Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.6, No.9, 2016

Unit-free strongly commuting Regular Rings

***Shabanali Safari Sabet**
Corresponding author, associate professor, Department of Mathematics, Islamic Azad University Central Tehran Branch,Tehran, Iran

Sakineh Saadati
Department of Mathematics, Islamic Azad University Central Tehran Branch,Tehran, Iran

Abstract
In this paper, ring R satisfying in the condition $xy = (yx)^2a(yx)^2$ for all $x; y \in R \setminus U$ with some a in R and is called Unit-free strongly commuting Regular Rings. We observe the structure of a Unit-free strongly commuting regular ring. In this paper shown that R is a Unit-free strongly commuting regular ring, then R is an abelian ring. we also proved that R is a Unit-free strongly commuting regular ring, then $J(R)$ _ $N(R)$ and shown that R is a local ring with $J(R)^2 = 0$ and also, we proved some main properties of the Unit-free strongly commuting regular rings and we give a necessary and su cient condition that a ring is Unit-free strongly commuting regular.
Keywords: *regular rings; Strongly commuting regular rings; Unit free commuting regular rings; reduced rings:*

1 Introduction

Let R be a ring. In 1936, Von Neumann de ned that an element x 2 R is regular if $x = xyx$, for some y 2 R, the ring R is regular if each element of R is regular. some properties of regular rings have been studied by Goodearl [4] and sher and snider [6]. A ring R is called -regular if for each x 2 R exists a positive integer n, depending on x, and y 2 R such that $x^n = x^nyx^n$, and is called strongly - regular if for each x 2 R exists a positive integer n, and y 2 R such that $x^n = x^{n+1}y$.

The strongly - regular has roles in ring theory as we see in [1], [8]. In 2004 Safari Sabet and Yamini [2] de ned that a ring is called commuting regular if for each $x; y$ 2 R there exists a 2 R such that $xy = yxayx$, see [2]. Then some results on commuting regular rings have been studied in [3, 5, 7, 8]. R is called Unit-free commuting regular ring if for any $x; y$ 2 R n U there exis.

a 2 R such that $xy = yxayx$. $N = N(R)$ the set of all nilpotent elements in R. $J = J(R)$, Jacobson radical of R and $Id(R)$, is the set of all idempotent elements in R.

we extend Unit-free commuting regular rings and introduce the concept of Unit-free strongly commuting Regular Rings as following:

2 Basic Properties of Unit-free strongly commuting Regular Rings

De nition 1. *R is called a Unit-free strongly commuting Regular Ring if for each x; y 2 R n U there exists a 2 R such that* $xy = (yx)^2 a(yx)^2$.

Proposition 1. *Let R be a Unit-free strongly commuting regular ring, then R is an abelian ring.*

Proof. Let e be an arbitary nontrivial idempotent in R and x 2 R. □

case(1) If x 2 R n U , then there exist a; b 2 R such that $ex = (xe)^2 a(xe)^2$, $xe = (ex)^2 b(ex)^2$ from these we have $exe = (xe)^2 a(xe)^2$, $exe = (ex)^2 b(ex)^2$ which implies that $ex = xe$ for every x 2 R n U .

case(2) If x 2 U the it not hard to see that $(ex \quad exe)^2 = 0$, so that $ex \quad exe$ 2= U .

consider the elements e and $ex \quad exe$.

As R is Unit-free strongly commuting Regular Ring, there exists a 2 R such that

$$e \qquad exe)\ e)^2\ a$$
$$(ex \quad exe) = ((ex \quad ((ex \qquad exe)\ e)^2 = 0:$$

Thus we conclude that $ex \quad exe = 0$ and so $ex = exe$. similarly $(xe \quad exe)^2 = 0$ and so there exists an element b 2 R such that

$$exe))^2\ b\ (e$$
$$(xe \quad exe)\ e = (e\ (xe \quad (xe \qquad exe))^2 = 0:$$

Thus $xe = exe$ and hence $ex = xe$, as required.

Lemma 1. *Let x be an element in the Jacobson radical* $J(R)$. *If for* y 2 R, $yx = y$ *that* $y = 0$

Proof. There exists t 2 R such that $x \quad + t \quad tx = x + t \quad xt = 0$
therefore, $0 = y(x + t \quad xt) = yx + yt \quad yxt = y + yt \quad yt = y$. □

Proposition 2. *Let R be a Unit-free strongly commuting regular ring, then* $J(R)$ _ $N(R)$.

Proof. Let x 2 $J(R)$ be an arbitary element.

since R is Unit-free commuting regular, it is for some a 2 R, $xy = (yx)^2 a (yx)^2$. if $x = y$ then $x^2 = x^4ax^4$, but in view of Lemma (1), $x^2 = 0$.

Therefore $J(R)$ _ $N(R)$ □

Lemma 2. *Let R be a local ring with* $J(R)^2 = 0$*. Then R is Unit-free strongly commuting regular ring.*

Proof. Let R be a local ring and $x; y$ 2= U .

Then xR _ $J(R)$ and yR _ $J(R)$, implies $x; y$ 2 $J(R)$.

As $J(R)^2 = 0$, we get $xy = 0 = yx$. and so $xy = (yx)^2c(yx)^2 = 0$ for every c 2 R, which implies that R is Unit-free strongly commuting regular ring. □

Proposition 3. *Let R be a Unit-free strongly commuting regular ring, then* $N = N(R)$*, is a nilpotent ideal. In fact* $RN = NR = N^2 = 0$

Proof. Let x 2 N then, there exists a 2 R such that $x^2 = x^4ax^4$. clearly, $x^4ax^2; x^2ax^4$ are idempotent elements because $(x^4ax^2)^2 = x^4ax^2x^4ax^2 = x^4ax^2$, $(x^2ax^4)^2 = x^2ax^4x^2ax^4 = x^2ax^4$ therefore, it is in the center of R. This shows that

$$\begin{aligned} x^2 &= x^4ax^4 = x^6ax^2 \\ &= x^{10}ax^2ax^2 \\ &= x^{14}(ax^2)^3a \\ &= \_\,\_\,\_ \end{aligned}$$

but x is a nilpotent element, so $x^2 = 0$.

Now suppose that y 2 R n U is an arbitary element.
Unit-free strongly commuting regularity of R implies
that;

$$\begin{aligned} xy &= (yx)^2a(yx)^2 \\ &= (yx)yxa(yx)^2 \end{aligned}$$

$$= \quad yx^2aybxay(yx)^2 = 0$$

For some a and b in R, this proves that $xy = 0$. Therefore $NR = 0$ and particular $N^2 = 0$. similarly, we can show that $RN = 0$. To complete the proof it su ces to note that for every $x; y$ 2 N , $(x + y)^2 = x^2 + xy + yx + y^2 = 0$. □

Proposition 4. *Let R be a Unit-free strongly commuting regular ring, if R is semiprime, that R is reduced.*

Proof. It is obvious by proposition (3) that $NRN = 0$,therefore $N = f0g$ since R is semiprime, so R is Reduced. □

Corollary 1. *Let R be a Unit-free commuting regular ring, if R is semiprime then it is semiprimitive.*

Proof. by propositions (4),(2) □

Corollary 2. *Let R be a Unit-free commuting regular ring, if R is reduced then it is semiprimitive.*

Proof. by propositions (4), corollary (1) □

Lemma 3. *suppose that R a domain Unit-free strongly commuting regular, then R is a diusion ring.*

Proof. since R is a Unit-free strongly commuting regular, therefore for each $0 \neq x$ 2 R. there exist a 2 R such that $x^2 = x^4ax^4$.

But $e = x^2ax^4$ is an idempotent element, this implise that $x^2 = x^2e$ and R being a domain we get $x = xe$.

For every y 2 R, $xy = xey$. so $y = ey$.
That is $e = 1_R$. but $x^2ax^4 = e$ implies that every x has an inverse. thus, R is a division ring. □

References

[1] AZumaya , *Strongly-regular ring*, J.Fac. Hokkaido Univ. 13(1954) 34-39.

[2] A. H. Yamini, Sh. A. Safari. Sabet, *Commuting regular rings*, Ant. J. Appl. Math. 14(4),(2004) 357-364.

[3] Ch. Huh, N. K. Kim, Y. Lee, *Example of strongly -regular rings*, Pure Appl. Algebra.

189(2004) 195-210.

[4] Goodearl, K. R, *Uon Neuman reglars, Krieger Pubishing Company*, 2nd edition (1991) .

[5] H. Doosti, L. Pour faraj, *On mimimal ideals of commuting regular rings and semigroups*, Int. J, Appl. Math. 19(2006) 201-216.

[6] Joe W. Fisher, Robertl. Sinder, *On the von Neumann reglarity of rings with regular prime factor rings*, Paci c J. Math. 54, No. 1, (1974) 135-144.

[7] M. Azadi. H. Doosti, L. Pour faraj, *Certain rings and semigroups examining the regularity property*, J. Math. state. Allied Fields. 2(1)(2008) 6pp.

[8] T. Y. Lam, *A rst course in noncommutative rings* , Graduate Texts in Mathematics. No. 137, Springer-Verlay. (1991) .

[9] S. Saadati, Sh.A. Safari Sabet, *Unit-Free Commuting Regular Rings*, Accepted in Turkish Journal Mathematics,June 2016.