L(d,2,1)-Labeling of Star and Sun Graphs

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Abstract

For positive integer d, L(d,2,1)-labeling of a graph G is a function f from $V(G)$ to the positive integers, $f: V(G) \longrightarrow \{1, 2,\dots\}$ such that $|f(u) - f(v)| \geq d$ if the distance between any 2 vertices u and v is 1 (D(u,v) =1), $|f(u) - f(v)| \geq 2$ if D(u,v) = 2, and $|f(u) - f(v)| \geq 1$ if D(u,v) = 3. The L(d,2,1)-labeling number $k_d(G)$ of a graph G is the smallest positive integer k_d such that G has an L(d,2,1)-labeling with k_d as the maximum label. This paper presents a general k_d-value of stars $K_{1,n}$ and k_d-value of sun graphs S_n for $d = 3$.
Keywords: L(d,2,1)-Labeling, distance, L(d,2,1)-Labeling number, sun graphs, stars.

1. Introduction

Let $G(V,E)$ be a finite, connected, simple and undirected graph, and let V and E denote the vertex set and edge set of G, respectively (Johnsonbaugh, 1986). Wallis (2001) defined a labeling (or valuation) of graph as follows: a labeling of a graph is a map that carries graph elements to the numbers (usually to the positive or non negative integers). The most common choices of domain are the set of all vertices and edges (such labeling is called total labeling), the vertex set alone (vertex labeling), or the edge set alone (edge labeling). In the recent development, although the domain is similar, the graph labeling can also be defined as different function (Galian, 2011).

The channel assignment problem is a problem to assign a channel (positive integer) to each radio station in a set of given stations such that there is no interference between stations and the span of the assigned channel is minimized. The level of interference between any two radio stations correlates with the geographic locations of the stations. Closer stations have a stronger interference, and thus there must be a greater difference between their assigned channels. Robert (1991) proposed a variation of the channel assignment problem in which the radio stations were considered either “close” or “very close”. “Close” stations means vertices with distance two apart on the graph, while “very close” stations means adjacent vertices on the graph.

Grig and Yeh (1992) defined an L(d_1, d_2)-labeling, that is a function $f: V(G) \rightarrow \{1, 2, \dots\}$ such that $|f(u) - f(v)| \geq d_i$ whenever the distance between u and v is i apart, $i \in \{1, 2\}$. The minimum span of any such labeling of G, denote $k_d(G)$, is a minimum largest label used in the labeling. Both Prakosa and Indriati (2009), and Kurniawan and Indriati (2009) have determined k_d-number of L(0,1), L(1,1) and L(1,2) –labelings for star, firecracker, banana-tree, caterpillar and T-pyramide. Practically, interference among channels may go beyond two level. L(3,2,1)-labeling extends from L(2,1)-labeling (Jia-zhuang and Zhen-dong, 2004). Clipperton (2008) has determined L(d,2,1)-labeling number for paths, cycles, complete graphs and complete bipartite graphs. In this paper, we study the L(d,2,1)-labeling, $d \geq 3$ and present a general k_d-value for stars and a k_d-value for suns with $d = 3$.

2. Definitions and Notations

The definitions and notations in this paper are adopted from those used by Wallis (2001) and Clipperton (2008).

Definition 1. *Let $G = (V,E)$ be a graph and f be a mapping $f: V \rightarrow N$. The distance between two such vertices is represented by $D(u,v)$ and the mapping of f is an L(d,2,1)-labeling of G if for all vertices $u, v \in V$,*

$$|f(u)-f(v)| \geq \begin{cases} d, \; if\; D(u,v)=1; \\ 2, \; if\; D(u,v)=2; \\ 1, \; if\; D(u,v)=3. \end{cases}$$

Definition 2. *The L(d,2,1)-number, k_d (G), of a graph G is the smallest natural number k_d such that G has an L(d,2,1)-labeling with k_d as the maximum label.*

Definition 3. *Star $K_{1,n}$ is an n-star, that is a complete bipartite graph with 1 vertex in one set and n vertices in the other set. Two vertices are adjacent if and only if they lie in different sets. The vertex of degree n is called the center, while the vertices of degree 1 are called leaves.*

Definition 4. *An n-sun, S_n, is a cycle C_n with an edge terminating in a vertex of degrees 1 attached to each of n-cycle vertices. The vertices have degree 1 are called leaves and the vertices at the n-cycle are called cycle vertices.*

3. Main Results

In this section, we present a general k_d-value for star $K_{1,n}$, and a k_d-value for sun S_n with d =3.

Theorem 1. *Let $K_{1,n}$ be stars with any positive integer n. Then, for $d \geq 3$, $k_d(K_{1,n}) = 2n + d - 1$.*

Proof. Let f be an L(d,2,1)-labeling. Suppose v is a center of star and v_1, $v_{2,\ldots,}$ v_n are consecutive leaves of star. If 1 is not used as a vertex label in an L(d,2,1)-labeling of a graph, then every vertex label can be decreased by one to obtain another L(d,2,1)-labeling of the graph. Therefore in a minimal L(d,2,1)-labeling, 1 will necessarily appear as a vertex label. There are 2 cases of the place of vertex with label 1.

1. Label 1 is at the center, $f(v) = 1$.

The distance of the center to the leaves is one, then there is a leaf v_i with label $\geq d+1$, for example $f(v_1) \geq d+1$. The distance from a leaf to each other is 2, therefore $f(v_2) \geq d+3$, $f(v_3) \geq d+5$, and the largest label $f(v_n) \geq 2n + d - 1$. Then, the minimum of the largest label, $k_d(K_{1,n}) = 2n + d - 1$.

2. Label 1 is at the leaf, for example $f(v_1) = 1$.

Then, we can give the further label to the other leaves or to the center.

a. If we give the further label to the other leaves, we have $f(v_2) \geq 3$, $f(v_3) \geq 5$, ..., $f(v_n) \geq 2n - 1$. After all the leaves are labeled, finally the center can be labeled with $f(v) \geq 2n + d - 1$. Then, the minimum of the largest label, $k_d(K_{1,n}) = 2n + d - 1$.

b. If we give the further label to the center, we have $f(v) \geq d+1$. Furthermore, we go back to the leaves again and we have $f(v_2) \geq 2d+1$, $f(v_3) \geq 2d+3$, ..., $f(v_n) \geq 2d + 2n - 3$ which is greater than $2n + d - 1$ for $d \geq 3$.

Therefore, if the label 1 is at the leaf, we choose the other leaf for further label until all of leaves are labeled, then, the last label is at the center. ■

Below, we present the k_d of sun graphs Sn with d = 3.

Theorem 2. *Let S_n be sun graphs with any positive integer $n \geq 3$. Then, for d =3, the minimum of the largest label of L(d,2,1)-labeling is*

1. *$k_d(S_n)=10$, for n = 3, 4, 6, 8,*
2. *$k_d(S_n)=11$, for $n\equiv 0 (mod\ 6)$, $n\equiv 2 (mod\ 6)$, $n\equiv 3 (mod\ 6)$, $n \geq 12$,*
3. *$k_d(S_n)=12$, for* $\begin{cases} n = 5 \text{ and } 9 \\ n \equiv 4 (mod\ 6), n \geq 10, n \equiv 5 (mod\ 6),\ n \geq 17 \end{cases}$
4. *$k_d(S_n)=13$, for n =11 and $n \equiv 1 (mod\ 6), n \geq 7$*

Proof. We define the L(d,2,1)-labeling, f, as has been mentioned in Definition 1. Suppose v_1, $v_{2,\ \ldots,}$ v_n be a consecutive cycle vertices. The leaf adjacent with v_n is v_{n+1}, while the leaf adjacent with v_1 is v_{n+2}, and the consecutive indexes of leaves are v_{n+1}, v_{n+2}, ..., v_{2n}. Let g be an f mapping with as minimally as possible label. To obtain the minimal possible label, firstly we consider to the vertices with longest distance, it means to the vertices with distance 3. After that or if there aren't those vertices, we consider to the vertices with shorter distance. The process will be continued until all of the vertices are labeled.

Case 1. For n = 3, 4, 6, 8.

We start the g-labeling with the label 1. There are two possibilities of g.

a. If the label 1 is in any cycle vertex, for example in v_1. Then we obtain

for n = 3: $\{g(v_1), g(v_2), ..., g(v_6)\} = \{1, 7, 10, 4, 5, 3\}$,
for n = 4: $\{g(v_1), g(v_2), ..., g(v_8)\} = \{1, 10, 7, 4, 9, 6, 3, 2\}$,
for n = 6: $\{g(v_1), g(v_2), ..., g(v_{12})\} = \{1, 6, 9, 4, 7, 10, 3, 4, 3, 2, 1, 2\}$,
for n = 8: $\{g(v_1), g(v_2), ..., g(v_{16})\} = \{1, 6, 9, 4, 7, 10, 5, 8, 3, 4, 3, 2, 1, 2, 1, 2\}$.

b. If the label 1 is in any leaf, for example in v_4, v_5, v_7, v_9 for n = 3, 4, 6, 8 respectively. We obtain
for n = 3: $\{g(v_1), g(v_2), ..., g(v_6)\} = \{8, 11, 5, 1, 2, 3\}$,
for n = 4: $\{g(v_1), g(v_2), ..., g(v_8)\} = \{5, 2, 11, 8, 1, 10, 7, 4\}$,
for n = 6: $\{g(v_1), g(v_2), ..., g(v_{12})\} = \{5, 10, 7, 4, 11, 8, 1, 2, 1, 2, 1, 2\}$,
for n = 8: $\{g(v_1), g(v_2), ..., g(v_{16})\} = \{5, 8, 11, 4, 9, 12, 7, 10, 1, 2, 1, 2, 1, 2, 1, 2\}$.

From those two possibilities, we conclude that the minimum of the largest label, $k_d(S_n)= 10$, which is occured from the g-labeling with the starting label 1 is in any cycle vertex.

Case 2. For $n \equiv 0(mod\ 6)$, $n \equiv 2(mod\ 6)$, $n \equiv 3(mod\ 6)$, $n \geq 12$.

The minimum of the largest label, $k_d(S_n)$, can be obtained from the g-labeling with the starting label 1 is in any cycle vertex, for example in v_1. The formula of the labeling is as follows.

$g(v_1) = 1$,

$g(v_{l+1}) = 3l + 3$, for $l = 1, 2$,

$$g(v_{6l-2}) = 4, \text{ for } l = \begin{cases} 1, 2, \dots, \frac{n}{6}, \text{ with } n \equiv 0(mod\ 6) \\ 1, 2, \dots, \frac{n+4}{6}, \text{with } n \equiv 2(mod\ 6) \\ 1, 2, \dots, \frac{n-3}{6}, \text{with } n \equiv 3(mod\ 6) \end{cases}$$

$$g(v_{6l-1}) = 7, \text{ for } l = \begin{cases} 1, 2, \dots, \frac{n}{6}, \text{ with } n \equiv 0(mod\ 6) \\ 1, 2, \dots, \frac{n-2}{6}, \text{with } n \equiv 2(mod\ 6) \\ 1, 2, \dots, \frac{n-3}{6}, \text{with } n \equiv 3(mod\ 6) \end{cases}$$

$$g(v_{6l}) = 10, \text{ for } l = \begin{cases} 1, 2, \dots, \frac{n}{6}, \text{ with } n \equiv 0(mod\ 6) \\ 1, 2, \dots, \frac{n-2}{6}, \text{with } n \equiv 2(mod\ 6) \\ 1, 2, \dots, \frac{n-3}{6}, \text{with } n \equiv 3(mod\ 6) \end{cases}$$

$$g(v_{6l+1}) = 5, \text{ for } l = \begin{cases} 1, 2, \dots, \frac{n}{6} - 1, \text{ with } n \equiv 0(mod\ 6) \\ 1, 2, \dots, \frac{n-2}{6}, \text{with } n \equiv 2(mod\ 6) \\ 1, 2, \dots, \frac{n-3}{6}, \text{with } n \equiv 3(mod\ 6) \end{cases}$$

$$g(v_{6l+2}) = 8, \text{ for } l = \begin{cases} 1, 2, \dots, \frac{n}{6}, \text{ with } n \equiv 0(mod\ 6) \\ 1, 2, \dots, \frac{n-2}{6}, \text{with } n \equiv 2(mod\ 6) \\ 1, 2, \dots, \frac{n-3}{6}, \text{with } n \equiv 3(mod\ 6) \end{cases}$$

$$g(v_{6l+3}) = 11, \text{ for } l = \begin{cases} 1, 2, \dots, \frac{n}{6} - 1, \text{ with } n \equiv 0(mod\ 6) \\ 1, 2, \dots, \frac{n-8}{6}, \text{with } n \equiv 2(mod\ 6) \\ 1, 2, \dots, \frac{n-3}{6}, \text{with } n \equiv 3(mod\ 6) \end{cases}$$

$$g(v_{n+l}) = \begin{cases} 3, \text{for } l = 1, 3, \text{with } n \equiv 0(mod\ 6), n \equiv 2(mod\ 6), n \equiv 3(mod\ 6) \\ 4, \text{ for } l = 2, \text{ with } n \equiv 3(mod\ 6) \end{cases}$$

$$g(v_{n+2l+i}) = 2, \text{ for } i = 2, \ l = \begin{cases} 1, 2, \dots, \frac{n-2}{2}, \text{ with } n \equiv 0(mod\ 6), n \equiv 2(mod\ 6) \\ 1, 2, \dots, \frac{n-3}{2}, \text{with } n \equiv 3(mod\ 6) \end{cases}$$

$$g(v_{n+2l+i}) = 1, \text{ for } i = 3, \ l = \begin{cases} 1, 2, \dots, \frac{n-4}{2}, \text{ with } n \equiv 0(mod\ 6), n \equiv 2(mod\ 6) \\ 1, 2, \dots, \frac{n-3}{2}, \text{with } n \equiv 3(mod\ 6) \end{cases}$$

From the labeling in Case 2, we conclude that the minimum of the largest label, $k_d(S_n)= 10$, which is occured from the g-labeling with the starting label 1 is in any cycle vertex.

Case 3a. For n = 5 and 9.

We start the g-labeling with the label 1. There are two possibilities of g.

a. If the label 1 is in any cycle vertex, for example in v_1. Then we obtain
   for n = 5: $\{g(v_1), g(v_2), ..., g(v_{10})\}$ = {1, 8, 12, 6, 10, 4, 5, 3, 2, 3},
   for n = 9: $\{g(v_1), g(v_2), ..., g(v_{18})\}$ = {1, 7, 10, 4, 8, 11, 5, 9, 12, 4, 5, 3, 2, 1, 2, 1, 2, 3},
b. If the label 1 is in any leaf, for example in v_6 or *in* v_{10} for n = 5 and 9, respectively. We obtain
   for n = 5: $\{g(v_1), g(v_2), ..., g(v_{10})\}$ = {5, 8, 12, 6, 10, 1, 2, 1, 2, 3},
   for n = 9: $\{g(v_1), g(v_2), ..., g(v_{18})\}$ = {5, 10, 7, 4, 12, 9, 6, 11, 8, 1, 2, 1, 2, 1, 2, 1, 2, 3},

From those two possibilities, we conclude that the minimum of the largest label, $k_d(S_n)= 12$, which is occured from the g-labeling with the starting label 1 is in any vertex for n = 5 and n = 9.

Case 3b. For $n \equiv 4 (mod\ 6)$, $n \geq 10$ and $n \equiv 5 (mod\ 6)$, $n \geq 17$.

The minimum of the largest label, $k_d(S_n)$, can be obtained from the g-labeling with the starting label 1 is in any cycle vertex, for example in v_1. The formula of the labeling is as follows.

$g(v_1) = 1$,

$g(v_{6l-4}) = 10$, for $l = 1, 2, \ldots, \frac{n-4}{6}$, with $n \equiv 4 (mod\ 6)$

$g(v_{6l-3}) = 7$, for $l = 1, 2, \ldots, \frac{n-4}{6}$, with $n \equiv 4 (mod\ 6)$

$$g(v_{6l-2}) = 4, \text{ for } l = \begin{cases} 1, 2, \ldots, \frac{n-4}{6}, \text{with } n \equiv 4 (mod\ 6) \\ 1, 2, \ldots, \frac{n-5}{6}, \text{with } n \equiv 5 (mod\ 6) \end{cases}$$

$g(v_{6l-1}) = 11$, for $l = 1, 2, \ldots, \frac{n-4}{6}$, with $n \equiv 4 (mod\ 6)$

$g(v_{6l}) = 8$, for $l = 1, 2, \ldots, \frac{n-4}{6}$, with $n \equiv 4 (mod\ 6$

$$g(v_{6l+1}) = 5, \text{ for } l = \begin{cases} 1, 2, \ldots, \frac{n-4}{6}, \text{with } n \equiv 4 (mod\ 6) \\ 1, 2, \ldots, \frac{n-5}{6}, \text{with } n \equiv 5 (mod\ 6) \end{cases}$$

$g(v_{n-l+1}) = 3l + 3$, for l = 1, 2, 3, with $n \equiv 4 (mod\ 6)$.

$g(v_{n+l}) = 3$, for l = 1, 3, with $n \equiv 4 (mod\ 6)$, $n \equiv 5 (mod\ 6)$.

$g(v_{n+2}) = 4$, with $n \equiv 4 (mod\ 6)$, $n \equiv 5 (mod\ 6)$.

$$g(v_{n+2l+i}) = \begin{cases} 2, \text{for } i = 2, l = \begin{cases} 1, 2, \ldots, \frac{n-3}{2}; \text{ with } n \equiv 5 (mod\ 6) \\ 1, 2, \ldots, \frac{n-2}{2}; \text{ with } n \equiv 4 (mod\ 6) \end{cases} \\ 1, \text{for } i = 3, l = \begin{cases} 1, 2, \ldots, \frac{n-3}{2}; \text{with } n \equiv 5 (mod\ 6) \\ 1, 2, \ldots, \frac{n-4}{2}, \text{with } n \equiv 4 (mod\ 6) \end{cases} \end{cases}$$

$g(v_{3l-1}) = l + 6$, for l = 1, 2, 3, with $n \equiv 5 (mod\ 6)$.

$g(v_{3l}) = l + 9$, for l = 1, 2, 3, with $n \equiv 5 (mod\ 6)$.

$$g(v_{6l+3i+2}) = \begin{cases} 7, \text{ for } i = 1, \ l = 1, 2, \ldots, \frac{n-11}{6}, \text{ with } n \equiv 5 (mod\ 6), \\ 8, \text{ for } i = 2, \ l = 1, 2, \ldots, \frac{n-11}{6}, \text{ with } n \equiv 5 (mod\ 6) \end{cases}$$

$$g(v_{6l+3i}) = \begin{cases} 10, \text{ for } i = 2, \ l = 1, 2, \ldots, \frac{n-11}{6}, \text{ with } n \equiv 5 (mod\ 6), \\ 11, \text{ for } i = 3, \ l = 1, 2, \ldots, \frac{n-11}{6}, \text{ with } n \equiv 5 (mod\ 6). \end{cases}$$

$g(v_{n+l-2}) = 3l + 3$, for l = 1, 2, with $n \equiv 5 (mod\ 6)$.

From the labeling in Case 3b, we conclude that the minimum of the largest label, $k_d(S_n)$= 12, which is occured from the g-labeling with the starting label 1 is in any cycle vertex.

Case 4. For n = 11 and $n \equiv 1 (mod\ 6)$, $n \geq 7$

<u>For n = 11</u>

We start the g-labeling with the label 1. There are two possibilities of g.

a. If the label 1 is in any cycle vertex, for example in v_1. Then we obtain $\{g(v_1), g(v_2), ..., g(v_{22})\}$ = {1, 7, 11, 4, 8, 12, 5, 9, 13, 6, 10, 3, 4, 3, 2, 1, 2, 1, 2, 1, 2, 1}.
b. If the label 1 is in any leaf, for example in v_{12}, we obtain $\{g(v_1), g(v_2), ..., g(v_{22})\}$ = {5, 8, 11, 4, 7, 12, 5, 9, 13, 7, 10, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 3}.

From those two possibilities, we conclude that the minimum of the largest label, $k_d(S_n)$= 13, which is occured from the g-labeling with the starting label 1 is in any vertex.

For $n \equiv 1 (mod\ 6), n \geq 7$

The minimum of the largest label, $k_d(S_n)$, can be obtained from the g-labeling with the starting label 1 is in any cycle vertex, for example in v_1. The formula of the labeling is as follows.

$g(v_1) = g(v_{n+2l+3}) = 1$, for $l = 1, 2, \dots, \frac{n-5}{2}$

$g(v_{n+2l+2}) = 2$, for $l = 1, 2, \dots, \frac{n-3}{2}$

$g(v_{nl-3l+6}) = 3$, for $l = 1, 2$

$g(v_{n+1}) = g(v_{6l+1}) = 4$, for $l = 1, 2, \dots, \frac{n-1}{6}$

$g(v_{n+2}) = g(v_{6l-2}) = 5$, for $l = 1, 2, \dots, \frac{n-7}{6}, n \geq 13$

$g(v_n) = g(v_{6l+3}) = 7$, for $l = 1, 2, \dots, \frac{n-7}{6}, n \geq 13$

$g(v_3) = g(v_{6l+6}) = 8$, for $l = 1, 2, \dots, \frac{n-13}{6}, n \geq 19$

$g(v_6) = 9$, for $n \neq 7$

$g(v_{n-1}) = g(v_{6l+2}) = 10$, for $l = 1, 2, \dots, \frac{n-7}{6}, n \geq 13$

$g(v_2) = g(v_{6l+5}) = 11$, for $l = 1, 2, \dots, \frac{n-13}{6}, n \geq 19$

$g(v_5) = 12$, for $n \neq 7$

$g(v_{n-2}) = 13$, for $n \neq 7$

From the labeling in Case 4, we conclude that the minimum of the largest label, $k_d(S_n)= 13$, which is occured from the g-labeling with the starting label 1 is in any cycle vertex.

From the above 4 cases, the complete proof has been done.

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