Special Second Order Non Symmetric Fitted Method for Singular Perturbation Problems

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Abstract

In this paper, we present a special second order non symmetric fitted difference method for solving singular perturbed two point boundary value problems having boundary layer at one end. We introduce a fitting factor in the special second order non symmetric finite difference scheme which takes care of the rapid changes occur that in the boundary layer. The value of this fitting factor is obtained from the theory of singular perturbations. The discrete invariant imbedding algorithm is used to solve the tridiagonal system obtained by the method. We discuss the existence and uniqueness of the discrete problem along with stability estimates and the convergence of the method. We present the maximum absolute errors in numerical results to illustrate the proposed method.

Keywords: Singularly perturbed two-point boundary value problem, Boundary layer, Fitting factor, Maximum absolute error

1. Introduction

During the last few years much progress has been made in the theory and in the computer implementation of the numerical treatment of singular perturbation problems. Typically, these problems arise very frequently in fluid mechanics, fluid dynamics, chemical reactor theory, elasticity, aero dynamics and other domains of the great world of fluid motion. The solution of this type of problem has a narrow region in which the solution changes rapidly and the outside solution changes smoothly. However, the area of singular perturbations is a field of increasing interest to applied mathematicians. Much progress has been made recently in developing finite element methods for solving singular perturbation problems. This type of problem was solved by Bellman (1964), Bender and Orszag (1978), Eckhaus (1973), Kevorkian and Cole (1981), Nayfeh (1973), O'Malley (1974), Van Dyke (1974), and numerically by Ascher and Weis (1984), Kadalbajoo, Reddy (1989) and Kadalbajoo and Patidar (2003), Lin and Su (1989), Roos (1986), Vulanovic (1991). It is well known that standard discretization methods for solving singular perturbation problems are unstable and fail to give accurate results when the perturbation parameter ε is small. Therefore, it is important to develop suitable numerical methods for these problems, whose accuracy does not depend on parameter ε as presented in Doolan *et al*. (1980). The fitted technique is one such tool to reach these goals in an optimum way. There are two possibilities to obtain small truncation error inside the boundary layer(s). The first is to choose a fine mesh there, whereas the second one is to choose a difference formula reflecting the behaviour of the solution(s) inside the boundary layer(s). Present work deals with the second approach. In this paper, we have presented a special second order non symmetric fitted difference method for solving singularly perturbed problems. We introduce a fitting factor in a special second order non symmetric finite difference scheme which takes care of the rapid changes occur that in the boundary layer. This fitting factor is obtained from the theory of singular perturbations. The discrete invariant imbedding algorithm is used to solve the tridiagonal system. The existence and uniqueness of the discrete problem along with stability estimates are discussed. We have discussed the convergence of the method. Maximum absolute errors in numerical results are presented to illustrate the proposed method.

2. Description of the method

2.1 Left-End Boundary Layer Problems

Consider a linearly singularly perturbed two point boundary value problem of the form:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \quad x \in [0,1] \tag{1}$$

with the boundary conditions $y(0) = \alpha$ (2a)

and $y(1) = \beta$ (2b)

where ε is a small positive parameter ($0 < \varepsilon << 1$) and α, β are given constants. We assume that *a(x), b(x)* and *f(x)* are sufficiently smooth functions and such that (1) with (2) has a unique solution in [0, 1]. Further more, we assume that $b(x) \leq 0$, $a(x) \geq M > 0$ throughout the interval [0, 1], where M is some positive constant. This assumption implies that boundary layer exists in the neighborhood of $x = 0$.
From the theory of singular perturbation s it is known that the solution of (1) - (2) is of the form

$$y(x) = y_0(x) + \frac{a(0)}{a(x)}(\alpha - y_0(0))e^{\int_0^x \left(\frac{a(x)}{\varepsilon} - \frac{b(x)}{a(x)}\right)dx} + O(\varepsilon) \tag{3}$$

where $y_0(x)$ is the solution of $a(x)y_0'(x) + b(x)y_0(x) = f(x)$, $y_0(1) = \beta$ (4)

By taking Taylor's series expansion for a(x) and b(x) about the point '0' and restricting to their first terms, (3) becomes,

$$y(x) = y_0(x) + (\alpha - y_0(0))e^{-\left(\frac{a(0)}{\varepsilon} - \frac{b(0)}{a(0)}\right)x} + O(\varepsilon) \tag{5}$$

Now we divide the interval [0, 1] into N equal parts with constant mesh length h. Let 0= $x_1, x_2, \ldots\ldots. x_N$ =1 be the mesh points. Then we have $x_i = ih : i = 0, 1, 2, \ldots, N$.

From (5), we have $y(ih) = y_0(ih) + (\alpha - y_0(0))e^{-\left(\frac{a(0)}{\varepsilon} - \frac{b(0)}{a(0)}\right)ih} + O(\varepsilon)$

therefore

$$\lim_{h\to 0} y(ih) = y_0(0) + (\alpha - y_0(0))e^{-\left(\frac{a^2(0) - \varepsilon b(0)}{a(0)}\right)i\rho} \tag{6}$$

where $\rho = \frac{h}{\varepsilon}$.

From finite differences, we have

$$y_{i-1} - 2y_i + y_{i+1} = h^2\left(y_i'' + \frac{h^2}{12}y_i^{(4)}\right) + O(h^6) \tag{7}$$

$$y_{i-1}'' - 2y_i'' + y_{i+1}'' = h^2 y_i^{(4)} + \frac{h^4}{12}y_i^{(6)} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Substituting $h^2 y_i^{(4)}$ from the above equation in (7), we have

$$y_{i-1} - 2y_i + y_{i+1} = \frac{h^2}{12}\left(y_{i-1}'' + 10y_i'' + y_{i+1}''\right) + O(h^6) \tag{8}$$

Now from the equation (1), we have

$$\varepsilon y_{i+1}'' = -a_{i+1}y_{i+1}'^{*} - b_{i+1}y_{i+1} + f_{i+1}$$

$$\varepsilon y_i'' = -a_i y_i' - b_i y_i + f_i \tag{9}$$

$$\varepsilon y_{i-1}'' = -a_{i-1}y_{i-1}'^{*} - b_{i-1}y_{i-1} + f_{i-1}$$

where we approximate $y_{i+1}'^{*}$ and $y_{i-1}'^{*}$ using non symmetric finite differences

$$y'^{*}_{i+1} = \frac{y_{i-1} - 4y_i + 3y_{i+1}}{2h} - hy''_i + O(h^2)$$

$$y'_i = \frac{y_{i+1} - y_{i-1}}{2h} + O(h^2) \tag{10}$$

$$y'^{*}_{i-1} = \frac{-3y_{i-1} + 4y_i - y_{i+1}}{2h} + hy''_i + O(h^2)$$

Substituting (9) and (10) in (8) and simplifying we get

$$\left(\varepsilon + \frac{ha_{i-1}}{12} - \frac{ha_{i+1}}{12}\right)\left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}\right) + \frac{a_{i-1}}{24h}\left(-3y_{i-1} + 4y_i - y_{i+1}\right) + \frac{10a_i}{24h}\left(y_{i+1} - y_{i-1}\right)$$

$$+ \frac{a_{i+1}}{24h}\left(y_{i-1} - 4y_i + 3y_{i+1}\right) + \frac{b_{i-1}}{12}y_{i-1} + \frac{10b_i}{12}y_i + \frac{b_{i+1}}{12}y_{i+1} = \frac{\left(f_{i-1} + 10f_i + f_{i+1}\right)}{12}$$

Now introducing the fitting factor $\sigma(\rho)$ in the above scheme

$$\left(\sigma(\rho)\varepsilon + \frac{ha_{i-1}}{12} - \frac{ha_{i+1}}{12}\right)\left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}\right) + \frac{a_{i-1}}{24h}\left(-3y_{i-1} + 4y_i - y_{i+1}\right) + \frac{10a_i}{24h}\left(y_{i+1} - y_{i-1}\right)$$

$$+ \frac{a_{i+1}}{24h}\left(y_{i-1} - 4y_i + 3y_{i+1}\right) + \frac{b_{i-1}}{12}y_{i-1} + \frac{10b_i}{12}y_i + \frac{b_{i+1}}{12}y_{i+1} = \frac{\left(f_{i-1} + 10f_i + f_{i+1}\right)}{12} \tag{11}$$

The fitting factor $\sigma(\rho)$ is to be determined in such a way that the solution of (11) converges uniformly to the solution of (1)-(2). Multiplying (11) by h and taking the limit as $h \to 0$, and by using equation (6), we get

$$\sigma = \rho\frac{a(0)}{2}\coth\left(\frac{\left(a^2(0) - \varepsilon b(0)\right)\rho}{2a(0)}\right) \tag{12}$$

which is a constant fitting factor.

The tridiagonal system of the equation (11) is given by

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \text{ for } i = 1,2,\ldots,N\text{-}1 \tag{13}$$

where

$$E_j = \frac{1}{h^2}\left(\varepsilon\sigma + \frac{ha_{i-1}}{12} - \frac{ha_{i+1}}{12}\right) - \frac{3a_{i-1}}{24h} + \frac{b_{i-1}}{12} - \frac{10a_i}{24h} + \frac{a_{i+1}}{24h}$$

$$F_j = \frac{2}{h^2}\left(\varepsilon\sigma + \frac{ha_{i-1}}{12} - \frac{ha_{i+1}}{12}\right) - \frac{4a_{i-1}}{24h} - \frac{10b_i}{12} + \frac{4a_{i+1}}{24h}$$

$$G_j = \frac{1}{h^2}\left(\varepsilon\sigma + \frac{ha_{i-1}}{12} - \frac{ha_{i+1}}{12}\right) - \frac{a_{i-1}}{24h} + \frac{b_{i+1}}{12} + \frac{10a_i}{24h} + \frac{3a_{i+1}}{24h}$$

$$H_j = \frac{1}{12}\left(f_{i-1} + 10f_i + f_{i+1}\right)$$

where σ is given by (12). We solve this tridiagonal system by the discrete invariant imbedding algorithm.

2.2 Right-end boundary layer problem

Finally, we discuss our method for singularly perturbed two point boundary value problems with right-end boundary layer of the underlying interval. To be specific, we consider a class of singular perturbation problem of the form:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x),\ x \in [0, 1] \tag{14}$$

with $y(0)=\alpha$ (15a)

and $y(1)= \beta$ (15b)

where ε is a small positive parameter (0 < ε <<1) and α, β are known constants. We assume that $a(x)$, $b(x)$ and $f(x)$ are sufficiently smooth functions in [0, 1]. We assume that $a(x) \le M < 0$ throughout the interval [0, 1], where M is some negative constant. This assumption merely implies that the boundary layer will be in the neighborhood of x =1. From the theory of singular perturbations the solution of (14)-(15) is of the form

$$y(x)=y_0(x)+\frac{a(1)}{a(x)}(\beta-y_0(1))e^{\int_x^1\left(\frac{a(x)}{\varepsilon}-\frac{b(x)}{a(x)}\right)dx}+O(\varepsilon) \tag{16}$$

where $y_0(x)$ is the solution of

$$a(x)y_0'(x)+b(x)y_0(x)=f(x),\ y_0(0)=\alpha\ . \tag{17}$$

By taking the Taylor's series expansion for $a(x)$ and $b(x)$ about the point '1' and restricting to their first terms, (16) becomes $y(x)=y_0(x)+(\beta-y_0(1))e^{\left(\frac{a(1)}{\varepsilon}-\frac{b(1)}{a(1)}\right)(1-x)}+O(\varepsilon)$ (18)

Now we divide the interval [0, 1] into N equal parts with constant mesh length h.

Let $0=x_1,x_2,.......x_N=1$ be the mesh points. Then we have $x_i=ih:i=0,1,2,\ldots,\mathrm{N}$.

From (18), we have $y(ih)=y_0(ih)+(\beta-y_0(1))e^{\left(\frac{a(1)}{\varepsilon}-\frac{b(1)}{a(1)}\right)(1-ih)}+O(\varepsilon)$.

Therefore $\lim_{h\to 0} y(ih)=y_0(0)+(\beta-y_0(1))e^{\left(\frac{a^2(1)-\varepsilon b(1)}{a(1)}\right)\left(\frac{1}{\varepsilon}-i\rho\right)}$ (19)

where $\rho=\frac{h}{\varepsilon}$.

Proceeding as in the left-end boundary layer problem, we get the fitting factor as $\sigma=\rho\frac{a(0)}{2}\coth\left(\frac{\left(a^2(1)-\varepsilon b(1)\right)\rho}{2a(1)}\right)$

which is a constant fitting factor. The tridiagonal system of the equation (11) is given by (13) where E_i,F_i,G_i and H_i are same as given in left-end boundary layer.

3. Stability and convergence analysis

Theorem 1. Under the assumptions $\varepsilon>0$, $a(x)\ge M>0$ and $b(x)<0$, $\forall x\in[0,1]$, the solution to the system of the difference equations (13), together with the given boundary conditions exists, is unique and satisfies $\|y\|_{h,\infty}\le 2M^{-1}\|f\|_{h,\infty}+\left(|\alpha|+|\beta|\right)$

where $\|\cdot\|_{h,\infty}$ is the discrete $l_\infty-norm$, given by $\|x\|_{h,\infty}=\max_{0\le i\le N}\{|x_i|\}$.

Proof. Let $L_h(.)$ denote the difference operator on left hand side of Eq. (13) and w_i be any mesh function satisfying $L_h(w_i)=f_i$

By rearranging the difference scheme (13) and using non-negativity of the coefficients

E_i, F_i and G_i, we obtain $F_i\,|w_i|\le|H_i|+E_i\,|w_{i-1}|+G_i\,|w_{i+1}|$

Now using the assumption $\varepsilon>0$ and $a_i\ge M$, the definition of l_∞-norm and manipulating the above inequality, we get

$$\sigma\,\varepsilon\frac{\left(|w_{i+1}|-2|w_i|+|w_{i-1}|\right)}{h^2}+\frac{M}{12}\left(\frac{-|w_{i+1}|+4|w_i|-3|w_{i-1}|}{2h}\right)+\frac{b_{i-1}}{12}|w_{i-1}|+\frac{10b_i}{12}|w_i|+\frac{b_{i+1}}{12}|w_{i+1}|$$
$$+\frac{10M}{12}\frac{\left(|w_{i+1}|-|w_{i-1}|\right)}{2h}+\frac{M}{12}\left(\frac{3|w_{i+1}|-4|w_i|+|w_{i-1}|}{2h}\right)+|H_i|\ge 0 \tag{20}$$

To prove the uniqueness and existence, let $\{u_i\},\{v_i\}$ be two sets of solution of the difference equation (13) satisfying boundary conditions. Then $w_i=u_i-v_i$ satisfies $L_h(w_i)=f_i$ where $f_i=0$ and $w_0=w_N=0$.

Summing (20) over $i=1,2,\ldots\ldots,N-1$, we obtain

$$
\begin{aligned}
&-\sigma\varepsilon\frac{|w_1|}{h^2}-\sigma\varepsilon\frac{|w_{N-1}|}{h^2}+\|a\|_{h,\infty}\frac{|w_1|}{24h}+\|a\|_{h,\infty}\frac{3|w_{N-1}|}{24h}+\frac{1}{12}\sum_{i=1}^{N-1}b_{i-1}|w_{i-1}|+\\
&\frac{10}{12}\sum_{i=1}^{N-1}b_i|w_i|+\frac{1}{12}\sum_{i=1}^{N-1}b_{i+1}|w_{i+1}|-\frac{10M}{24h}|w_1|-\frac{10M}{24h}|w_{N-2}|-\frac{3M}{24h}|w_1|-\frac{M}{24h}|w_{N-1}|+\sum_{i=1}^{N-1}|H_i|\ge 0
\end{aligned}
\tag{21}
$$

Since $\varepsilon>0,\ \|a\|_{h,\infty}\ge 0, b_i<0$ and $|w_i|\ge 0\ \ \forall i, i=1,2,\ldots\ldots,N-1,$ therefore for inequality (21) to hold, we must have $w_i=0\ \ \forall i,\ i=1,2,\ldots..N-1$.

This implies the uniqueness of the solution of the tridiagonal system of difference equations (13). For linear equations, the existence is implied by uniqueness. Now to establish the estimate, let $w_i=y_i-l_i$,
where y_i satisfies difference equations (13), the boundary conditions and $l_i=(1-ih)\alpha+(ih)\beta,$
then $w_0=w_N=0,$ and $w_i,\ i=1,2,\ldots N-1$
$L_h(w_i)=f_i$
Now let $|w_n|=\|w\|_{h,\infty}\ge|w_i|, i=0,1,\ldots\ldots,N.$
Then summing (20) from $i=n$ to N-1 and using the assumption on *a(x),* which gives

$$
\begin{aligned}
&-\sigma\varepsilon\frac{(|w_n|-|w_{n-1}|)}{h^2}-\sigma\varepsilon\frac{|w_{N-1}|}{h^2}+\frac{M}{12}\left(\frac{3|w_{N-1}|+|w_n|-3|w_{n-1}|}{2h}\right)+\frac{1}{12}\sum_{i=n}^{N-1}b_{i-1}|w_{i-1}|+\\
&\frac{10}{12}\sum_{i=n}^{N-1}b_i|w_i|+\frac{1}{12}\sum_{i=n}^{N-1}b_{i+1}|w_{i+1}|+\frac{10M}{12}\left(\frac{|w_{N-1}|-|w_n|-|w_{n-1}|}{2h}\right)\\
&+\frac{M}{12}\left(\frac{-|w_{N-1}|-3|w_n|+|w_{n-1}|}{2h}\right)+\sum_{i=n}^{N-1}|H_i|\ge 0
\end{aligned}
\tag{22}
$$

Inequality (22), together with the condition on *b(x)* implies that

$$\frac{M}{2}|w_n|\le h\sum_{i=n}^{N-1}|f_i|\le h\sum_{i=0}^{N}|f_i|\le\|f\|_{h,\infty},$$

i.e., we have

$$|w_n|\le 2M^{-1}\|f\|_{h,\infty} \tag{23}$$

Also, we have $y_i=w_i+l_i$

$$\|y\|_{h,\infty}=\max_{0\le i\le N}\{|y_i|\}\le\|w\|_{h,\infty}+\|l\|_{h,\infty}\le|w_n|+\|l\|_{h,\infty}. \tag{24}$$

Now to complete the estimate, we have to find out the bound on l_i

$\|l\|_{h,\infty}=\max_{0\le i\le N}\{|l_i|\}\le\max_{0\le i\le N}\{|(1-ih)||\alpha|+|ih||\beta|\}\le\max_{0\le i\le N}\{(1-ih)|\alpha|+(ih)|\beta|\},$ i.e., we have

$$\|\,l\,\|_{h,\infty}\le|\alpha|+|\beta|. \tag{25}$$

From Eqs. (23) – (25), we obtain the estimate $\|y\|_{h,\infty}\le 2M^{-1}\|f\|_{h,\infty}+(|\alpha|+|\beta|)$.

This theorem implies that the solution to the system of the difference equations (18) are uniformly bounded, independent of mesh size h and the perturbation parameter ε . Thus the scheme is stable for all step sizes.

Corollary 1. Under the conditions for theorem 1, the error $e_i=y(x_i)-y_i$ between the solution *y(x)* of the continues problem and the solution y_i of the discretized problem, with boundary conditions, satisfies the estimate

$\|e\|_{h,\infty}\le 2M^{-1}\|\tau\|_{h,\infty}$, where

$$|\tau_i|\le\max_{x_{i-1}\le x\le x_{i+1}}\left\{\frac{\sigma h^2\varepsilon^2}{12}\left|y^{(4)}(x)\right|\right\}+\max_{x_{i-1}\le x\le x_{i+1}}\left\{\frac{ah^2}{72}\left|y^{(3)}(x)\right|\right\}+\max_{x_{i-1}\le x\le x_{i+1}}\left\{\frac{10ah^2}{72}\left|y^{(3)}(x)\right|\right\}+\max_{x_{i-1}\le x\le x_{i+1}}\left\{\frac{ah^2}{72}\left|y^{(3)}(x)\right|\right\}$$

Proof. Truncation error τ_i in the difference scheme is given by

$$\tau_i = \sigma\varepsilon\left\{\left(\frac{y_{i+1}-y_i+y_{i-1}}{h^2}\right)-y_i''\right\}+\frac{a_{i-1}}{12}\left(\frac{-3y_{i-1}+4y_i-y_{i+1}}{2h}+hy_i''-y_{i-1}'\right)+\frac{10a_i}{12}\left(\frac{y_{i+1}-y_{i-1}}{2h}-y_i'\right)$$

$$+\frac{a_{i+1}}{12}\left(\frac{y_{i-1}-4y_i+3y_{i+1}}{2h}-hy_i''-y_{i+1}'\right)$$

$$|\tau_i| \le \max_{x_{i-1}\le x\le x_{i+1}}\left\{\frac{\sigma h^2\varepsilon^2}{12}\left|y^{(4)}(x)\right|\right\}+\max_{x_{i-1}\le x\le x_{i+1}}\left\{\frac{ah^2}{72}\left|y^{(3)}(x)\right|\right\}+\max_{x_{i-1}\le x\le x_{i+1}}\left\{\frac{10ah^2}{72}\left|y^{(3)}(x)\right|\right\}+\max_{x_{i-1}\le x\le x_{i+1}}\left\{\frac{ah^2}{72}\left|y^{(3)}(x)\right|\right\} \quad (26)$$

One can easily show that the error e_i, satisfies

$L_h(e(x_i)) = L_h(y(x_i)) - L_h(y_i) = \tau_i,\ i = 1,2,\dots,N-1$

And $e_0 = e_N = 0$.

Then Theorem 1 implies that

$$\|e\|_{h,\infty} \le 2M^{-1}\|\tau\|_{h,\infty}. \quad (27)$$

The estimate (26) establishes the convergence of the difference scheme for the fixed values of the parameter ε.

Theorem 2. Under the assumptions $\varepsilon > 0$, $a(x) \le M < 0$ and $b(x) < 0, \forall x \in [0,1]$, the solution to the system of the difference equations (13), together with the given boundary conditions exists, is unique and satisfies

$$\|y\|_{h,\infty} \le 2M^{-1}\|f\|_{h,\infty} + (|\alpha| + |\beta|).$$

The proof of estimate can be done on similar lines as we did in theorem 1.

4. Numerical Examples

To demonstrate the applicability of the method we have applied the method to three linear singular perturbation problems with left-end boundary layer and two linear singular perturbation problems with right-end boundary layer. These examples have been chosen because they have been widely discussed in literature and because approximate solutions are available for comparison. The numerical solutions are compared with the exact solutions and maximum absolute errors with and without fitting factor are presented to support the given method.

Example 1. Consider the following homogeneous singular perturbation problem from Bender and Orszag [4] $\varepsilon y''(x) + y'(x) - y(x) = 0$, $x\in[0,1]$ with $y(0) = 1$ and $y(1) = 1$.

Clearly this problem has a boundary layer at x = 0 i.e., at the left end of the underlying interval.

The exact solution is given by

$y(x) = \dfrac{[(e^{m_2}-1)e^{m_1 x} + (1-e^{m_1})e^{m_2 x}]}{[e^{m_2}-e^{m_1}]}$ where $m_1 = (-1+\sqrt{1+4\varepsilon})/(2\varepsilon)$ and $m_2 = (-1-\sqrt{1+4\varepsilon})/(2\varepsilon)$

The maximum absolute errors are presented in table 1 for different values of ε with and without fitting factor.

Example 2. Now consider the following non-homogeneous singular perturbation problem from fluid dynamics for fluid of small viscosity $\varepsilon y''(x) + y'(x) = 1 + 2x$; $x\in[0,1]$ with $y(0)=0$ and $y(1)=1$. The exact solution is given by $y(x) = x(x+1-2\varepsilon) + \dfrac{(2\varepsilon-1)(1-e^{-x/\varepsilon})}{(1-e^{-1/\varepsilon})}$. The maximum absolute errors are presented in table 2 for different values of ε with and without fitting factor.

Example 3. Finally we consider the following variable coefficient singular perturbation problem from Kevorkian and Cole [3] $\varepsilon y'' + \left(1-\dfrac{x}{2}\right)y' - \dfrac{1}{2}y = 0$, $x\in[0,1]$ with y(0)=0 and y(1)=1.

We have chosen to use uniformly valid approximation (which is obtained by the method given by Nayfeh [2] as our 'exact' solution: $y(x) = \dfrac{1}{2-x} - \dfrac{1}{2}e^{-\left(\frac{x-x^2/4}{\varepsilon}\right)}$

The maximum absolute errors are presented in table 3 for different values of ε with and without fitting factor.

Example 4. Consider the following singular perturbation problem $\varepsilon y''(x) - y'(x) = 0$; $x \in [0,1]$ with $y(0) = 1$ and $y(1) = 0$. Clearly, this problem has a boundary layer at x=1. i.e., at the right end of the underlying interval.

The exact solution is given by $y(x) = \dfrac{\left(e^{(x-1)/\varepsilon} - 1\right)}{\left(e^{-1/\varepsilon} - 1\right)}$. The maximum absolute errors are presented in table 4 for different values of ε with and without fitting factor.

Example 5. Now we consider the following singular perturbation problem

$\varepsilon y''(x) - y'(x) - (1+\varepsilon)y(x) = 0$, $x \in [0,1]$ with $y(0) = 1+\exp(-(1+\varepsilon)/\varepsilon)$ and $y(1) = 1+1/e$.

The exact solution is given by $y(x) = \exp\left(1+\varepsilon\right)\left(x-1\right)/\varepsilon + \exp(-x)$

The maximum absolute errors are presented in table 5 for different values of ε with and without fitting factor.

5. Discussions and conclusions

We have described a special second order fitted difference method for solving singularly perturbed two point boundary value problems. We have introduced a fitting factor in a special second order finite difference scheme which takes care of the rapid changes occur that in the boundary layer. This fitting factor is obtained from the theory of singular perturbations. Thomas algorithm is used to solve the tridiagonal system of the fitted method. The existence and uniqueness of the discrete problem along with stability estimates are discussed. We have presented maximum absolute errors for the standard examples chosen from the literature and also presented maximum absolute errors for the some of the examples with and without fitting factor to show the efficiency of the method when $\varepsilon << h$. The computational rate of convergence is also obtained by using the double mesh principle [4] defined below.

Let $Z_h = \max_j \left| y_j^h - y_j^{h/2} \right|$, $j = 0, 1, 2, \ldots, N-1$ where y_j^h is the computed solution on the mesh $\{x_j\}_0^N$ at the nodal point x_j where $x_j = x_{j-1} + h,\ j = 1, 2, \ldots N$ and $y_j^{h/2}$ is the computed solution at the nodal point x_j on the mesh $\{x_j\}_0^{2N}$ where $x_j = x_{j-1} + h/2,\ j = 1(1)2N$. In the same way we can define $Z_{h/2}$ by replacing h by $h/2$ and N by $2N$ i.e., $Z_{h/2} = \max_j \left| y_j^{h/2} - y_j^{h/4} \right|$, $j = 0, 1, 2, \ldots, 2N-1$.

The computed order of convergence is defined as $\text{Order} = \dfrac{\log Z_h - \log Z_{h/2}}{\log(2)}$. We have taken $h = 2^{-3}$ for finding the computed order of convergence and results are shown in Table 6.

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Table 1. The maximum absolute errors in solution of example 1

h	$\varepsilon=10^{-3}$		$\varepsilon=10^{-5}$	
	with fitting factor	without fitting factor	with fitting factor	without fitting factor
1/8	2.0222e-002	7.8221e-001	2.0572e-002	8.8255e-001
1/16	1.0633e-002	6.5875e-001	1.0993e-002	8.9995e-001
1/32	5.2691e-003	5.4804e-001	5.6320e-003	8.9962e-001
1/64	2.4803e-003	4.8335e-001	2.8438e-003	8.6571e-001
1/128	1.0641e-003	3.7260e-001	1.4269e-003	7.6293e-001

Table 2. The maximum absolute errors in solution of example 2

h	$\varepsilon=10^{-3}$		$\varepsilon=10^{-5}$	
	with fitting factor	without fitting factor	with fitting factor	without fitting factor
1/8	1.0762e-001	15.4146	1.0936e-001	1.562(+3)
1/16	5.6719e-002	4.0430	5.8575e-002	390.4989
1/32	2.8336e-002	1.8208	3.0254e-002	97.6052
1/64	1.3412e-002	1.5446	1.5361e-002	24.4346
1/128	6.1757e-003	1.1839	7.7316e-003	6.2991

Table 3: The maximum absolute errors in solution of example 3

h	$\varepsilon=10^{-3}$		$\varepsilon=10^{-5}$	
	with fitting factor	without fitting factor	with fitting factor	without fitting factor
1/8	3.9346e-002	2.3495	3.9346e-002	6.8742
1/16	2.1781e-002	7.1315e-001	2.1781e-002	6.8494
1/32	1.1552e-002	4.4515e-001	1.1552e-002	5.8527
1/64	5.9684e-003	3.8685e-001	5.9682e-003	3.4058
1/128	3.4448e-003	2.9529e-001	3.0365e-003	9.6955e-001

Table 4: The maximum absolute errors in solution of example 4

h	$\varepsilon=10^{-3}$		$\varepsilon=10^{-5}$	
	with fitting factor	without fitting factor	with fitting factor	without fitting factor
1/8	0	7.7150e+000	0	7.8113e+002
1/16	0	2.0235e+000	0	1.9525e+002
1/32	1.1102e-016	9.1132e-001	0	4.8803e+001
1/64	1.1102e-016	7.7305e-001	0	1.2217e+001
1/128	2.7867e-014	5.9276e-001	0	3.1496e+000

Table 5: The maximum absolute errors in solution of example 5

h	$\varepsilon=10^{-3}$		$\varepsilon=10^{-5}$	
	with fitting factor	without fitting factor	with fitting factor	without fitting factor
1/8	2.0279e-002	1.2380e+000	2.0605e-002	1.3968e+000
1/16	1.0653e-002	1.0426e+000	1.1002e-002	1.4239e+000
1/32	5.2770e-003	8.6750e-001	5.6344e-003	1.4232e+000
1/64	2.4835e-003	7.6510e-001	2.8444e-003	1.3695e+000
1/128	1.0651e-003	5.8979e-001	1.4271e-003	1.2069e+000

Table 6. Numerical order of convergence for examples 1-5

	h	$h/2$	z_h	Order of convergence
Example 1	2^{-3}	2^{-4}	1.4166e-003	
	2^{-4}	2^{-5}	6.8540e-004	1.0474
Example 2	2^{-3}	2^{-4}	7.6904e-003	
	2^{-4}	2^{-5}	3.8452e-003	1.0000
Example 3	2^{-3}	2^{-4}	3.0871e-003	
	2^{-4}	2^{-5}	1.3966e-003	1.1444
Example 4	2^{-3}	2^{-4}	2.7645e-014	
	2^{-4}	2^{-5}	2.5757e-014	1.0202e-001
Example 5	2^{-3}	2^{-4}	1.4187e-003	
	2^{-4}	2^{-5}	6.8606e-004	1.0481