^{\theta})\sum_{i=1}^{n} x_i^{\theta} + b\right)$. So the posterior distribution of $(\alpha/\underline{x}) \sim G(\alpha_2, \beta_2)$

5.1.1 Estimation under Quadratic loss function

By using quadratic loss function $l(\hat{\alpha},\alpha) = \left(\frac{\alpha-\hat{\alpha}}{\alpha}\right)^2$ the risk function is given by

$$R(\hat{\alpha},\alpha) = \int_0^{\infty}\left(\frac{\alpha-\hat{\alpha}}{\alpha}\right)^2 \frac{\beta_2^{\,n+a}}{\Gamma(n+a)}\alpha^{n+a-1}e^{-\alpha\beta_2}\,d\alpha$$

$$R(\hat{\alpha},\alpha) = \frac{\beta_2^{\,n+a}}{\Gamma(n+a)}\left[\int_0^{\infty}\alpha^{n+a-1}e^{-\alpha\beta_2}\,d\alpha \;+\hat{\alpha}^2\int_0^{\infty}\alpha^{n+a-2-1}e^{-\alpha\beta_2}\,d\alpha - 2\hat{\alpha}\int_0^{\infty}\alpha^{n+a-1-1}e^{-\alpha\beta_2}\,d\alpha\right]$$

$$R(\hat{\alpha},\alpha) = \frac{\beta_2^{\,n+a}}{\Gamma(n+a)}\left[\frac{\Gamma(n+a)}{\beta_2^{\,n+a}} + \hat{\alpha}^2\frac{\Gamma(n+a-2)}{\beta_2^{\,n+a-2}} - 2\hat{\alpha}\frac{\Gamma(n+a-1)}{\beta_2^{\,n+a-1}}\right]$$

Now solving $\frac{\partial R(\hat{\alpha},\alpha)}{\partial\hat{\alpha}} = 0$, we obtain the Baye's estimator as

$$\hat{\alpha}_{2Q} = \frac{(n+a-2)}{\beta_2} \tag{5.3}$$

5.1.2 Estimation under Entropy Loss Function

By using entropy loss function $L(\delta) = c[\delta - \log\delta - 1]$ for some constant c the risk function is given by

$$R(\hat{\alpha},\alpha) = \int_0^{\infty} c\left(\delta - \log(\delta) - 1\right)p_2(\alpha\,|\,\underline{x})\,d\alpha$$

$$R(\hat{\alpha},\alpha) = \int_0^{\infty} c\left(\frac{\hat{\alpha}}{\alpha} - \log\left(\frac{\hat{\alpha}}{\alpha}\right) - 1\right)\frac{\beta_2^{\,n+a}}{\Gamma(n+a)}\alpha^{n+a-1}e^{-\alpha\beta_2}\,d\alpha$$

$$R(\hat{\alpha},\alpha)=c\frac{\beta_2{}^{n+a}}{\Gamma(n+a)}\left[\hat{\alpha}\int_0^\infty \alpha^{n+a-1-1}e^{-\alpha\beta_2}d\alpha-\ln(\hat{\alpha})\int_0^\infty \alpha^{n+a-1}e^{-\alpha\beta_2}d\alpha+\int_0^\infty \ln(\alpha)\alpha^{n+a-1}e^{-\alpha\beta_2}d\alpha-\int_0^\infty \alpha^{n+a-1}e^{-\alpha\beta_2}d\alpha\right]$$

$$R(\hat{\alpha},\alpha)=c\frac{\beta_2{}^{n+a}}{\Gamma(n+a)}\left[\hat{\alpha}\frac{\Gamma(n+a-1)}{\beta_2{}^{n+a-1}}-\ln(\hat{\alpha})\frac{\Gamma(n+a)}{\beta_2{}^{n+a}}+\frac{\Gamma'(n+a)}{\beta_2{}^{n+a}}-\frac{\Gamma(n+a)}{\beta_2{}^{n+a}}\right]$$

Now solving $\frac{\partial R(\hat{\alpha},\alpha)}{\partial\hat{\alpha}}=0$, we obtain the Baye's estimator as

$$\hat{\alpha}_{2E}=\frac{(n+a-1)}{\beta_2} \tag{5.4}$$

5.2 Bayes estimator of $S(x)$

By using posterior distribution function (5.2), we can found the survival function such that

$$\hat{S}_{2G}(x)=\int_0^\infty\left\{e^{-(\alpha x^\theta+\alpha(\lambda x)^\theta)}\right\}p_2(\alpha/\underline{x})d\alpha$$

$$\hat{S}_{2G}(x)=\int_0^\infty\left\{e^{-\alpha(1+\lambda^\theta)x^\theta}\right\}\frac{\beta_2{}^{n+a}}{\Gamma(n+a)}\alpha^{n+a-1}e^{-\alpha\beta_2}\;d\alpha$$

$$\hat{S}_{2G}(x)=\frac{\beta_2{}^{n+a}}{\Gamma(n+a)}\int_0^\infty \alpha^{n+a-1}e^{-\alpha(\beta_2+(1+\lambda^\theta)x^\theta)}d\alpha$$

$$\hat{S}_{2G}(x)=\left(\frac{\beta_2}{\beta_2+(1+\lambda^\theta)x^\theta}\right)^{n+a} \tag{5.5}$$

Where β_2 has been defined above

Numerical Example: In this section, we analyze the data set from Bjerkedal (1960) represents the survival times, in days of guinea pigs injected with different doses of tubercle bacilli. The data set consists of 72 observations and are listed below: 12, 15, 22, 24 ,24, 32, 32,33, 34, 38, 38, 43,44, 48, 52, 53, 54, 54, 55, 56, 57, 58, 58, 59, 60, 60, 60, 60,61, 62, 63, 65, 65, 67, 68,70, 70, 72, 73, 75, 76,76, 81, 83, 84, 85, 87, 91, 95,96, 98, 99, 109, 110, 121, 127, 129, 131, 143, 146, 146, 175, 175, 211, 233, 258,258, 263, 297, 341, 341, 376.

Table 1: Bayes Estimates of α under Jeffreys Prior

θ	λ	MLE	QLF	ELF
0.5	1.0	0.053431	0.051947	0.052689
1.0	1.5	0.004007	0.003896	0.003952
1.5	2.0	0.000216	0.000209	0.000213

Table 2: Bayes Estimates of α under gamma prior

θ	λ	a	b	MLE	QLF	ELF
0.5	1.0	0.2	1.2	0.053431	0.052049	0.052791
1.0	1.5	0.2	1.2	0.004007	0.003907	0.003962
1.5	2.0	0.2	1.2	0.000216	0.000210	0.000213

Below are given the tables computing risks of the α estimates.

Table 3: Bayes Risk of λ under Jeffrey's Prior

θ	λ	QLF	ELF	
			C=0.5	C=1.0
0.5	1.0	0.014084	3.606524	7.213047
1.0	1.5	0.014084	4.901673	9.803346
1.5	2.0	0.014084	6.362591	12.725182

Table 4: Bayes Risk of λ under gamma prior

θ	λ	a	b	QLF	ELF	
					C=0.5	C=1.0
0.5	1.0	0.2	1.2	0.014045	3.606959	7.213918
1.0	1.5	0.2	1.2	0.014045	4.901696	9.803393
1.5	2.0	0.2	1.2	0.014045	6.362583	12.725166

Conclusion:

It is observed from table 1 to 4 the comparison of Bayes posterior risk under entropy loss function and quadratic loss function using Jeffrey's prior and gamma prior has been made through which we can conclude that Quadratic loss function is more preferable loss function and among the priors gamma prior provides least posterior risk within each loss function. Hence we conclude that gamma prior is best for weighted weibull distribution.

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